Electronic Supplementary Materials

For https://doi.org/10.1631/jzus.A2400538

A two-stage framework for automated operational modal identification using OPTICS-KNN-based clustering

Yi CHEN[1,2], Wenwei FU[3,4], Yaozhi LUO[1], Yanbin SHEN[1,4], Hui YANG[2], Shiying WANG[5]

[1]*College of Civil Engineering and Architecture, Zhejiang University, Hangzhou 310058, China*

[2]*Zhejiang Communications Construction Group Co., LTD., Hangzhou 310051, China*

[3]*School of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215011, China*

[4]*Future City Laboratory, Innovation Center of Yangtze River Delta, Zhejiang University, Jiaxing 314102, China*

[5]*Hangzhou Institute of Communications Planning Design & Research, Hangzhou 310030, China*

Section S1

Stochastic subspace identification (SSI), a time-domain OMA method based on the state space model, is characterized by its quick implementation, high precision, and assumption of white noise as ambient stimulation. This method has been widely used in OMA over the last several decades, including data-driven SSI and covariance-driven SSI (SSI-COV) [47,50]. The latter is utilized for OMA in this paper due to its ability to reduce the dimensions of the output matrix and computational cost. The discrete-time state-space equation defines the dynamic motion of a linear and stationary *n*-degree-of-freedom system, which is the underlying assumption of SSI-COV. Its detailed theoretical principles are described below:

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{\omega}_k \\ \boldsymbol{y}_k &= \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{v}_k \end{aligned} \tag{S1}$$

where $k \in [1, \cdots, s]$ denotes the sampling interval within samples *s*; $\boldsymbol{x}_k \in \mathbb{R}^{n\times1}$ and $\boldsymbol{y}_k \in \mathbb{R}^{l\times1}$ are the discrete-time state vector and sampled output vector at the time interval *k* with *n* DOFs, respectively; *l* denotes the number of the measurement points; $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is the discrete state matrix with the model order $n = 2N$, which describes the relationship between the current state of the system and its future state; $\boldsymbol{C} \in \mathbb{R}^{l\times n}$ denotes the discrete output matrix which describes how the state variables of the system contribute to the output measurements. $\boldsymbol{\omega}_k \in \mathbb{R}^{n\times1}$ denotes the white noise caused by disturbances and model inaccuracies; $\boldsymbol{v}_k \in \mathbb{R}^{l\times1}$ is the measurement of the white noise from sensor inaccuracies. In practical applications, SSI-COV can be generally summarized in the following steps: (a) using techniques such as detrending and filtering to preprocess the output data to remove noise and outliers. (b) creating the block Hankel matrix $\boldsymbol{H}$ from the preprocessed data. (c) performing singular value decomposition (SVD) on the block Hankel matrix to estimate the observability matrix $\boldsymbol{O}_i$. (d) investigating the shifted observability matrix $\boldsymbol{O}_i^{\uparrow}$, $\boldsymbol{O}_i^{\downarrow}$ and reversed controllability matrix $\boldsymbol{\Gamma}$. (e) identifying modal parameters through $\boldsymbol{A}$ and $\boldsymbol{C}$ deduced from the shifted observability matrix and reversed controllability matrix. The SSI-COV begins with a block Hankel matrix that may be partitioned into two sub-matrices through a symmetrical division form.

$$\boldsymbol{Y}_p = \frac{1}{\sqrt{j}} \begin{bmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_j \\ \boldsymbol{y}_2 & \boldsymbol{y}_3 & \cdots & \boldsymbol{y}_{j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{y}_i & \boldsymbol{y}_{i+1} & \cdots & \boldsymbol{y}_{i+j-1} \end{bmatrix} \tag{S2}$$

$$\boldsymbol{Y}_f = \frac{1}{\sqrt{j}} \begin{bmatrix} \boldsymbol{y}_{i+1} & \boldsymbol{y}_{i+2} & \cdots & \boldsymbol{y}_{i+j} \\ \boldsymbol{y}_{i+2} & \boldsymbol{y}_{i+3} & \cdots & \boldsymbol{y}_{i+j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{y}_{2i} & \boldsymbol{y}_{2i+1} & \cdots & \boldsymbol{y}_{2i+j-1} \end{bmatrix} \tag{S3}$$

where $\boldsymbol{y}_i \in \mathbb{R}^{l\times 1}$ denotes the measured response vector; $\boldsymbol{Y}_p \in \mathbb{R}^{li\times j}$ and $\boldsymbol{Y}_f \in \mathbb{R}^{li\times j}$ denote the past and future matrices, respectively; i is the user-defined time lag; j is the block columns and conventionally set as:

$$j = s - 2i + 1 \tag{S4}$$

The block Hankel matrix $\boldsymbol{H} \in \mathbb{R}^{li\times li}$ can be rearranged as follows:

$$\boldsymbol{H} = \boldsymbol{Y}_f \boldsymbol{Y}_p^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{R}_i & \boldsymbol{R}_{i-1} & \cdots & \boldsymbol{R}_1 \\ \boldsymbol{R}_{i+1} & \boldsymbol{R}_i & \cdots & \boldsymbol{R}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{R}_{2i-1} & \boldsymbol{R}_{2i-2} & \cdots & \boldsymbol{R}_i \end{bmatrix} \tag{S5}$$

where $\boldsymbol{R}_i = E\left(\boldsymbol{y}_k \boldsymbol{y}_{k-i}^{\mathrm{T}}\right) \in \mathbb{R}^{l\times l}$ denotes the covariance matrices between the past and future matrices; $\mathrm{E}(\cdot)$ denotes the expectation operator. The block Hankel matrix can be obtained as below:

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{C}^{\mathrm{T}} \\ (\boldsymbol{CA})^{\mathrm{T}} \\ \vdots \\ \left(\boldsymbol{CA}^{i-1}\right)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}^{i-1}\boldsymbol{G} & \cdots & \boldsymbol{AG} & \boldsymbol{G} \end{bmatrix} = \boldsymbol{O}_i \boldsymbol{\Gamma}_i \tag{S6}$$

where $\boldsymbol{G} = \mathrm{E}\left(\boldsymbol{x}_{k+1} \boldsymbol{y}_k^{\mathrm{T}}\right)$ is the next state-output covariance matrix; Then, the singular value decomposition is applied to perform the factorization of the block Hankel matrix as below:

$$\boldsymbol{H} = \boldsymbol{USV}^{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{S}_1 & \boldsymbol{0}_{n\times(li-n)} \\ \boldsymbol{0}_{(li-n)\times n} & \boldsymbol{S}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_1^{\mathrm{T}} \\ \boldsymbol{V}_2^{\mathrm{T}} \end{bmatrix} \approx \boldsymbol{U}_1 \boldsymbol{S}_1 \boldsymbol{V}_1^{\boldsymbol{T}} \tag{S7}$$

where $\boldsymbol{U}_1 \in \mathbb{R}^{li\times n}$ and $\boldsymbol{V}_1 \in \mathbb{R}^{li\times n}$ denote the left and right singular vectors, respectively; $\boldsymbol{S}_1 \in \mathbb{R}^{n\times n}$ is the n non-zero singular values matrix contains singular values in descending order; $\boldsymbol{S}_2 \in \mathbb{R}^{(li-n)\times(li-n)}$ is a series of small singular values that are usually omitted which is often related to the noise term. In the comparison of Eqs. (6-7), the following expressions for the observability matrix $\boldsymbol{O}_i \in \mathbb{R}^{li\times n}$, reversed controllability matrix $\boldsymbol{\Gamma}_i \in \mathbb{R}^{n\times li}$, and state matrix $\boldsymbol{A}$ are given by:

$$\boldsymbol{O}_i = \boldsymbol{U}_1 \boldsymbol{S}_1^{1/2}, \quad \boldsymbol{\Gamma}_i = \boldsymbol{S}_1^{1/2} \boldsymbol{V}_1^{\mathrm{T}} \tag{S8}$$

$$\boldsymbol{A} = \left(\boldsymbol{O}_i^{\uparrow}\right)^{\dagger} \boldsymbol{O}_i^{\downarrow} \tag{S9}$$

where $\left(\cdot\right)^{\dagger}$ denotes the Moore-Penrose pseudoinverse; $\boldsymbol{O}_i^{\uparrow}$ and $\boldsymbol{O}_i^{\downarrow}$ can be calculated by removing the last and the first l rows of $\boldsymbol{O}_i$, respectively. Once $\boldsymbol{A}$ obtained above, the eigenvalue decomposition is computed.

$$\left(\boldsymbol{A} - \lambda_p \boldsymbol{I}_n\right)\phi_p = \boldsymbol{0}_n \tag{S10}$$

where ϕ_p denotes the p eigenvectors of $\boldsymbol{A}$, while the corresponding eigenvalues λ_p are collected which denotes the generic pole. Finally, the natural frequencies f_p and damping ratios ξ_p are computed below while the mode shapes $\boldsymbol{\psi}_p \in \mathbb{C}^l$ can be derived by multiplying by the output matrix $\boldsymbol{C}$.

$$f_p = \frac{f_s \left|\ln\left(\lambda_p\right)\right|}{2\pi}, \quad \xi_p = \frac{\mathrm{Re}\left[\ln\left(\lambda_p\right)\right]}{\left|\ln\left(\lambda_p\right)\right|}, \quad \boldsymbol{\psi}_p = \boldsymbol{C}\phi_p \tag{S11}$$

where the f_s denotes the sampling rate; $\mathrm{Re}(\cdot)$ denotes the real operator for complex variables.

As mentioned in the key elements above, the explicit mathematical formulation of SSI-COV makes it particularly well-suited for AOMA [31]. Appropriate parameter settings mainly involve model order n and time lag i should be determined before its implementation, which will significantly affect the computational efficiency and accuracy of modal identification. Unfortunately, no consensus formulation for optimal estimation of those parameters exists as of yet. A conservative strategy is to over-specify the model order that can cover the weakly excited modes in practical applications. Over-specification invariably creates spurious modes that must be isolated from the physical poles making it complicated to estimate the modal parameter. On the other hand, A key determinant of the success of modal parameter identification is the selection of the parameter *i*. Identification may fail if the duration of the time history signal is shorter than the period of the first structural mode. There is more information about the structural vibration in longer signals. The value of *i*, on the other hand, has no upper limit. A larger i leads to the size increase of the block Hankel matrix, causing a greater computational burden. In the presence of operational noise, larger *i* may create more spurious modes in the SSI output. Therefore, careful selection of *i* is necessary to control the number of spurious modes considering the natural frequency of the first mode. In the context of the symmetric partition of the Hankel matrix as described in Eqs. (2-3), several researchers have widely adopted a rule of thumb to address this issue [34]:

$$i \geq n_c \frac{f_s}{2 f_0} \tag{S12}$$

where f_s and f_0 represent the sampling frequency of measured data and the natural frequency of the first mode, respectively; n_c denotes the number of cycles, which is commonly set to one.