**Electronic supplementary materials**
For https://doi.org/10.1631/jzus.A2500172

# Fixed-time robust attitude tracking control for high-speed aircraft: a precise funnel-guided approach

Kai AN, Wei HUANG, Shuangxi LIU

*Advanced Propulsion Technology Laboratory, National University of Defense Technology, Changsha 410073, China*

## Section S1

The detailed expression of $\overline{G}$ in Section 2.2 is

$$\begin{aligned} \overline{G}_\alpha &= m_0 g_0 \cos\gamma \cos\mu \\ \overline{G}_\beta &= m_0 g_0 \cos\gamma \sin\mu \\ \overline{G}_\mu &= -m_0 g_0 \cos\gamma \tan\beta \cos\mu \end{aligned} \tag{S1}$$

where $g_0$ is the gravitational acceleration.

The specific expressions for some notations in Section 2.2 Eq. (S3) are given as follows

$$\begin{aligned} \mathbf{g}_\Omega &= \begin{bmatrix} 1 & -\sin\alpha\tan\beta & -\cos\alpha\tan\beta \\ 0 & -\cos\alpha & \sin\alpha \\ 0 & \sin\alpha\sec\beta & \cos\alpha\sec\beta \end{bmatrix} \\ \mathbf{g}_\omega &= \mathbf{J}^{-1}\overline{q}S \begin{bmatrix} g_{q\delta_e} & g_{q\delta_a} & g_{q\delta_r} \\ g_{p\delta_e} & g_{r\delta_a} & g_{r\delta_r} \\ g_{p\delta_r} & g_{p\delta_a} & g_{p\delta_r} \end{bmatrix} \end{aligned} \tag{S2}$$

$$\begin{aligned} g_{q\delta_e} &= cC_m^{\delta_e} + x_{cg}(C_D^{\delta_e}\sin\alpha + C_L^{\delta_e}\cos\alpha) \\ g_{r\delta_e} &= bC_n^{\delta_e} - x_{cg}C_Y^{\delta_e} \\ g_{p\delta_e} &= bC_l^{\delta_e} \\ g_{r\delta_a} &= cC_m^{\delta_a} + x_{cg}(C_D^{\delta_a}\sin\alpha + C_L^{\delta_a}\cos\alpha) \\ g_{r\delta_a} &= bC_n^{\delta_a} - x_{cg}C_Y^{\delta_a} \\ g_{p\delta_a} &= bC_l^{\delta_a} \\ g_{r\delta_r} &= cC_m^{\delta_r} + x_{cg}C_D^{\delta_r}\sin\alpha \\ g_{r\delta_r} &= bC_n^{\delta_r} - x_{cg}C_Y^{\delta_r} \\ g_{p\delta_r} &= bC_l^{\delta_r} \end{aligned} \tag{S3}$$

The detailed expressions of $\mathbf{f}_\omega = \left[f_q, f_r, f_p\right]^{\mathrm{T}}$ in Section 2.2 are given as follows

$$\begin{aligned}
f_q &= \frac{(J_z - J_x)pr}{J_y} + \frac{\bar{q}cS(C_m^\alpha + C_m^q qc/(2V))}{J_y} + \frac{x_{cg}\bar{q}S(C_D^\alpha \sin\alpha + C_L^\alpha \cos\alpha)}{J_y} \\
f_r &= \frac{(J_x - J_y)pq}{J_z} - \frac{x_{cg}\bar{q}SC_Y^\beta}{J_z} + \frac{\bar{q}bS(C_n^\beta \beta + C_n^p pb/(2V) + C_n^r rb/(2V))}{J_z} \\
f_p &= \frac{(J_y - J_z)qr}{J_x} + \frac{(C_l^p rb/(2V))}{J_x} + \frac{\bar{q}bS(C_l^\beta \beta + C_l^p pb/(2V))}{J_x}
\end{aligned} \tag{S4}$$

where the dynamic derivative coefficients for pitch channel are $C_m^\alpha, C_D^\alpha, C_L^\alpha, C_m^\alpha$ for yaw channel are $C_Y^\beta, C_n^\beta, C_n^p, C_n^r$ and for roll channel are $C_l^p, C_l^\beta$. The aerodynamic derivatives in terms of control surface are also defined as $C_i^{\delta_e}, C_i^{\delta_a}, C_i^{\delta_r}$ $i = L, D, Y, l, m, n$ , which all can be found in (Keshmiri et al., 2004).

## Section S2

Originally, an unknown continuous nonlinear function $f(\mathbf{x}) \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^M$ can be represented by a neural network consisting of optimal weights $W^* \in \mathbb{R}^N$ and a number of Gaussian basis functions $h(\mathbf{x})$, which is defined as

$$f(\mathbf{x}) = W^{*T} h(\mathbf{x}) + \epsilon(\mathbf{x}) \tag{S5}$$

where $\epsilon(\mathbf{x})$ is the NN approximation error, $h(\mathbf{x})$ is given by

$$h_j(\mathbf{x}) = \exp\left(-\frac{\|\mathbf{x} - C_j\|^2}{2n_j^2}\right) \tag{S6}$$

where $h_j(\mathbf{x})$ is the output of the $j$-th neuron in the hidden layer. $x_i, i = 1,2,\ldots,\mathcal{M}$, $\boldsymbol{C} \in \mathbb{R}^{M \times N}$, $\boldsymbol{n} \in \mathbb{R}^N$, $j = 1,2,\ldots,\mathcal{N}$.

**The proof process of Theorem 1:**

*Proof.* A Lyapunov candidate function is selected as

$$V_0 = \frac{1}{2}\Delta\mathcal{Z}^{\mathrm{T}}\mathbf{P}\Delta\mathcal{Z} + \frac{1}{2}\widetilde{\mathbf{W}}^{\mathrm{T}}\widetilde{\mathbf{W}} \tag{S7}$$

then differentiating $V_0$ along the observation error $\Delta\mathcal{Z}$ yields

$$\begin{aligned}
\dot{V}_0 &= \frac{1}{2}\Delta\mathcal{Z}^{\mathrm{T}}\mathbf{P}\Delta\dot{\mathcal{Z}} + \frac{1}{2}\Delta\dot{\mathcal{Z}}^{\mathrm{T}}\mathbf{P}\Delta\mathcal{Z} + \widetilde{\mathbf{W}}^{\mathrm{T}}\widetilde{\mathbf{W}} \\
&= \frac{1}{2}\Delta\mathcal{Z}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{PA})\Delta\mathcal{Z} + \Delta\mathcal{Z}^{\mathrm{T}}\mathbf{P}(\widetilde{\mathbf{W}}^{\mathrm{T}}h(\mathcal{Z}) \\
&\quad +\epsilon(\mathcal{Z}) - \boldsymbol{C}_1\mathbf{sig}^{a_1}\Delta\mathcal{Z} - \boldsymbol{C}_2\mathbf{sig}^{a_2}\Delta\mathcal{Z} - \boldsymbol{C}_3\Delta\mathcal{Z}) - \widetilde{\mathbf{W}}^{\mathrm{T}}\widetilde{\mathbf{W}} \\
&= -\frac{1}{2}\Delta\mathcal{Z}^{\mathrm{T}}(\mathbf{Q} + 2\mathbf{P}\boldsymbol{C}_3)\Delta\mathcal{Z} \\
&\quad +\Delta\mathcal{Z}^{\mathrm{T}}\mathbf{P}\epsilon(\mathcal{Z}) + \widetilde{\mathbf{W}}^{\mathrm{T}}\widetilde{\mathbf{W}} \\
&\quad -\textstyle\sum_{i=1}^q c_{1,i}\, p_{i,i}|\Delta\mathcal{Z}_i|^{a_1+1} - \sum_{i=1}^q c_{2,i}\, p_{i,i}|\Delta\mathcal{Z}_i|^{a_2+1}
\end{aligned} \tag{S8}$$

where $\mathbf{Q} = \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{PA}$. In alignment with the Assumption 1, Lemma 3 and Lemma 4, we can obtain the following result

$$\begin{aligned}
V_0 &\leq -\frac{1}{2}\lambda_{\min}(\mathbf{Q} + 2\mathbf{P}\boldsymbol{C}_3) \parallel \Delta\mathcal{Z} \parallel^2 + \parallel \Delta\mathcal{Z} \parallel \epsilon_N \\
&\quad +\frac{1}{2}\mathbf{W}^{\mathrm{T}}\mathbf{W} - \frac{1}{2}\widetilde{\mathbf{W}}^{\mathrm{T}}\widetilde{\mathbf{W}} \\
&\quad -\lambda_{\min}(\boldsymbol{C}_1)\left(\textstyle\sum_{i=1}^q p_{i,i}\, |\Delta\mathcal{Z}_i|^2\right)^{\frac{a_1+1}{2}} \\
&\quad -\lambda_{\min}(\boldsymbol{C}_2)\left(\textstyle\sum_{i=1}^q p_{i,i}\, |\Delta\mathcal{Z}_i|^2\right)^{\frac{a_2+1}{2}} \\
&\leq -\frac{1}{2}\lambda_{\min}(\mathbf{Q} + 2\mathbf{P}\boldsymbol{C}_3) \parallel \Delta\mathcal{Z} \parallel^2 + \parallel \Delta\mathcal{Z} \parallel \epsilon_N \\
&\quad +\frac{1}{2}\mathbf{W}^{\mathrm{T}}\mathbf{W} - \frac{1}{2}\widetilde{\mathbf{w}}^{\mathrm{T}}\widetilde{\mathbf{W}} \\
&\quad -\min\{\lambda_{\min}(\boldsymbol{C}_1), \lambda_{\min}(\boldsymbol{C}_2)\} \cdot \left((\Delta\mathcal{Z}^{\mathrm{T}}\mathbf{P}\Delta\mathcal{Z})^{\frac{a_1+1}{2}} + (\Delta\mathcal{Z}^{\mathrm{T}}\mathbf{P}\Delta\mathcal{Z})^{\frac{a_2+1}{2}}\right)
\end{aligned} \tag{S9}$$

Considering the Assumption 1, Eq. (S9) can be reformulated accordingly

$$\begin{aligned}\dot{V}_0 \leq & -\tfrac{1}{2}\lambda_{\min}(\mathbf{Q}+2\mathbf{P}\boldsymbol{C}_3) \parallel \Delta\mathcal{Z} \parallel^2 + \parallel \Delta\mathcal{Z} \parallel \epsilon_N \\ & -\tfrac{1}{2}W_N^2 - \tfrac{1}{2}\widetilde{\mathbf{W}}^{\mathrm{T}}\widetilde{\mathbf{W}} \\ & -\min\{\lambda_{\min}(\boldsymbol{C}_1), \lambda_{\min}(\boldsymbol{C}_2)\}\Delta\mathcal{Z}^{\mathrm{T}}\mathbf{P}\Delta\mathcal{Z} \\ \leq & -\bar{c}V_0 + \Psi \end{aligned} \tag{S10}$$

where

$$\Psi = -\tfrac{1}{2}\lambda_{\min}(\mathbf{Q}+2\mathbf{P}\boldsymbol{C}_3) \parallel \Delta\mathcal{Z} \parallel^2 + \parallel \Delta\mathcal{Z} \parallel \epsilon_N - \tfrac{1}{2}W_N^2$$

is a bounded constant. $\bar{c} = 2\min\left\{\frac{1}{2}, \lambda_{\min}(\boldsymbol{C}_1), \lambda_{\min}(\boldsymbol{C}_2)\right\}$. In accordance with Lyapunov stability theory, the observer system has asymptotic stability when $t \to \infty$, $V_0 \to 0$ . Then, the disturbance approximation error $\boldsymbol{\varsigma}_{\mathbf{z}} = \widetilde{\mathbf{W}}^{\mathrm{T}}h(\mathcal{Z}) + \epsilon(\mathcal{Z})$ can be verified to be continuously bounded, satisfying $\parallel \boldsymbol{\varsigma}_{\mathbf{z}} \parallel \leq \boldsymbol{\varsigma}_{\max}$.

Then the equation $\Delta\dot{\mathcal{Z}}$ can be rewritten as

$$\Delta\dot{\mathcal{Z}} = (\mathbf{A} - \boldsymbol{C}_3)\Delta\mathcal{Z} - \boldsymbol{C}_1\mathbf{sig}^{a_1}\Delta\mathcal{Z} - \boldsymbol{C}_2\mathbf{sig}^{a_2}\Delta\mathcal{Z} + \boldsymbol{\varsigma}_{\max} \tag{S11}$$

For the states $\mathbf{z}_\Omega$, the Lemma 2 can guarantee that observation errors and weight errors $\widetilde{\mathbf{W}}$ are fixed-time stable with the settling time $T_{0,\Omega}$ for every angle errors as follows:

$$\begin{aligned} T_{0,\Omega} \leq T_{i,\text{upper}} = & \max\left\{\frac{\ln(1+\frac{c_{3,i}}{\delta c_{1,i}})}{c_{3,i}(1-a_1)}, \frac{\ln(1+\frac{\delta c_{3,i}}{c_{1,i}})}{c_{3,i}\delta(1-a_1)}\right\} \\ & +\max\left\{\frac{\ln(1+\frac{c_{3,i}}{\delta c_{2,i}})}{c_{3,i}(a_2-1)}, \frac{\ln(1+\frac{\delta c_{3,i}}{c_{2,i}})}{\delta c_{3,i}(a_2-1)}\right\} \end{aligned} \tag{S12}$$

where $c_{3,i} = 1 - d_n$, $\delta = 0.5$.

For the states $\mathbf{z}_\omega$, $c_{3,i} = 0$, the condition of Lemma 1 is satisfied, then the fixed settling time $T_{0,\omega}$ is:

$$T_{0,\omega} \leq T_{i,\text{upper}} = \frac{1}{c_{1,i}\delta(1-a_1)} + \frac{1}{c_{2,i}\delta(a_2-1)} \tag{S13}$$

In summary, the total settling time $T_{0,\max}$ is calculated by $T_{0,\max} = \max\{T_{0,\Omega}, T_{0,\omega}\}$. The proof has been done.

# Section S3

$$\begin{aligned} T_d &= \frac{(s_{\mathcal{U}} - s_{\mathcal{L}})}{(s_{i,e} - s_{\mathcal{L}})(s_{\mathcal{U}} - s_{i,e})} \\ T_c &= \frac{s_{i,e}(\dot{s}_{\mathcal{L}} - \dot{s}_{\mathcal{U}}) - \dot{s}_{\mathcal{L}}s_{\mathcal{U}} + \dot{s}_{\mathcal{U}}s_{\mathcal{L}}}{(s_{i,e} - s_{\mathcal{L}})(s_{\mathcal{U}} - s_{i,e})} \end{aligned} \tag{S14}$$

$$\begin{aligned} T_{d,\Omega} &= T_d \\ T_{c,\Omega} &= T_c - T_d\dot{\mathbf{e}}_{op} \end{aligned} \tag{S15}$$

# Section S4

Establish the derivative of $\mathbf{S}_\Omega$ with respect to $t$ and substituting $\dot{\overline{\mathbf{e}}}_\Omega$ into $\dot{\mathbf{S}}_\Omega$ yields

$$\begin{aligned} \dot{\mathbf{S}}_\Omega &= T_{d,\Omega}\dot{\mathbf{e}}_\Omega + T_{c,\Omega} + c_0\overline{\mathbf{e}}_\Omega + k_1\mathbf{sig}^{b_1}\overline{\mathbf{e}}_\Omega + k_2\mathbf{sig}^{b_2}\overline{\mathbf{e}}_\Omega \\ &= T_{d,\Omega}(\mathbf{g}_\Omega\boldsymbol{\omega} + \Delta_\Omega) + T_{c,\Omega} + c_0\overline{\mathbf{e}}_\Omega + k_1\mathbf{sig}^{b_1}\overline{\mathbf{e}}_\Omega + k_2\mathbf{sig}^{b_2}\overline{\mathbf{e}}_\Omega \end{aligned} \tag{S16}$$

The first-order Taylor series expansion of $\mathbf{S}_\Omega$ in discretized time node $(t + h_s)$ is given

$$\mathbf{S}_\Omega(t + h_s) = \mathbf{S}_\Omega(t) + \frac{\mathrm{d}\mathbf{S}_\Omega(t)}{\mathrm{d}t}h_s + \mathcal{O}(h_s) \tag{S17}$$

where $h_s$ is the time step, $O(h_s)$ is the high-order residual term. Neglect the residual expression $O(h_s)$ , and we can get a proportional relationship between the control signal $\frac{\mathrm{d}\mathbf{S}_\Omega(t)}{\mathrm{d}t}$ and control error estimation $\Delta\mathbf{S}_\Omega = \mathbf{S}_\Omega(t+h_s) - \mathbf{S}_\Omega(t)$ at time step $h_s$ in advance. Thus, the derivative behavior in the control signal ( can theoretically foresee estimated state errors and suppress them effectively.

**The proof process of Lemma 5:**

*Proof.* Substituting Eq. (S18) in Section 3.3 into $\dot{\mathbf{S}}_\Omega$, then we can get

$$\dot{\mathbf{S}}_\Omega = \widetilde{\Delta}_\Omega - k_3\mathbf{sig}^{b_3}\mathbf{S}_\Omega - k_4\mathbf{sig}^{b_4}\mathbf{S}_\Omega - k_5\tanh\frac{\mathbf{S}_\Omega}{w} - k_\Omega\exp^{|\tanh(\dot{\mathbf{S}}_\Omega)|}\dot{\mathbf{S}}_\Omega \tag{S18}$$

After that, the above equation can be rewritten as

$$\dot{\mathbf{S}}_\Omega = \frac{-k_3\mathbf{sig}^{b_3}\mathbf{S}_\Omega - k_4\mathbf{sig}^{b_4}\mathbf{S}_\Omega - k_5\tanh\frac{\mathbf{S}_\Omega}{w} + \widetilde{\Delta}_\Omega}{\left(1+k_\Omega\exp^{|\tanh(\dot{\mathbf{S}}_\Omega)|}\right)} \tag{S19}$$

A Lyapunov candidate function is selected as

$$V_1 = \frac{1}{2}\mathbf{S}_\Omega^{\mathrm{T}}\mathbf{S}_\Omega \tag{S20}$$

Differentiating $V_1$ and the following equation can be obtained

$$\begin{aligned}\dot{V}_1 &= \mathbf{S}_\Omega^{\mathrm{T}}\dot{\mathbf{S}}_\Omega \\ &= \frac{1}{\left(1+k_\Omega\exp^{|\tanh(\mathbf{S}_\Omega)|}\right)}(-k_3\mathbf{sig}^{b_3+1}\mathbf{S}_\Omega \\ &\quad -k_4\mathbf{sig}^{b_4+1}\mathbf{S}_\Omega - k_5\mathbf{S}_\Omega^{\mathrm{T}}\tanh\frac{\mathbf{S}_\Omega}{w} + \mathbf{S}_\Omega^{\mathrm{T}}\widetilde{\Delta_\Omega})\end{aligned} \tag{S21}$$

According to Remark 1, the estimated disturbance term satisfies $\mathbf{S}_\Omega^{\mathrm{T}}\widetilde{\Delta}_\Omega \leq \| \mathbf{S}_\Omega \| D_\Omega$. The function $\tanh(\cdot)$ directly denotes the sign of independent variable $\mathbf{S}_\Omega$, and the parameter $w$ determines the upbound of this adaptive term. Because of the parameter range of $\mathcal{K}_\Omega$, the range of denominator function can be obtained by $(1+\mathcal{K}_\Omega) \in (1+k_\Omega, 1+2.718k_\Omega)$, then the following inequality can be deduced

$$\begin{aligned}\dot{V}_1 &\leq \frac{1}{1+k_\Omega}(-k_3\mathbf{sig}^{b_3+1}\mathbf{S}_\Omega - k_4\mathbf{sig}^{b_4+1}\mathbf{S}_\Omega - k_5 \| \mathbf{S}_\Omega \| + \| \mathbf{S}_\Omega \| D_\Omega) \\ &\leq \frac{1}{1+k_\Omega}\left(-k_3(\mathbf{S}_\Omega^{\mathrm{T}}\mathbf{S}_\Omega)^{\frac{b_3+1}{2}} - k_4\left(\mathbf{S}_\Omega^{\mathrm{T}}\mathbf{S}_\Omega\right)^{\frac{b_4+1}{2}} + \| \mathbf{S}_\Omega \| (D_\Omega - k_5)\right) \\ &= -K_3V_1^{\frac{b_3+1}{2}} - K_4V_1^{\frac{b_4+1}{2}} + \xi_\Omega\end{aligned} \tag{S22}$$

where $K_3 = \frac{k_3}{1+k_\Omega}2^{\frac{b_3+1}{2}}$, $K_4 = \frac{k_4}{1+k_\Omega}2^{\frac{b_4+1}{2}}$, $\xi_\Omega = \frac{\|\mathbf{S}_\Omega\|(D_\Omega - k_5)}{1+k_\Omega} > 0$.

According to Lemma 1, the designed sliding mode manifold $\mathbf{S}_\Omega$ will reach the equilibrium hyperplane $\mathbf{S}_\Omega = 0$ in a fixed time $T_\Omega$, where $T_\Omega$ is given by

$$T_\Omega \leq \frac{2}{K_3(b_3-1)} + \frac{2}{K_4(1-b_4)} \tag{S23}$$

The proof is completed.

It should be noted that the convergence time $T_\Omega \geq T_{0,\max}$ in Section S3.1 Eq. (S13). The stability demonstration of the attitude angle sliding mode manifold explains that the first stage of sliding mode motion has been accomplished. Then the system states will continue to move along this manifold until all the angles reach equilibrium points.

**The proof process of Lemma 6:**

*Proof.* Similar to the proof process of Lemma 5, the quadratic Lyapunov function is given by

$$V_2 = \frac{1}{2}\mathbf{S}_\omega^{\mathrm{T}}\mathbf{S}_\omega \tag{S24}$$

and then derive the derivative of $V_2$

$$
\begin{aligned}
\dot{V}_2 &= \mathbf{S}_\omega^{\mathrm{T}} \dot{\mathbf{S}}_\omega \\
&= \frac{1}{\left(1+k_\omega \exp^{\left|\tanh(\hat{\mathbf{s}}_\omega)\right|}\right)} (-k_6 \mathbf{sig}^{c_3+1} \mathbf{S}_\omega \\
&\quad -k_7 \mathbf{sig}^{c_4+1} \mathbf{S}_\omega - k_8 \mathbf{S}_\omega^{\mathrm{T}} \tanh \frac{\mathbf{S}_\omega}{w} + \mathbf{S}_\omega^{\mathrm{T}} \widetilde{\Delta}_\omega) \\
&\leq \frac{1}{1+k_\omega} (-k_6 \mathbf{sig}^{c_3+1} \mathbf{S}_\omega - k_7 \mathbf{sig}^{c_4+1} \mathbf{S}_\omega \\
&\quad \| \mathbf{S}_\omega \| (D_\omega - k_8)) \\
&= -K_6 V_2^{\frac{c_3+1}{2}} - K_7 V_2^{\frac{c_4+1}{2}} + \xi_\omega
\end{aligned} \tag{S25}
$$

where the approximated error $\xi_\omega = \frac{\|\mathbf{S}_\omega\|(D_\omega - k_8)}{1+k_\omega} > 0$, $K_6 = \frac{k_6}{1+k_\omega} 2^{\frac{c_3+1}{2}}$, $K_7 = \frac{k_7}{1+k_\omega} 2^{\frac{c_4+1}{2}}$. According to Lemma 1 and Remark 1, the fixed convergence time $T_\omega$ can be deduced as

$$
T_\omega \leq \frac{2}{K_6(c_3-1)} + \frac{2}{K_7(1-c_4)} \tag{S26}
$$

It also should be noted that the convergence time $T_\omega \geq T_{0,\max}$ in Section S3.1 Eq. (S13).

## Section S5

**The proof process of Theorem 2:**

*Proof.* Select a Lyapunov function

$$
V_3 = \frac{1}{2} \mathbf{e}_\Omega^{\mathrm{T}} \mathbf{e}_\Omega + \frac{1}{2} \mathbf{e}_\omega^{\mathrm{T}} \mathbf{e}_\omega + \frac{1}{2} \widetilde{\Delta}_\Omega^{\mathrm{T}} \widetilde{\Delta}_\Omega + \frac{1}{2} \widetilde{\Delta}_\omega^{\mathrm{T}} \widetilde{\Delta}_\omega \tag{S27}
$$

According to the time derivative of $V_3$, and transformed attitude error system $\dot{\mathbf{e}}_\Omega$ and $\dot{\mathbf{e}}_\omega$, then we can obtain

$$
\begin{aligned}
\dot{V}_3 = {} & \bar{\mathbf{e}}_\Omega^{\mathrm{T}} \left( T_{d,\Omega} (\mathbf{g}_\omega \boldsymbol{\omega} + \Delta_\Omega) + T_{c,\Omega} \right) \\
& + \mathbf{e}_\omega^{\mathrm{T}} (\mathbf{f}_\omega + \mathbf{g}_\omega \mathbf{u} + \Delta_\omega) \\
& + \widetilde{\Delta}_\Omega^{\mathrm{T}} \widetilde{\Delta}_\Omega + \widetilde{\Delta}_\omega^{\mathrm{T}} \widetilde{\Delta}_\omega
\end{aligned} \tag{S28}
$$

Substituting the proposed controllers $\boldsymbol{\omega}^*$ and $\mathbf{u}^*$ into Eq. (S28) and $\mathbf{S}_\Omega = 0$, $\mathbf{S}_\omega = 0$ are satisfied, we can obtain

$$
\begin{aligned}
\dot{V}_3 = {} & -c_0 \mathbf{e}_\Omega^{\mathrm{T}} \bar{\mathbf{e}}_\Omega - k_1 \textstyle\sum_{i=1}^{3} |\, \bar{\mathbf{e}}_{\Omega,i}|^{b_1+1} - k_2 \textstyle\sum_{i=1}^{3} |\, \bar{\mathbf{e}}_{\Omega,i}|^{b_2+1} \\
& -c_\omega \mathbf{e}_\omega^{\mathrm{T}} \mathbf{e}_\omega - l_1 \textstyle\sum_{i=1}^{3} |\, \mathbf{e}_{\omega,i}|^{c_1+1} - l_2 \textstyle\sum_{i=1}^{3} |\, \mathbf{e}_{\omega,i}|^{c_2+1} \\
& + \mathbf{e}_\Omega^{\mathrm{T}} \widetilde{\Delta}_\Omega + \mathbf{e}_\omega^{\mathrm{T}} \widetilde{\Delta}_\omega + \widetilde{\Delta}_\Omega^{\mathrm{T}} \widetilde{\Delta}_\Omega + \widetilde{\Delta}_\omega^{\mathrm{T}} \widetilde{\Delta}_\omega
\end{aligned} \tag{S29}
$$

According to Lemma 3 and Lemma 4, the following relationship is hold,

$$
\begin{aligned}
\dot{V}_3 \leq {} & -\left(c_0 - \frac{1}{2}\right) \bar{\mathbf{e}}_\Omega^{\mathrm{T}} \bar{\mathbf{e}}_\Omega - k_1 \left( \bar{\mathbf{e}}_\Omega^{\mathrm{T}} \bar{\mathbf{e}}_\Omega \right)^{\frac{b_1+1}{2}} - k_2 \left( \bar{\mathbf{e}}_\Omega^{\mathrm{T}} \bar{\mathbf{e}}_\Omega \right)^{\frac{b_2+1}{2}} - \left(c_\omega - \frac{1}{2}\right) \mathbf{e}_\omega^{\mathrm{T}} \mathbf{e}_\omega \\
& -l_1 (\mathbf{e}_\omega^{\mathrm{T}} \mathbf{e}_\omega)^{\frac{c_1+1}{2}} - l_2 (\mathbf{e}_\omega^{\mathrm{T}} \mathbf{e}_\omega)^{\frac{c_2+1}{2}} + \widetilde{\Delta}_\Omega^{\mathrm{T}} \widetilde{\Delta}_\Omega + \widetilde{\Delta}_\omega^{\mathrm{T}} \widetilde{\Delta}_\omega + \frac{1}{2} \widetilde{\Delta}_\Omega^{\mathrm{T}} \widetilde{\Delta}_\Omega + \frac{1}{2} \widetilde{\Delta}_\omega^{\mathrm{T}} \widetilde{\Delta}_\omega
\end{aligned} \tag{S30}
$$

There are two exponential combination situations that need to be discussed respectively,

1) If the conditions $b = b_1 = c_1$ and $c = b_2 = c_2$ are satisfied, considering Remark 1 and Lemma 3, then

$$
\begin{aligned}
\dot{V}_3 & \leq -M_1 \left( \bar{\mathbf{e}}_\Omega^{\mathrm{T}} \bar{\mathbf{e}}_\Omega + \mathbf{e}_\omega^{\mathrm{T}} \mathbf{e}_\omega \right) - M_2 \left( \bar{\mathbf{e}}_\Omega^{\mathrm{T}} \bar{\mathbf{e}}_\Omega + \mathbf{e}_\omega^{\mathrm{T}} \mathbf{e}_\omega \right)^{\frac{b+1}{2}} - M_3 \left( \bar{\mathbf{e}}_\Omega^{\mathrm{T}} \bar{\mathbf{e}}_\Omega + \mathbf{e}_\omega^{\mathrm{T}} \mathbf{e}_\omega \right)^{\frac{c+1}{2}} + \xi_D \\
& \leq -\overline{M}_1 V_3 - \overline{M}_2 V_3^{\frac{b+1}{2}} - \overline{M}_3 V_3^{\frac{c+1}{2}} + \xi_D
\end{aligned} \tag{S31}
$$

where

$$
M_1 = \min\left\{ \left(c_0 - \frac{1}{2}\right), \left(c_\omega - \frac{1}{2}\right) \right\},
$$

$$
M_2 = \min\{k_1, l_1\},
$$

$$M_3 = \min\{k_2, l_2\}, \xi_D = D_\Omega^2 + D_\omega^2 + D_\Omega^2 + D_\omega^2$$

and $\overline{M}_1 = 2M_1, \overline{M}_2 = 2^{\frac{b+1}{2}} M_2, \overline{M}_3 = 2^{\frac{c+1}{2}} M_3$.

According to Lemma 2, the semi-global practical fixed-time stability (SPFS) is achieved, and the convergence time is deduced by

$$T_3 \leq T_{3,\max} = 2\max\left\{\frac{\ln\left(1+\frac{\overline{M}_1}{\delta\overline{M}_2}\right)}{\overline{M}_1(1-b)}, \frac{\ln\left(1+\frac{\delta\overline{M}_1}{\overline{M}_2}\right)}{\delta\overline{M}_1(1-b)}\right\} + 2\max\left\{\frac{\ln(1+\frac{\overline{M}_1}{\delta\overline{M}_3})}{\overline{M}_1(c-1)}, \frac{\ln(1+\frac{\delta\overline{M}_1}{\overline{M}_3})}{\delta\overline{M}_1(c-1)}\right\} \tag{S32}$$

where $\delta = 0.5$, and the errors will converge to a sufficiently small residual set $e_i$

$$e_i \leq \min\left\{\frac{\xi_D}{\overline{M}_1(1-\delta)}, \left[\frac{\xi_D}{\overline{M}_2(1-\delta)}\right]^{\frac{2}{b+1}} \cdot \left[\frac{\xi_D}{\overline{M}_3(1-\delta)}\right]^{\frac{2}{c+1}}\right\} \tag{S33}$$

2) If $b_1 \neq c_1$ and $b_2 \neq c_2$ , then

$$\begin{aligned}
\dot{V}_3 \leq & -M_1\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega + \mathbf{e}_\omega^{\mathrm{T}}\mathbf{e}_\omega\right) - k_1\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega\right)^{\frac{b_1+1}{2}} - k_1\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega\right)^{\frac{c_1+1}{2}} + k_1\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega\right)^{\frac{c_1+1}{2}} \\
& -k_2\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega\right)^{\frac{b_2+1}{2}} - k_2\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega\right)^{\frac{c_2+1}{2}} + k_2\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega\right)^{\frac{c_2+1}{2}} - l_1(\mathbf{e}_\omega^{\mathrm{T}}\mathbf{e}_\omega)^{\frac{c_1+1}{2}} - l_2(\mathbf{e}_\omega^{\mathrm{T}}\mathbf{e}_\omega)^{\frac{c_2+1}{2}} + \xi_D \\
\leq & -M_1\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega + \mathbf{e}_\omega^{\mathrm{T}}\mathbf{e}_\omega\right) - M_2\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega + \mathbf{e}_\omega^{\mathrm{T}}\mathbf{e}_\omega\right)^{\frac{b_1+1}{2}} - M_3\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega + \mathbf{e}_\omega^{\mathrm{T}}\mathbf{e}_\omega\right)^{\frac{c_1+1}{2}} + \xi_D \\
& +k_1\left(\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega\right)^{\frac{c_1+1}{2}} - \left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega\right)^{\frac{b_2+1}{2}}\right) + k_2\left(\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega\right)^{\frac{c_2+1}{2}} - \left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega\right)^{\frac{b_2+1}{2}}\right)
\end{aligned} \tag{S34}$$

Setting $m_1 = \frac{c_1+1}{2}, n_1 = \frac{b_1+1}{2}, m_2 = \frac{c_2+1}{2}, n_2 = \frac{b_2+1}{2}$, then

$$\begin{aligned}
\dot{V}_3 \leq & -M_1\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega + \mathbf{e}_\omega^{\mathrm{T}}\mathbf{e}_\omega\right) - M_2\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega + \mathbf{e}_\omega^{\mathrm{T}}\mathbf{e}_\omega\right)^{\frac{c_1+1}{2}} \\
& -M_3\left(\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega + \mathbf{e}_\omega^{\mathrm{T}}\mathbf{e}_\omega\right)^{\frac{c_2+1}{2}} + \xi_D \\
\leq & -N_1 V_3 - N_2 V_3^{\frac{c_1+1}{2}} - N_3 V_3^{\frac{c_2+1}{2}} + \xi_D
\end{aligned} \tag{S35}$$

where $N_1 = 2M_1$, $N_2 = 2^{\frac{c_1+1}{2}} M_2$, $N_3 = 2^{\frac{c_2+1}{2}} M_3$.

There are two special cases that can make the above inequality hold: 1°: the term $\overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega > 1$, $0 < m_1 < n_1 < 1$ and $1 < m_2 < n_2$. 2°: the conditions $0 < \overline{\mathbf{e}}_\Omega^{\mathrm{T}}\overline{\mathbf{e}}_\Omega < 1$, $0 < n_1 < m_1 < 1$ and $1 < n_2 < m_2$ are satisfied. Then the error states $\overline{\mathbf{e}}_\Omega$ and $\mathbf{e}_\omega$ satisfy SPFS and the convergence time can be derived by

$$\begin{aligned}
T_3 \leq T_{3,\max} = & 2\max\left\{\frac{\ln(1+\frac{N_1}{\delta N_2})}{N_1(1-c_1)}, \frac{\ln(1+\frac{\delta N_1}{N_2})}{\delta N_1(1-c_1)}\right\} \\
& +2\max\left\{\frac{\ln(1+\frac{N_1}{\delta N_3})}{N_1(c_2-1)}, \frac{\ln(1+\frac{\delta N_1}{N_3})}{\delta N_1(c_2-1)}\right\}
\end{aligned} \tag{S36}$$

where $\delta = 0.5$, similarly, the sufficiently small residual set $e_i$ can be provided by

$$e_i \leq \min\left\{\frac{\xi_D}{N_1(1-\delta)}, \left[\frac{\xi_D}{N_2(1-\delta)}\right]^{\frac{c_1^2}{c_1+1}}, \left[\frac{\xi_D}{N_3(1-\delta)}\right]^{\frac{c_2^2}{c_2+1}}\right\} \tag{S37}$$

Then the upper bound of total convergence time of closed-loop system $T_{all}$ satisfies

$$T_{all} = \max\{T_\Omega, T_\omega\} + T_3 \tag{S38}$$

The proof is completed.