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Supplementary materials for

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Proof of Theorem 1

Consider a Lyapunov function as follows:

$$V(t)=\Upsilon(t)V_{\sigma(t)}(t)+(1-\Upsilon(t))V_{\sigma(t)\varrho(t)}(t), \tag{S1}$$

where $\Upsilon(t)\in\{0,1\}$,

$$\begin{cases}\sigma(t)=\varrho(t), & \Upsilon(t)=1,\\ \sigma(t)\neq\varrho(t), & \Upsilon(t)=0,\end{cases}$$

$$\begin{aligned}V_{\sigma(t)}(t)&=x^T(t)P_{\sigma(t)}x(t)+\int_{t-\varepsilon_m}^{t}e^{\alpha_i(t-\kappa)}x^T(\kappa)H^1_{\sigma(t)}x(\kappa)d\kappa\\&+\int_{t-\varepsilon_M}^{t-\varepsilon_m}e^{\alpha_i(t-\kappa)}x^T(\kappa)H^2_{\sigma(t)}x(\kappa)d\kappa+\varepsilon_m\int_{t-\varepsilon_m}^{t}\int_{\varsigma}^{t}e^{\alpha_i(t-\kappa)}\dot{x}^T(\kappa)M^1_{\sigma(t)}\dot{x}(\kappa)d\kappa d\varsigma\\&+(\varepsilon_m-\varepsilon_M)\int_{t-\varepsilon_M}^{t-\varepsilon_m}\int_{\varsigma}^{t}e^{\alpha_i(t-\kappa)}\dot{x}^T(\kappa)M^2_{\sigma(t)}\dot{x}(\kappa)d\kappa d\varsigma,\end{aligned} \tag{S2}$$

$$\begin{aligned}V_{\sigma(t),\varrho(t)}(t)&=x^T(t)P_{\sigma(t),\varrho(t)}x(t)+\int_{t-\varepsilon_m}^{t}e^{\gamma_{i,j}(t-\kappa)}x^T(\kappa)H^1_{\sigma(t),\varrho(t)}x(\kappa)d\kappa\\&+\int_{t-\varepsilon_M}^{t-\varepsilon_m}e^{\gamma_{i,j}(t-\kappa)}x^T(\kappa)H^2_{\sigma(t),\varrho(t)}x(\kappa)d\kappa+\varepsilon_m\int_{t-\varepsilon_m}^{t}\int_{\varsigma}^{t}e^{\gamma_{i,j}(t-\kappa)}\dot{x}^T(\kappa)M^1_{\sigma(t),\varrho(t)}\dot{x}(\kappa)d\kappa d\varsigma\\&+(\varepsilon_m-\varepsilon_M)\int_{t-\varepsilon_M}^{t-\varepsilon_m}\int_{\varsigma}^{t}e^{-\gamma_{i,j}(t-\kappa)}\dot{x}^T(\kappa)M^2_{\sigma(t),\varrho(t)}\dot{x}(\kappa)d\kappa d\varsigma.\end{aligned} \tag{S3}$$

Case I: When $\sigma(t)=\varrho(t)=i,\forall i\in\mathcal{W}$, the modes of the system are synchronized with the modes of controller. Then, by calculating the derivative of $V_i(t)$ along Eq. (S2), we obtain

$$\begin{aligned}\dot{V}_i(t)&=x^T(t)(A_i^TP_i+P_iA_i+H_i^1-\alpha_iP_i)x(t)+2x^T(t)P_i\tilde{B}_iK_ix(t-\varepsilon(t))+2x^T(t)P_iE_i\omega(t)\\&-2x^T(t)P_i\tilde{B}_iK_ie(s_{c,v}k)+e^{\alpha_i\varepsilon_m}x^T(t-\varepsilon_m)H_i^2x(t-\varepsilon_m)-e^{\alpha_i\varepsilon_M}x^T(t-\varepsilon_M)H_i^2x(t-\varepsilon_M)\\&-e^{\alpha_i\varepsilon_m}x^T(t-\varepsilon_m)H_i^1x(t-\varepsilon_m)+(\varepsilon_M-\varepsilon_m)^2\dot{x}^T(t)M_i^2\dot{x}(t)+\varepsilon_m^2\dot{x}^T(t)M_i^1\dot{x}(t)\\&-\varepsilon_m\int_{t-\varepsilon_m}^{t}\dot{x}(\kappa)^TM_i^1\dot{x}(\kappa)d\kappa-(\varepsilon_M-\varepsilon_m)\int_{t-\varepsilon_M}^{t-\varepsilon_m}e^{\alpha_i(t-\kappa)}\dot{x}(\kappa)^TM_i^2\dot{x}(\kappa)d\kappa+\alpha_iV_i(t).\end{aligned} \tag{S4}$$

Applying Jensen's inequality, we have

$$-\varepsilon_m\int_{t-\varepsilon_m}^{t}\dot{x}(\kappa)^TM_i^1\dot{x}(\kappa)d\kappa\leq-[x(t)-x(t-\varepsilon_m)]^TM_i^1[x(t)-x(t-\varepsilon_m)]. \tag{S5}$$

Because $\begin{bmatrix} M_i^2 & Z_i \\ Z_i^T & M_i^2 \end{bmatrix} > 0$, by using Park's method, we get

$$\begin{aligned} &-(\varepsilon_M-\varepsilon_m)\int_{t-\varepsilon_M}^{t-\varepsilon_m} e^{\alpha_i(t-\kappa)}\dot{x}(\kappa)^T M_i^2 \dot{x}(\kappa)d\kappa \\ &\le -e^{\alpha_i\varepsilon_m}\begin{bmatrix} x^T(t-\varepsilon_m)-x^T(t-\varepsilon(t)) \\ x^T(t-\varepsilon(t))-x^T(t-\varepsilon_M) \end{bmatrix}^T \begin{bmatrix} M_i^2 & Z_i \\ Z_i^T & M_i^2 \end{bmatrix} \begin{bmatrix} x(t-\varepsilon_m)-x(t-\varepsilon(t)) \\ x(t-\varepsilon(t))-x(t-\varepsilon_M) \end{bmatrix}. \end{aligned} \tag{S6}$$

Combine Eqs. (2) and (S4)–(S6), we have

$$\begin{aligned} &\dot{V}_i(t)+x(t)^TQ_ix(t)+u(t)^TD_iu(t) \\ =&x^T(t)(A_i^TP_i+P_iA_i+H_i^1-\alpha_iP_i-M_i^1+Q_i)x(t)+2x^T(t)P_i\tilde{B}_iK_ix(t-\varepsilon(t))+2x^T(t)P_iE_i\omega(t) \\ &-2x^T(t)P_i\tilde{B}_iK_ie(s_{c,v}k)+2x^T(t)M_i^1x(t-\varepsilon_m)-e^{\alpha_i\varepsilon_m}x^T(t-\varepsilon(t))(2M_i^2-Z_i-Z_i^T)x(t-\varepsilon(t)) \\ &+2e^{\alpha_i\varepsilon_m}x^T(t-\varepsilon(t))(M_i^2-Z_i)x(t-\varepsilon_M)+2e^{\alpha_i\varepsilon_m}x^T(t-\varepsilon_m)(M_i^2-Z_i)x(t-\varepsilon(t)) \\ &+x^T(t-\varepsilon_m)[e^{\alpha_i\varepsilon_m}(H_i^2-H_i^1-M_i^2)-M_i^1]x(t-\varepsilon_m)+2e^{\alpha_i\varepsilon_m}x^T(t-\varepsilon_m)Z_ix(t-\varepsilon_M) \\ &+x^T(t-\varepsilon_M)(e^{\alpha_i\varepsilon_M}H_i^2-e^{\alpha_i\varepsilon_m}M_i^2)x(t-\varepsilon_M)+\varepsilon_m^2\dot{x}^T(t)M_i^1\dot{x}(t)+(\varepsilon_M-\varepsilon_m)^2\dot{x}^T(t)M_i^2\dot{x}(t) \\ &-e^T(s_{c,v}k)K_i^TD_iK_ix(t-\varepsilon(t))-x^T(t-\varepsilon(t))K_i^TD_iK_ie(s_{c,v}k) \\ &+x^T(t-\varepsilon(t))K_i^TD_iK_ix(t-\varepsilon(t))+\alpha_iV_i(t)+e^T(s_{c,v}k)K_i^TD_iK_ie(s_{c,v}k) \\ \le&\digamma^T(t)[\mho_0+\Re(\varepsilon_m^2M_i^1+(\varepsilon_M-\varepsilon_m)^2M_i^2)\Re^T]\digamma(t)+\alpha_iV_i(t)+\omega^T(t)\omega(t), \end{aligned} \tag{S7}$$

where

$$\begin{aligned} \Re &= [A_i \quad \tilde{B}_iK_i \quad 0 \quad 0 \quad E_i \quad -\tilde{B}_iK_i]^T, \\ \digamma(t) &= \mathrm{col}\{x(t),\ x(t-\varepsilon(t)),\ x(t-\varepsilon_m),\ x(t-\varepsilon_M),\ \omega(t),\ e(s_{c,v}k)\}. \end{aligned}$$

inequality (11) is treated by Schur's lemma, we can get

$$\mho_0+\Re(\varepsilon_m^2M_i^1+(\varepsilon_M-\varepsilon_m)^2M_i^2)\Re^T<0. \tag{S8}$$

This combined with inequality (S7) shows

$$\dot{V}_i(t)\le\alpha_iV_i(t)+\omega^T(t)\omega(t)-\hbar_i, \tag{S9}$$

where $\hbar_i=x(t)^TQ_ix(t)+u(t)^TD_iu(t)$.

Since Q_i and D_i are positive definite matrices, inequality (S9) implies

$$\dot{V}_i(t)\le\alpha_iV_i(t)+\omega^T(t)\omega(t). \tag{S10}$$

Case II: When $\sigma(t)\neq\varrho(t)$, a mode mismatch will occur between the system and the controller. Selecting constant $\gamma_{i,j}$ for the LKF (S1), let $\sigma(t)=i$, $\varrho(t)=j$, and we can then provide a proof similar to Case I and inequality (12), resulting in the following outcome:

$$\dot{V}_{i,j}(t)\le\gamma_{i,j}V_{i,j}(t)+\omega^T(t)\omega(t)-\hbar_{i,j}. \tag{S11}$$

Since $Q_{i,j}$ and $D_{i,j}$ are positive definite matrices, inequality (S11) implies

$$\dot{V}_{i,j}(t)\le\gamma_{i,j}V_{i,j}(t)+\omega^T(t)\omega(t). \tag{S12}$$

Integrating inequality (S12) from t_h to t gives

$$V_{\sigma(t_h),\varrho(t_h)}(t)<e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-t_h)}V_{\sigma(t_h),\varrho(t_h)}(t_h)+\int_{t_h}^{t}e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-\kappa)}\omega(\kappa)^T\omega(\kappa)d\kappa. \tag{S13}$$

From ETS (2), we can learn that the mode of subsystem t_h is detected within a momentary $\tilde{t}_k = [\frac{t_h}{k}]k$. But, due to the impact of network-induced delays, the controller is unable to switch instantly until $\hat{t}_h \triangleq \tilde{t}_h + \varepsilon_{\tilde{t}_h}$. Therefore, $\forall h \in \mathbb{N}$, when $t \in [\hat{t}_h, t_{h+1}), \sigma(t) = \varrho(t)$; when $t \in [t_h, \hat{t}_h), \sigma(t) \neq \varrho(t)$.

Note that $\sigma(\hat{t}_h) = \sigma(t_h), \varrho(\hat{t}_h) = \varrho(t_{h+1})$. From inequalities (15), (16), (S10), and (S12), $\forall t \in [t_h, \hat{t}_h)$, we get

$$\begin{aligned}
V(t) &= V_{\sigma(t_h),\varrho(t_h)}(t) \\
&\leq e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-t_h)} V_{\sigma(t_h),\varrho(t_h)}(t_h) + \int_{t_h}^{t} e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-\kappa)} \omega(\kappa)^T \omega(\kappa) d\kappa \\
&\leq e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-t_h)} \mu_{\sigma(t_h),\varrho(t_h)} V_{\sigma(t_h),\varrho(t_h)}(t_h) + \int_{t_h}^{t} e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-\kappa)} \omega(\kappa)^T \omega(\kappa) d\kappa \\
&\quad + \mu_{\sigma(t_h),\varrho(t_h)} \int_{t_h}^{t} e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-\kappa)} \omega(\kappa)^T \omega(\kappa) d\kappa \\
&\leq e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-t_h)} \mu_{\sigma(t_h),\varrho(t_h)} e^{\alpha_{\sigma(t_{h-1})}(t_h-\hat{t}_{h-1})} V_{\sigma(t_{h-1}),\varrho(t_h)}(\hat{t}_{h-1}) \\
&\quad + \int_{t_h}^{t} e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-\kappa)} \omega(\kappa)^T \omega(\kappa) d\kappa + \mu_{\sigma(t_h),\varrho(t_h)} \int_{t_h}^{t} e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-\kappa)} \omega(\kappa)^T \omega(\kappa) d\kappa \\
&\quad + \mu_{\sigma(t_h),\sigma(t_h)} \int_{\hat{t}_{h-1}}^{t_h} e^{\alpha_{\sigma(t_{h-1})}(t-\kappa)} \omega(\kappa)^T \omega(\kappa) d\kappa \\
&\leq e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-t_h)} \mu_{\sigma(t_h),\varrho(t_h)} e^{\alpha_{\sigma(t_{h-1})}(t_h-\hat{t}_{h-1})} \upsilon_{\sigma(t_{h-1}),\varrho(t_{h-1})} V_{\sigma(t_{h-1}),\varrho(t_{h-1})}(\hat{t}_{h-1}) \\
&\quad + \int_{t_h}^{t} e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-\kappa)} \omega(\kappa)^T \omega(\kappa) d\kappa + \mu_{\sigma(t_h),\sigma(t_h)} \int_{t_h}^{t} e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-\kappa)} \omega(\kappa)^T \omega(\kappa) d\kappa \\
&\quad + \mu_{\sigma(t_h),\varrho(t_h)} \int_{\hat{t}_{h-1}}^{t_h} e^{\alpha_{\sigma(t_{h-1})}(t-\kappa)} \omega(\kappa)^T \omega(\kappa) d\kappa \\
&\quad + \mu_{\sigma(t_h),\varrho(t_h)} \upsilon_{\sigma(t_{h-1}),\varrho(t_{h-1})} \int_{\hat{t}_{h-1}}^{t_h} e^{\alpha_{\sigma(t_{h-1})}(t-\kappa)} \omega(\kappa)^T \omega(\kappa) d\kappa \\
&\leq \quad \cdots \\
&\leq \prod_{k=1}^{h} \mu_{\sigma(t_k),\varrho(t_k)} \prod_{k=1}^{h-1} e^{\gamma_{\sigma(t_k),\varrho(t_k)}\varepsilon(t_k)} e^{\gamma_{\sigma(t_k),\varrho(t_k)}(t-t_k)} \prod_{k=0}^{h-1} \upsilon_{\sigma(t_k),\varrho(t_k)} \prod_{k=1}^{h-1} e^{\alpha_{\sigma(t_k)}(t_{k+1}-\hat{t}_k)} \\
&\quad \cdot e^{\alpha_{\sigma(0)}(t_1-0)} (V_{\sigma(0),\varrho(0)}(0) + \int_0^t \omega(\kappa)^T \omega(\kappa) d\kappa).
\end{aligned} \tag{S14}$$

Let $\ln\upsilon_{\sigma(t_0),\varrho(t_0)} \leq \ln\upsilon_{i,j}$. By calculation, we have

$$\begin{aligned}
&\prod_{k=1}^{h} \mu_{\sigma(t_k),\varrho(t_k)} \prod_{k=0}^{h-1} \upsilon_{\sigma(t_k),\varrho(t_k)} = \exp\left(\sum_{k=1}^{h} \ln\mu_{\sigma(t_k),\varrho(t_k)}\right) \exp\left(\sum_{k=0}^{h-1} \ln\upsilon_{\sigma(t_k),\varrho(t_k)}\right) \\
&= \exp\left(\sum_{k=1}^{h} \sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}} \sum_{(i,j)\in\Theta_u} \ln\mu_{i,j}\right) \exp\left(\sum_{k=1}^{h-1} \sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}} \sum_{(i,j)\in\Theta_s} \ln\upsilon_{i,j}\right) \exp\left(\sum_{k=1}^{h} \ln\upsilon_{\sigma(0),\varrho(0)}\right).
\end{aligned} \tag{S15}$$

Considering the stability and instability of the system that can occur when asynchronous switching is considered, the formula (S15) can be reconfigured as

$$\begin{aligned}
&\exp\left(\sum_{k=1}^{h}\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_u}\ln\mu_{i,j}\right)\exp\left(\sum_{k=1}^{h-1}\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_s}\ln\upsilon_{i,j}\right)\exp\left(\sum_{k=1}^{h}\ln\upsilon_{\sigma(0),\varrho(0)}\right)\\
&\leq\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_u}N^{\sigma}_{i,j}(0,t)\ln\mu_{i,j}+\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_s}N^{\sigma}_{i,j}(0,t)\ln\upsilon_{i,j}\right).
\end{aligned}\tag{S16}$$

Similarly, we have

$$\begin{aligned}
&\prod_{k=1}^{h-1}\exp\left(\gamma_{\sigma(t_k),\varrho(t_k)}(t_{k+1}-\hat{t}_k)\right)\exp\left(\gamma_{\sigma(t_h),\varrho(t_h)}(t-t_h)\right)\prod_{k=1}^{h-1}\exp\left(\alpha_{\sigma(t_k)}\varepsilon(t_k)\right)\exp\left(\alpha_{\sigma(0)}(t_1-0)\right)\\
&=\exp\left(\sum_{k=1}^{h-1}\gamma_{\sigma(t_k),\varrho(t_k)}(t_{k+1}-\hat{t}_k)+\gamma_{\sigma(t_h),\varrho(t_h)}(t-t_h)\right)\exp\left(\sum_{k=1}^{h-1}\alpha_{\sigma(t_k)}\varepsilon(t_k)+\alpha_{\sigma(0)}(t_1-0)\right)\\
&=\exp\left(\sum_{(i,j)\in\Theta_s,\sigma(t_k)=i}\alpha_i T_i(0,t)+\sum_{\sigma(t_k)=i,\varrho(t_k)=j,(i,j)\in\Theta_{u\downarrow}}\gamma_{i,j}T_i(0,t)+\sum_{\sigma(t_k)=i,\varrho(t_k)=j,(i,j)\in\Theta_{u\uparrow}}\gamma_{i,j}T_i(0,t)\right)\\
&=\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_s}\alpha_i T_i(0,t)+\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\downarrow}}\gamma_{i,j}T_i(0,t)+\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\uparrow}}\gamma_{i,j}T_i(0,t)\right),
\end{aligned}\tag{S17}$$

where on the interval $[0,t)$, $T_{i,j}(0,t)$ and $N^{\sigma}_{i,j}$ stand for the total runtime and switching numbers of subsystem i whenever the subsystem i switches from subsystem j. Substituting (S16) and (S17) into (S14), and combine (6), (7) and (9) as

$$\begin{aligned}
&V_{\sigma(t),\varrho(t)}(t)\\
&\leq\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_s}\left(N^{\sigma}_{i,j}(0,t)\ln\upsilon_{i,j}+\alpha_i T_{i,j}(0,t)\right)\right)\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\downarrow}}\left(N^{\sigma}_{i,j}(0,t)\ln\mu_{i,j}+\gamma_{i,j}T_{i,j}(0,t)\right)\right)\\
&\quad\times\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\uparrow}}\left(N^{\sigma}_{i,j}(0,t)\ln\mu_{i,j}+\gamma_{i,j}T_{i,j}(0,t)\right)\right)(V_{\sigma(0),\varrho(0)}(0)+d)\\
&\leq\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_s}\left(\left(\underline{N}^{0}_{i,j}(0,t)+\frac{T_{i,j}(0,t)}{\tau^{a}_{i,j}}\right)\ln\upsilon_{i,j}+\alpha_i T_{i,j}(0,t)\right)\right)\\
&\quad\times\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\downarrow}}\left(\left(\underline{N}^{0}_{i,j}(0,t)+\frac{T_{i,j}(0,t)}{d^{a}_{i,j}}\right)\ln\mu_{i,j}+\gamma_{i,j}T_{i,j}(0,t)\right)\right)\\
&\quad\times\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\uparrow}}\left(\left(\overline{N}^{0}_{i,j}(0,t)+\frac{T_{i,j}(0,t)}{d^{a}_{i,j}}\right)\ln\mu_{i,j}+\gamma_{i,j}T_{i,j}(0,t)\right)\right)(V_{\sigma(0),\varrho(0)}(0)+d)
\end{aligned}$$

$$
\begin{aligned}
&\leq \exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_s}\underline{N}^0_{i,j}\ln\upsilon_{i,j}+\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\downarrow}}\underline{N}^0_{i,j}\ln\mu_{i,j}+\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\uparrow}}\overline{N}^0_{i,j}\ln\mu_{i,j}\right)\\
&\quad\times\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_s}\left(\frac{\ln\upsilon_{i,j}}{\tau^a_{i,j}}+\alpha_i\right)T_{i,j}(0,t)\right)\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\downarrow}}\left(\frac{\ln\mu_{i,j}}{d^a_{i,j}}+\gamma_{i,j}\right)T_{i,j}(0,t)\right)\\
&\quad\times\exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\uparrow}}\left(\frac{\ln\mu_{i,j}}{d^a_{i,j}}+\gamma_{i,j}\right)T_{i,j}(0,t)\right)\times(V_{\sigma(0),\varrho(0)}(0)+d).
\end{aligned}
\tag{S18}
$$

According to Definition 2, Eqs. (9), (S1), and (S18), we can get

$$
x^T(t)Gx(t)\leq\frac{1}{\lambda_1}V(t)\leq\frac{\rho}{\lambda_1}e^{\varpi_{i,j}}\times(V_{\sigma(t)}(x(0))+d)
\tag{S19}
$$

and

$$
V_{\sigma(0)}(0)\leq\vartheta_i,
\tag{S20}
$$

where

$$
\begin{aligned}
\rho&=\exp\Bigg(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_s}\underline{N}^0_{i,j}\ln\upsilon_{i,j}+\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\downarrow}}\underline{N}^0_{i,j}\ln\mu_{i,j}+\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\uparrow}}\overline{N}^0_{i,j}\ln\mu_{i,j}\Bigg),\\
\varpi_{i,j}&=\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_s}\left(\frac{\ln\upsilon_{i,j}}{\tau^a_{i,j}}+\alpha_{max}\right)T_{i,j}(0,t)+\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\downarrow}}\left(\frac{\ln\mu_{i,j}}{d^a_{i,j}}+\gamma_{max}\right)T_{i,j}(0,t)\\
&\quad+\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}}\sum_{(i,j)\in\Theta_{u\uparrow}}\left(\frac{\ln\mu_{i,j}}{d^a_{i,j}}+\gamma_{min}\right)T_{i,j}(0,t),\\
\vartheta_i&=\lambda_2c_1\Bigg[1-\frac{1}{\alpha_i}-\frac{\varepsilon_m}{\alpha_i}+\frac{1}{\alpha_i}e^{\alpha_i\varepsilon_M}-\frac{(\varepsilon_M-\varepsilon_m)^2}{\alpha_i}+\frac{\varepsilon_m}{\alpha_i^2}e^{\alpha_i\varepsilon_m}-\frac{\varepsilon_M-\varepsilon_m}{\alpha_i^2}(e^{\alpha_i\varepsilon_m}-e^{\alpha_i\varepsilon_M})-\frac{\varepsilon^2}{\alpha_i}\Bigg].
\end{aligned}
$$

Hence, combining Eqs. (S19), (S20), and (18), we have $x^T(t)Gx(t)<c_2$, which implies the system (5) is FTB with respect to (c_1,c_2,G,d,T_f).

Then, we show how to implement the upper bound of the cost function. Integrating Eq. (S11) from t_h to t gives

$$
\begin{aligned}
&V_{\sigma(t_h),\varrho(t_h)}(t)\\
&<e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-t_h)}V_{\sigma(t_h),\varrho(t_h)}(t_h)+\int_{t_h}^{t}e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-\kappa)}\omega(\kappa)^T\omega(\kappa)d\kappa\\
&\quad-\int_{t_h}^{t}e^{\gamma_{\sigma(t_h),\varrho(t_h)}(t-\kappa)}\hbar_{\sigma(t_h),\varrho(t_h)}d\kappa.
\end{aligned}
\tag{S21}
$$

From Eqs. (S11)–(S20), and combining Case I and Case II, we can get

$$
\begin{aligned}
V(t) &= V_{\sigma(t_h),\varrho(t_h)}(t) \\
&\le \exp\Bigg(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}} \sum_{(i,j)\in\Theta_s} \underline{N}^0_{i,j}\ln\upsilon_{i,j} + \sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}} \sum_{(i,j)\in\Theta_{u\downarrow}} \underline{N}^0_{i,j}\ln\mu_{i,j} + \sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}} \sum_{(i,j)\in\Theta_{u\uparrow}} \overline{N}^0_{i,j}\ln\mu_{i,j} \Bigg) \\
&\quad \times \exp\left(\sum_{\substack{(i,j)\in\Theta_s,\\ \sigma(t_k)=i}} \sum_{\varrho(t_k)=j} \left(\frac{\ln\upsilon_{i,j}}{\tau^a_{i,j}} + \alpha_{max} \right) T_{i,j}(0,t) \right) \exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}} \sum_{(i,j)\in\Theta_{u\downarrow}} \left(\frac{\ln\mu_{i,j}}{d^a_{i,j}} + \gamma_{max} \right) T_{i,j}(0,t) \right) \\
&\quad \times \exp\left(\sum_{\substack{\varrho(t_k)=j,\\ \sigma(t_k)=i}} \sum_{(i,j)\in\Theta_{u\uparrow}} \left(\frac{\ln\mu_{i,j}}{d^a_{i,j}} + \gamma_{min} \right) T_{i,j}(0,t) \right) (V_{\sigma(0),\varrho(0)}(0) + d - J). \qquad \text{(S22)}
\end{aligned}
$$

Since $\rho e^{\varpi_{i,j}} > 0, V(t) > 0$, combined with Eq. (S22), we see

$$\rho e^{\varpi_{i,j}} J \le \rho e^{\varpi_{i,j}} J + V(t) \le \rho e^{\varpi_{i,j}} \times (V_{\sigma(0),\varrho(0)}(0) + d).$$

And then, there holds

$$J \le \vartheta_i + d \le \vartheta_{max} + d \triangleq J^*, \qquad \text{(S23)}$$

where

$$
\begin{aligned}
\vartheta_{max} = \lambda_2 c_1 \Bigg[& 1 - \frac{1}{\alpha_{max}} - \frac{\varepsilon^2}{\alpha_{max}} - \frac{\varepsilon_m}{\alpha_{max}} - \frac{(\varepsilon_M - \varepsilon_m)^2}{\alpha_{max}} + \frac{1}{\alpha_{max}} e^{\alpha_{max}\varepsilon_M} + \frac{\varepsilon_m}{\alpha^2_{max}} e^{\alpha_{max}\varepsilon_m} \\
& - \frac{\varepsilon_M - \varepsilon_m}{\alpha^2_{max}} (e^{\alpha_{max}\varepsilon_m} - e^{\alpha_{max}\varepsilon_M}) \Bigg], \quad \alpha_{max} = \max_{i\in\mathcal{W}}\{\alpha_i\}. \quad \square
\end{aligned}
$$