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Supplementary materials for

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1 Proof of Theorem 1

From Eqs. (13) and (15), the filtering error covariance matrices of unknown inputs and system states can be derived as follows:

$$\begin{aligned}
&P_{i,d}(l_t)\\
=&\mathbb{E}\{\tilde{d}_i(l_t)\tilde{d}_i^{\mathrm{T}}(l_t)\}\\
=&\mathbb{E}\{(-L_i(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{A}(l_t)\tilde{x}_i(l_t|l_t)+\mathcal{C}_i(l_{t+1})\bar{\mathcal{A}}(l_t)\vec{\omega}(l_t)+\nu_i(l_{t+1})\\
&-\sigma_i(l_{t+1})))(-L_i(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{A}(l_t)\tilde{x}_i(l_t|l_t)\\
&+\mathcal{C}_i(l_{t+1})\bar{\mathcal{A}}(l_t)\vec{\omega}(l_t)+\nu_i(l_{t+1})-\sigma_i(l_{t+1})))^{\mathrm{T}}\}\\
=&(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\mathbb{E}\{\tilde{x}_i(l_t|l_t)\tilde{x}_i^{\mathrm{T}}(l_t|l_t)\}\mathcal{A}^{\mathrm{T}}(l_t)(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}}\\
&+(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\mathbb{E}\{\vec{\omega}(l_t)\vec{\omega}^{\mathrm{T}}(l_t)\}\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}}\\
&+L_i(l_{t+1})\mathbb{E}\{\nu_i(l_{t+1})\nu_i^{\mathrm{T}}(l_{t+1})\}L_i^{\mathrm{T}}(l_{t+1})+L_i(l_{t+1})\mathbb{E}\{\sigma_i(l_{t+1})\sigma_i^{\mathrm{T}}(l_{t+1})\}L_i^{\mathrm{T}}(l_{t+1})\\
&+\varUpsilon_1+\varUpsilon_1^{\mathrm{T}}+\varUpsilon_2+\varUpsilon_2^{\mathrm{T}}+\varUpsilon_3+\varUpsilon_3^{\mathrm{T}}+\varUpsilon_4+\varUpsilon_4^{\mathrm{T}}+\varUpsilon_5+\varUpsilon_5^{\mathrm{T}}+\varUpsilon_6+\varUpsilon_6^{\mathrm{T}},
\end{aligned}\tag{S1}$$

and

$$\begin{aligned}
&P_{i,x}(l_{t+1}|l_{t+1})\\
=&\mathbb{E}\{\tilde{x}_i(l_{t+1}|l_{t+1})\tilde{x}_i^{\mathrm{T}}(l_{t+1}|l_{t+1})\}\\
=&\mathbb{E}\{((I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))(\mathcal{A}(l_t)\tilde{x}_i(l_t|l_t)+\bar{\mathcal{A}}(l_t)\vec{\omega}(l_t))-N_i(l_{t+1})(\nu_i(l_{t+1})\\
&-\sigma_i(l_{t+1})))((I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))(\mathcal{A}(l_t)\tilde{x}_i(l_t|l_t)\\
&+\bar{\mathcal{A}}(l_t)\vec{\omega}(l_t))-N_i(l_{t+1})(\nu_i(l_{t+1})-\sigma_i(l_{t+1})))^{\mathrm{T}}\}\\
=&(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\mathbb{E}\{\tilde{x}_i(l_t|l_t)\tilde{x}_i^{\mathrm{T}}(l_t|l_t)\}\mathcal{A}^{\mathrm{T}}(l_t)(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}}\\
&+(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\mathbb{E}\{\vec{\omega}(l_t)\vec{\omega}^{\mathrm{T}}(l_t)\}\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}}\\
&+N_i(l_{t+1})\mathbb{E}\{\nu_i(l_{t+1})\nu_i^{\mathrm{T}}(l_{t+1})\}N_i^{\mathrm{T}}(l_{t+1})+N_i(l_{t+1})\mathbb{E}\{\sigma_i(l_{t+1})\sigma_i^{\mathrm{T}}(l_{t+1})\}N_i^{\mathrm{T}}(l_{t+1})\\
&+\varUpsilon_7+\varUpsilon_7^{\mathrm{T}}+\varUpsilon_8+\varUpsilon_8^{\mathrm{T}}+\varUpsilon_9+\varUpsilon_9^{\mathrm{T}}+\varUpsilon_{10}+\varUpsilon_{10}^{\mathrm{T}}+\varUpsilon_{11}+\varUpsilon_{11}^{\mathrm{T}}+\varUpsilon_{12}+\varUpsilon_{12}^{\mathrm{T}},
\end{aligned}\tag{S2}$$

where

$$\begin{aligned}
\Upsilon_1 \triangleq&(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\mathbb{E}\{\tilde{x}_i(l_t|l_t)\vec{\omega}^{\mathrm{T}}(l_t)\}\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}},\\
\Upsilon_2 \triangleq&(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\mathbb{E}\{\tilde{x}_i(l_t|l_t)\nu_i^{\mathrm{T}}(l_{t+1})\}L_i^{\mathrm{T}}(l_{t+1}),\\
\Upsilon_3 \triangleq&-(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\mathbb{E}\{\tilde{x}_i(l_t|l_t)\sigma_i^{\mathrm{T}}(l_{t+1})\}L_i^{\mathrm{T}}(l_{t+1}),\\
\Upsilon_4 \triangleq&(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\mathbb{E}\{\vec{\omega}(l_t)\nu_i^{\mathrm{T}}(l_{t+1})\}L_i^{\mathrm{T}}(l_{t+1}),\\
\Upsilon_5 \triangleq&-(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\mathbb{E}\{\vec{\omega}(l_t)\sigma_i^{\mathrm{T}}(l_{t+1})\}L_i^{\mathrm{T}}(l_{t+1}),\\
\Upsilon_6 \triangleq&-L_i(l_{t+1})\mathbb{E}\{\nu_i(l_{t+1})\sigma_i^{\mathrm{T}}(l_{t+1})\}L_i^{\mathrm{T}}(l_{t+1}),\\
\Upsilon_7 \triangleq&(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\mathbb{E}\{\tilde{x}_i(l_t|l_t)\vec{\omega}^{\mathrm{T}}(l_t)\}\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}},\\
\Upsilon_8 \triangleq&-(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\mathbb{E}\{\tilde{x}_i(l_t|l_t)\nu_i^{\mathrm{T}}(l_{t+1})\}N_i^{\mathrm{T}}(l_{t+1}),\\
\Upsilon_9 \triangleq&(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\mathbb{E}\{\tilde{x}_i(l_t|l_t)\sigma_i^{\mathrm{T}}(l_{t+1})\}N_i^{\mathrm{T}}(l_{t+1}),\\
\Upsilon_{10} \triangleq&-(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\mathbb{E}\{\vec{\omega}(l_t)\nu_i^{\mathrm{T}}(l_{t+1})\}N_i^{\mathrm{T}}(l_{t+1}),\\
\Upsilon_{11} \triangleq&(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\mathbb{E}\{\vec{\omega}(l_t)\sigma_i^{\mathrm{T}}(l_{t+1})\}N_i^{\mathrm{T}}(l_{t+1}),\\
\Upsilon_{12} \triangleq&-N_i(l_{t+1})\mathbb{E}\{\nu_i(l_{t+1})\sigma_i^{\mathrm{T}}(l_{t+1})\}N_i^{\mathrm{T}}(l_{t+1}).
\end{aligned}$$

It follows immediately from

$$\begin{aligned}
&\mathbb{E}\{\tilde{x}_i(l_t|l_t)\vec{\omega}^{\mathrm{T}}(l_t)\}=0,\\
&\mathbb{E}\{\tilde{x}_i(l_t|l_t)\nu_i^{\mathrm{T}}(l_{t+1})\}=0,\\
&\mathbb{E}\{\vec{\omega}(l_t)\nu_i^{\mathrm{T}}(l_{t+1})\}=0,\\
&\mathbb{E}\{\vec{\omega}(l_t)\sigma_i^{\mathrm{T}}(l_{t+1})\}=0,
\end{aligned} \tag{S3}$$

that $\Upsilon_1=0,\Upsilon_2=0,\Upsilon_4=0,\Upsilon_5=0,\Upsilon_7=0,\Upsilon_8=0,\Upsilon_{10}=0,\Upsilon_{11}=0$. Nevertheless, the two other cross terms $\mathbb{E}\{\tilde{x}_i(l_t|l_t)\sigma_i^{\mathrm{T}}(l_{t+1})\}$ and $\mathbb{E}\{\nu_i(l_{t+1})\sigma_i^{\mathrm{T}}(l_{t+1})\}$ are not equal to zero, which requires further discussion.

According to the definitions of $\sigma_i(l_{t+1})$ and δ_i, one can easily obtain

$$\begin{aligned}
&\mathbb{E}\{\nu_i(l_{t+1})\sigma_i^{\mathrm{T}}(l_{t+1})\}\\
=&\mathbb{E}\{\nu_i(l_{t+1})(y_i(l_{t+1})-y_i(s_{\tau_j^i}))^{\mathrm{T}}\}\\
=&\mathbb{E}\{\nu_i(l_{t+1})(\mathcal{C}_i(l_{t+1})x(l_{t+1})+\nu_i(l_{t+1})-y_i(s_{\tau_j^i}))^{\mathrm{T}}\}\\
=&\delta_i V_i(l_{t+1}).
\end{aligned}$$

Besides, in light of the adaptive event-triggered mechanism, it is not difficult to find that

$$\sigma_i(l_{t+1})\sigma_i^{\mathrm{T}}(l_{t+1})\leq\sigma_i^{\mathrm{T}}(l_{t+1})\sigma_i(l_{t+1})I\leq\bar{\rho}I. \tag{S4}$$

By means of Lemma 1, one obtains

$$\begin{aligned}
\Upsilon_3+\Upsilon_3^{\mathrm{T}}\leq&\xi_i(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)P_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}}\\
&+\xi_i^{-1}\bar{\rho}L_i(l_{t+1})L_i^{\mathrm{T}}(l_{t+1})\\
\Upsilon_6+\Upsilon_6^{\mathrm{T}}\leq&-2\delta_i L_i(l_{t+1})V_i(l_{t+1})L_i^{\mathrm{T}}(l_{t+1}),\\
\Upsilon_9+\Upsilon_9^{\mathrm{T}}\leq&\varepsilon_i(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)P_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}}\\
&+\varepsilon_i^{-1}\bar{\rho}N_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1})\\
\Upsilon_{12}+\Upsilon_{12}^{\mathrm{T}}\leq&-2\delta_i N_i(l_{t+1})V_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1}).
\end{aligned}$$

Taking them into account, one further acquires

$$\begin{aligned}
P_{i,d}(l_t) \leq &(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)P_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} \\
&+ (L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} \\
&+ L_i(l_{t+1})V_i(l_{t+1})L_i^{\mathrm{T}}(l_{t+1}) + \bar{\rho}L_i(l_{t+1})L_i^{\mathrm{T}}(l_{t+1}) \\
&+ \Upsilon_3 + \Upsilon_3^{\mathrm{T}} + \Upsilon_6 + \Upsilon_6^{\mathrm{T}} \\
\leq &(1+\xi_i)(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\Xi_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} \\
&+ (L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(L_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} \\
&+ (1-2\delta_i)L_i(l_{t+1})V_i(l_{t+1})L_i^{\mathrm{T}}(l_{t+1}) + (1+\xi_i^{-1})\bar{\rho}L_i(l_{t+1})L_i^{\mathrm{T}}(l_{t+1}) \\
\triangleq &\Xi_{i,d}(l_t)
\end{aligned}$$

and

$$\begin{aligned}
P_{i,x}(l_{t+1}|l_{t+1}) \leq &(I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)P_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)(I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} \\
&+ (I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} \\
&+ N_i(l_{t+1})V_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1}) + \bar{\rho}N_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1}) \\
&+ \Upsilon_9 + \Upsilon_9^{\mathrm{T}} + \Upsilon_{12} + \Upsilon_{12}^{\mathrm{T}} \\
\leq &(1+\varepsilon_i)(I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\Xi_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)(I - N_i(l_{t+1}) \\
&\cdot \mathcal{C}_i(l_{t+1}))^{\mathrm{T}} + (I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t) \\
&\cdot (I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} + (1-2\delta_i)N_i(l_{t+1})V_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1}) \\
&+ (1+\varepsilon_i^{-1})\bar{\rho}N_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1}) \\
\triangleq &\Xi_{i,x}(l_{t+1}|l_{t+1}),
\end{aligned}$$

which mean

$$\begin{aligned}
P_{i,d}(l_t) &\leq \Xi_{i,d}(l_t), \\
P_{i,x}(l_{t+1}|l_{t+1}) &\leq \Xi_{i,x}(l_{t+1}|l_{t+1}).
\end{aligned} \tag{S5}$$

The proof is now completed.

2 Proof of Theorem 2

It follows from Eq. (17) that the upper bound $\Xi_{i,d}(l_t)$ of the filtering error covariance of unknown inputs can be rewritten into a more compact form:

$$\Xi_{i,d}(l_t) = L_i(l_{t+1})\Theta_i(l_{t+1})L_i(l_{t+1})^{\mathrm{T}}. \tag{S6}$$

In what follows, under constraint (13), a Lagrange multiplier $\Lambda_i(l_{t+1})$ is introduced to construct a new function

$$\begin{aligned}
&\mathcal{H}(L_i(l_{t+1}), \Lambda_i(l_{t+1})) \\
=&L_i(l_{t+1})\Theta_i(l_{t+1})L_i(l_{t+1})^{\mathrm{T}} + \Lambda_i(l_{t+1})(I - L_i(l_{t+1})\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}} \\
&+ (I - L_i(l_{t+1})\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))\Lambda_i^{\mathrm{T}}(l_{t+1}).
\end{aligned} \tag{S7}$$

Using the completing-the-square method, one can further achieve

$$\begin{aligned}&\mathcal{H}(L_i(l_{t+1}),\varLambda_i(l_{t+1}))\\=&L_i(l_{t+1})\varTheta_i(l_{t+1})L_i(l_{t+1})^{\mathrm{T}}-\varLambda_i(l_{t+1})(L_i(l_{t+1})\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\\&-(L_i(l_{t+1})\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))\varLambda_i^{\mathrm{T}}(l_{t+1})+\varLambda_i(l_{t+1})+\varLambda_i^{\mathrm{T}}(l_{t+1})\\&+\varLambda_i(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\varTheta_i^{-1}(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))\varLambda_i^{\mathrm{T}}(l_{t+1})\\&-\varLambda_i(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\varTheta_i^{-1}(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))\varLambda_i^{\mathrm{T}}(l_{t+1})\\=&(L_i(l_{t+1})-\varLambda_i(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\varTheta_i^{-1}(l_{t+1}))\varTheta_i(l_{t+1})(L_i(l_{t+1})\\&-\varLambda_i(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\varTheta_i^{-1}(l_{t+1}))^{\mathrm{T}}+\varLambda_i(l_{t+1})+\varLambda_i^{\mathrm{T}}(l_{t+1})\\&-\varLambda_i(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\varTheta_i^{-1}(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))\varLambda_i^{\mathrm{T}}(l_{t+1}).\end{aligned}$$

It can be seen that $\mathcal{H}(L_i(l_{t+1}),\varLambda_i(l_{t+1}))$ is minimized when the gain $L_i(l_{t+1})$ is taken as

$$L_i(l_{t+1})=\varLambda_i(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\varTheta_i^{-1}(l_{t+1}).$$

Substituting it into constraint (14), we have

$$\varLambda_i(l_{t+1})=((\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\varTheta_i^{-1}(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t)))^{-1}.$$

Obviously, the upper bound (17) takes the minimum value when the inequality in problem (25) holds.

On the other hand, $\varXi_{i,x}(l_{t+1}|l_{t+1})$ can be rewritten as

$$\begin{aligned}&\varXi_{i,x}(l_{t+1}|l_{t+1})\\=&(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))((1+\varepsilon_i)\mathcal{A}(l_t)\varXi_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)+\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\\&\cdot\bar{\mathcal{A}}^{\mathrm{T}}(l_t))(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}}+N_i(l_{t+1})((1-2\delta_i)V_i(l_{t+1})\\&+(1+\varepsilon_i^{-1})\bar{\rho}I)N_i^{\mathrm{T}}(l_{t+1})\\=&(1+\varepsilon_i)\mathcal{A}(l_t)\varXi_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)+\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t)-N_i(l_{t+1})\mathcal{C}_i(l_{t+1})\\&\cdot((1+\varepsilon_i)\mathcal{A}(l_t)\varXi_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)+\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t))\\&-((1+\varepsilon_i)\mathcal{A}(l_t)\varXi_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)+\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t))\mathcal{C}_i^{\mathrm{T}}(l_{t+1})N_i^{\mathrm{T}}(l_{t+1})\\&+N_i(l_{t+1})\mathcal{C}_i(l_{t+1})((1+\varepsilon_i)\mathcal{A}(l_t)\varXi_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)\\&+\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t))\mathcal{C}_i^{\mathrm{T}}(l_{t+1})N_i^{\mathrm{T}}(l_{t+1})\\&+N_i(l_{t+1})((1-2\delta_i)V_i(l_{t+1})+(1+\varepsilon_i^{-1})\bar{\rho}I)N_i^{\mathrm{T}}(l_{t+1})\\=&\varPhi_i(l_t)-N_i(l_{t+1})\mathcal{C}_i(l_{t+1})\varPhi_i(l_t)-\varPhi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})N_i^{\mathrm{T}}(l_{t+1})\\&+N_i(l_{t+1})\varOmega_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1}).\end{aligned}$$

Following the same methodology for function $\mathcal{H}(L_i(l_{t+1}),\varLambda_i(s_{k+1}))$, one has

$$\begin{aligned}&\varXi_{i,x}(l_{t+1}|l_{t+1})\\=&\,(N_i(l_{t+1})-\varPhi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\varOmega_i^{-1}(l_{t+1}))\varOmega_i(l_{t+1})(N_i(l_{t+1})\\&-\varPhi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\varOmega_i^{-1}(l_{t+1}))^{\mathrm{T}}+\varPhi_i(l_t)-\varPhi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\\&\cdot\varOmega_i^{-1}(l_{t+1})\mathcal{C}_i(l_{t+1})\varPhi_i(l_t).\end{aligned}$$

It is easy to see that $\varXi_{i,x}(l_{t+1}|l_{t+1})$ is minimized when

$$N_i(l_{t+1})=\varPhi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\varOmega_i^{-1}(l_{t+1}).\tag{S8}$$

Noting the definition of $N_i(l_{t+1})$ in Eq. (S8) and the value of $L_i(s_{k+1})$ in Eq. (19), one has

$$\begin{aligned}&\varPhi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\varOmega_i^{-1}(l_{t+1})\\=&K_i(l_{t+1})+(I-K_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{B}(l_t)L_i(l_{t+1}).\end{aligned}\tag{S9}$$

Furthermore, we have

$$\begin{aligned}&K_i(l_{t+1})\\=&(\Phi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\Omega_i^{-1}(l_{t+1})-\mathcal{B}(l_t)L_i(l_{t+1}))(I-\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t)L_i(l_{t+1}))^{\dagger}\\=&\big\{\Phi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\Omega_i^{-1}(l_{t+1})-\mathcal{B}(l_t)(\Theta_i^{cb}(l_{t+1}))^{-1}(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\\&\cdot\Theta_i^{-1}(l_{t+1})\big\}(I-\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t)(\Theta_i^{cb}(l_{t+1}))^{-1}(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\Theta_i^{-1}(l_{t+1})))^{\dagger}.\end{aligned}$$

Finally, it is not difficult to see that the upper bound $\Xi_{i,x}(l_{t+1}|l_{t+1})$ is minimized when the above equation holds. The proof of Theorem 2 is completed.

3 Proof of Theorem 3

For convenience, we first define the following notations:

$$\begin{aligned}\mathcal{A}_{i,1}(l_{t+1})=&(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t),\\\mathcal{A}_{i,2}(l_{t+1})=&(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t),\\f_i(l_{t+1})=&\mathcal{A}_{i,2}(l_{t+1})\vec{\omega}(l_t)-N_i(l_{t+1})(\nu_i(l_{t+1})-\sigma_i(l_{t+1})),\\Z(l_{t+1})=&(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(I-N_i(l_{t+1})\\&\cdot\mathcal{C}_i(l_{t+1}))^{\mathrm{T}}+(1-2\delta_i)N_i(l_{t+1})V_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1})\\&+(1+\varepsilon_i^{-1})\bar{\rho}N_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1}).\end{aligned}$$

Obviously, the filtering error dynamics (16) can be rewritten as

$$\tilde{x}_i(l_{t+1}|l_{t+1})=\mathcal{A}_{i,1}(l_{t+1})\tilde{x}_i(l_t|l_t)+f_i(l_{t+1}),\tag{S10}$$

and the upper bound in Eq. (18) can be rewritten as

$$\begin{aligned}\Xi_{i,x}(l_{t+1}|l_{t+1})\triangleq&(1+\varepsilon_i)(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\Xi_{i,x}(l_t|l_t)\mathcal{A}^{\mathrm{T}}(l_t)\\&\cdot(I-N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}}+Z(l_{t+1})\\=&(1+\varepsilon_i)\mathcal{A}_{i,1}(l_t)\Xi_{i,x}(l_t|l_t)\mathcal{A}_{i,1}^{\mathrm{T}}(l_t)+Z(l_{t+1}).\end{aligned}\tag{S11}$$

In what follows, we introduce a quadratic function

$$V_{i,k}(\tilde{x}_i(l_t|l_t))=\tilde{x}_i^{\mathrm{T}}(l_t|l_t)\Xi_{i,x}^{-1}(l_t|l_t)\tilde{x}_i(l_t|l_t).$$

Then we have

$$\begin{aligned}V_{i,k}(\tilde{x}_i(l_{t+1}|l_{t+1}))=&\tilde{x}_i^{\mathrm{T}}(l_{t+1}|l_{t+1})\Xi_{i,x}^{-1}(l_{t+1}|l_{t+1})\tilde{x}_i(l_{t+1}|l_{t+1})\\=&(\mathcal{A}_{i,1}(l_t)\tilde{x}_i(l_t|l_t)+f_i(l_{t+1}))^{\mathrm{T}}\Xi_{i,x}^{-1}(l_{t+1}|l_{t+1})(\mathcal{A}_{i,1}(l_t)\tilde{x}_i(l_t|l_t)+f_i(l_{t+1})).\end{aligned}$$

According to Eq. (19), Eq. (20), and inequality (22), we have

$$\begin{aligned}&\parallel L_i(l_{t+1})\parallel\\=&\parallel((\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\Theta_i^{-1}(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t)))^{-1}\\&\cdot(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\Theta_i^{-1}(l_{t+1})\parallel\\\leq&\frac{\bar{\theta}\bar{c}\bar{b}}{\underline{c}^2\underline{b}^2\underline{\theta}}\triangleq\bar{l},\\&\parallel L_i(l_{t+1})\parallel\\=&\parallel((\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\Theta_i^{-1}(l_{t+1})(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t)))^{-1}\\&\cdot(\mathcal{C}_i(l_{t+1})\mathcal{B}(l_t))^{\mathrm{T}}\Theta_i^{-1}(l_{t+1})\parallel\\\geq&\frac{\underline{\theta}\underline{c}\underline{b}}{\bar{c}^2\bar{b}^2\bar{\theta}}\triangleq\underline{l},\end{aligned}\tag{S12}$$

and

$$\begin{aligned}
&\parallel K_i(l_{t+1}) \parallel \\
=&\parallel (\Phi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\Omega_i^{-1}(l_{t+1}) - \mathcal{B}(l_t)L_i(l_{t+1})) \\
&\cdot (I - \mathcal{C}_i(l_{t+1})\mathcal{B}(l_t)L_i(l_{t+1}))^{\dagger} \parallel \\
\leq&\parallel (\Phi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\Omega_i^{-1}(l_{t+1}) - \mathcal{B}(l_t)L_i(l_{t+1})) \parallel \\
&\cdot \parallel (I - \mathcal{C}_i(l_{t+1})\mathcal{B}(l_t)L_i(l_{t+1}))^{\dagger} \parallel \\
\leq&(\parallel \Phi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\Omega_i^{-1}(l_{t+1}) \parallel + \parallel \mathcal{B}(l_t)L_i(l_{t+1}) \parallel) \\
&\cdot \parallel (I - \mathcal{C}_i(l_{t+1})\mathcal{B}(l_t)L_i(l_{t+1}))^{\dagger} \parallel \\
\leq&\left(\frac{\bar{\theta}\bar{c}}{\underline{\varpi}} + \bar{b}\bar{l}\right)(1 + \underline{cbl})^{-1} \triangleq \bar{k}.
\end{aligned} \tag{S13}$$

Noting the expression of $N_i(l_{t+1})$ in Eq. (S8), we can obtain

$$\begin{aligned}
&\parallel N_i(l_{t+1}) \parallel \\
=&\parallel \Phi_i(l_t)\mathcal{C}_i^{\mathrm{T}}(l_{t+1})\Omega_i^{-1}(l_{t+1}) \parallel \\
\leq&\frac{\bar{\phi}\bar{c}}{\underline{\varpi}}.
\end{aligned} \tag{S14}$$

On the other hand, one can derive

$$\begin{aligned}
&\parallel \mathcal{A}_{i,1}(l_t)\mathcal{A}_{i,1}^{\mathrm{T}}(l_t) \parallel \\
=&\parallel (I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\mathcal{A}(l_t)\mathcal{A}^{\mathrm{T}}(l_t)(I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} \parallel \\
\leq&(1 + \frac{\bar{\phi}^2\bar{c}^4}{\underline{\varpi}^2})\bar{\alpha}_1^2 \\
=&\tilde{a}_1^2 \\
&\parallel \mathcal{A}_{i,2}(l_t)\mathcal{A}_{i,2}^{\mathrm{T}}(l_t) \parallel \\
=&\parallel (I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} \parallel \\
\leq&(1 + \frac{\bar{\phi}^2\bar{c}^4}{\underline{\varpi}^2})\bar{\alpha}_2^2 \\
=&\tilde{a}_2^2, \\
&\parallel \mathcal{A}_{i,2}(l_t)\mathcal{A}_{i,2}^{\mathrm{T}}(l_t) \parallel \\
=&\parallel (I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} \parallel \\
\geq&(\underline{\alpha}_2 - \frac{\bar{\phi}\bar{c}^2\bar{\alpha}_2}{\underline{\varpi}})^2 \\
=&\underline{a}_2^2, \\
&\parallel Z(l_{t+1}) \parallel \\
=&\parallel (I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\bar{W}(l_t)\bar{\mathcal{A}}^{\mathrm{T}}(l_t)(I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))^{\mathrm{T}} \\
&+ (1 - 2\delta_i)N_i(l_{t+1})V_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1}) + (1 + \varepsilon_i^{-1})\bar{\rho}N_i(l_{t+1})N_i^{\mathrm{T}}(l_{t+1}) \parallel \\
\leq&\tilde{a}_2^2 + ((1 - 2\delta_i)\bar{\nu} + (1 + \varepsilon_i^{-1})\bar{\rho})\frac{\bar{\phi}^2\bar{c}^2}{\underline{\varpi}^2} \\
=&\bar{z}.
\end{aligned} \tag{S15}$$

Then, one has

$$\begin{aligned}
&\mathbb{E}\{f_i^{\mathrm{T}}(l_{t+1})f_i(l_{t+1})\} \\
=&\mathbb{E}\{(\mathcal{A}_{i,2}(l_t)\vec{\omega}(l_t) - N_i(l_{t+1})(\nu_i(l_{t+1}) - \sigma_i(l_{t+1})))^{\mathrm{T}}(\mathcal{A}_{i,2}(l_t)\vec{\omega}(l_t) \\
&- N_i(l_{t+1})(\nu_i(l_{t+1}) - \sigma_i(l_{t+1})))\} \\
\leq&\mathbb{E}\{((I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\vec{\omega}(l_t))^{\mathrm{T}}((I - N_i(l_{t+1})\mathcal{C}_i(l_{t+1}))\bar{\mathcal{A}}(l_t)\vec{\omega}(l_t))\} \\
&+ \mathbb{E}\{(N_i(l_{t+1})\nu_i(l_{t+1}))^{\mathrm{T}}(N_i(l_{t+1})\nu_i(l_{t+1}))\} \\
&+ \mathbb{E}\{(N_i(l_{t+1})\sigma_i(l_{t+1}))^{\mathrm{T}}(N_i(l_{t+1})\sigma_i(l_{t+1}))\} \\
\leq&\tilde{a}_2^2 n_x\bar{\omega} + \frac{\bar{\phi}^2\bar{c}^2}{\underline{\varpi}^2}(\bar{\nu} + \bar{\rho} + 2\delta_i\bar{\nu}) \\
=&\bar{f}.
\end{aligned} \tag{S16}$$

For any scalar $\epsilon_i > 0$, it is not difficult to derive that

$$\begin{aligned}
&\mathbb{E}\{V_{i,k+1}(\tilde{x}_i(l_{t+1}|l_{t+1}) \mid \tilde{x}_i(l_t|l_t))\} - (1+\epsilon_i)V_{i,k}(\tilde{x}_i(l_t|l_t)) \\
=&\mathbb{E}\{(\mathcal{A}_{i,1}(l_t)\tilde{x}_i(l_t|l_t) + f_i(l_{t+1}))^{\mathrm{T}}\mathit{\Xi}_{i,x}^{-1}(l_{t+1}|l_{t+1})(\mathcal{A}_{i,1}(l_t)\tilde{x}_i(l_t|l_t) \\
&+ f_i(l_{t+1}))\} - (1+\epsilon_i)\tilde{x}_i^{\mathrm{T}}(l_t|l_t)\mathit{\Xi}_{i,x}^{-1}(l_t|l_t)\tilde{x}_i(l_t|l_t) \\
=&\mathbb{E}\{\tilde{x}_i^{\mathrm{T}}(l_t|l_t)\mathcal{A}_{i,1}^{\mathrm{T}}(l_t)\mathit{\Xi}_{i,x}^{-1}(l_{t+1}|l_{t+1})\mathcal{A}_{i,1}(l_t)\tilde{x}_i(l_t|l_t)\} + \mathbb{E}\{f_i^{\mathrm{T}}(l_{t+1})\mathit{\Xi}_{i,x}^{-1}(l_{t+1}|l_{t+1})f_i(l_{t+1})\} \\
&+ \mathbb{E}\{\tilde{x}_i^{\mathrm{T}}(l_t|l_t)\mathcal{A}_{i,1}^{\mathrm{T}}(l_t)\mathit{\Xi}_{i,x}^{-1}(l_{t+1}|l_{t+1})f_i(l_{t+1})\} + \mathbb{E}\{f_i^{\mathrm{T}}(l_{t+1})\mathit{\Xi}_{i,x}^{-1}(l_{t+1}|l_{t+1}) \\
&\cdot \mathcal{A}_{i,1}(l_t)\tilde{x}_i(l_t|l_t)\} - (1+\epsilon_i)\tilde{x}_i^{\mathrm{T}}(l_t|l_t)\mathit{\Xi}_{i,x}^{-1}(l_t|l_t)\tilde{x}_i(l_t|l_t) \\
\leq&(1+\epsilon_i)\mathbb{E}\{\tilde{x}_i^{\mathrm{T}}(l_t|l_t)(\mathcal{A}_{i,1}^{\mathrm{T}}(l_t)\mathit{\Xi}_{i,x}^{-1}(l_{t+1}|l_{t+1})\mathcal{A}_{i,1}(l_t) \\
&- \mathit{\Xi}_{i,x}^{-1}(l_t|l_t))\tilde{x}_i(l_t|l_t)\} + (1+\epsilon_i^{-1})\mathbb{E}\{f_i^{\mathrm{T}}(l_{t+1})\mathit{\Xi}_{i,x}^{-1}(l_{t+1}|l_{t+1})f_i(l_{t+1})\}.
\end{aligned}$$

Using the matrix inversion lemma, the matrix in the first term in the above equation can be handled as follows:

$$\begin{aligned}
&\mathcal{A}_{i,1}^{\mathrm{T}}(l_t)\mathit{\Xi}_{i,x}^{-1}(l_{t+1}|l_{t+1})\mathcal{A}_{i,1}(l_t) - \mathit{\Xi}_{i,x}^{-1}(l_t|l_t) \\
=&\mathcal{A}_{i,1}^{\mathrm{T}}(l_t)((1+\varepsilon_i)\mathcal{A}_{i,1}(l_t)\mathit{\Xi}_{i,x}(l_t|l_t)\mathcal{A}_{i,1}^{\mathrm{T}}(l_t) + Z(l_{t+1}))^{-1}\mathcal{A}_{i,1}(l_t) - \mathit{\Xi}_{i,x}^{-1}(l_t|l_t) \\
\leq&\mathcal{A}_{i,1}^{\mathrm{T}}(l_t)(\mathcal{A}_{i,1}(l_t)\mathit{\Xi}_i(l_t)\mathcal{A}_{i,1}^{\mathrm{T}}(l_t) + Z(l_{t+1}))^{-1}\mathcal{A}_{i,1}(l_t) - \mathit{\Xi}_{i,x}^{-1}(l_t|l_t) \\
=&-(\mathit{\Xi}_{i,x}(l_t|l_t) + \mathit{\Xi}_{i,x}(l_t|l_t)\mathcal{A}_{i,1}^{\mathrm{T}}(l_t)Z^{-1}(l_{t+1})\mathcal{A}_{i,1}(l_t)\mathit{\Xi}_{i,x}(l_t|l_t))^{-1} \\
=&-(I + \mathcal{A}_{i,1}^{\mathrm{T}}(l_t)Z^{-1}(l_{t+1})\mathcal{A}_{i,1}(l_t)\mathit{\Xi}_{i,x}(l_t|l_t))^{-1}\mathit{\Xi}_{i,x}^{-1}(l_t|l_t) \\
\leq&-\left(1 + \frac{\tilde{a}_1^2\bar{p}}{\underline{z}}\right)^{-1}\mathit{\Xi}_{i,x}^{-1}(l_t|l_t),
\end{aligned} \tag{S17}$$

which means

$$\begin{aligned}
&\mathbb{E}\{V_{i,k+1}(\tilde{x}_i(l_{t+1}|l_{t+1})) \mid \tilde{x}_i(l_t|l_t))\} - (1+\epsilon_i)V_{i,k}(\tilde{x}_i(l_t|l_t)) \\
\leq&-(1+\epsilon_i)(1 + \frac{\tilde{a}_1^2\bar{p}}{\underline{z}})^{-1}V_{i,k}(\tilde{x}_i(l_t|l_t)) + (1+\epsilon_i^{-1})\frac{\bar{f}}{\underline{p}}.
\end{aligned} \tag{S18}$$

Such an inequality can be rearranged by

$$\begin{aligned}
&\mathbb{E}\{V_{i,k+1}(\tilde{x}_i(l_{t+1}|l_{t+1})) \mid \tilde{x}_i(l_t|l_t))\} \\
\leq&\chi V_{i,k}(\tilde{x}_i(l_t|l_t)) + \eta \\
\leq&0,
\end{aligned} \tag{S19}$$

where

$$\chi = (1+\epsilon_i)(1-(1+\frac{\tilde{a}_1^2\bar{p}}{\underline{z}})^{-1}),$$
$$\eta = (1+\epsilon_i^{-1})\frac{\bar{f}}{\underline{p}}.$$

Obviously, there exists a positive scalar ϵ_i such that $0 < \chi < 1$. Subsequently, we can obtain from inequality (S19) that

$$\begin{aligned}
&\mathbb{E}\{\| \tilde{x}_i(l_{t+1}|l_{t+1}) \|^2\} \\
=&\mathbb{E}\{\tilde{x}_i^{\mathrm{T}}(l_{t+1}|l_{t+1})\Xi_{i,x}^{-1}(l_{t+1}|l_{t+1}) \\
&\cdot \Xi_{i,x}(l_{t+1}|l_{t+1})\tilde{x}_i(l_{t+1}|l_{t+1})\} \\
\leq&\bar{p}\mathbb{E}\{V_{i,k+1}(\tilde{x}_i(l_{t+1}|l_{t+1}))\} \\
\leq&\frac{\bar{p}}{\underline{p}}\chi^{k+1}\mathbb{E}\{\| \tilde{x}_i(0) \|^2\} + \bar{p}\eta\sum_{l=0}^{\infty}\chi^l \\
\leq&\frac{\bar{p}}{\underline{p}}\chi^{k+1}\mathbb{E}\{\| \tilde{x}_i(0) \|^2\} + \frac{\bar{p}\eta}{1-\chi}.
\end{aligned}$$

Based on the above discussions and Definition 1, we can know that the filtering error $\tilde{x}_i(l_t|l_t)$ is exponentially bounded in the mean square sense. The proof of Theorem 3 is completed.