Frontiers of Information Technology & Electronic Engineering
www.jzus.zju.edu.cn; engineering.cae.cn; www.springerlink.com
ISSN 2095-9184 (print); ISSN 2095-9230 (online)
E-mail: jzus@zju.edu.cn

Supplementary materials for

Weibin CHEN, Yangyang CHEN, Ya ZHANG, 2022. Finite-time coordinated path-following control of leader-following multi-agent systems. *Front Inform Technol Electron Eng*, 23(10):1511-1521.
https://doi.org/10.1631/FITEE.2100476

1 Theorem 4.2 in Bhat and Bernstein (2000)

Theorem 4.2 Consider the system $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$ and $f(0) = 0$. Let $V : \mathcal{D} \to \mathbb{R}$ be a positive definite continuous function. Suppose that there exist real numbers $c > 0$, $\alpha \in (0, 1)$, and an open neighborhood $\mathcal{V} \subseteq \mathcal{D}$ of the origin, such that $\dot{V}(x) + c(V(x))^\alpha \leq 0$, $x \in \mathcal{V}\backslash\{0\}$. Then the origin $x = 0$ is finite-time stable; i.e., there exists a finite time $T > 0$, $T \leq \frac{(V(0))^{1-\alpha}}{c(1-\alpha)}$, such that $V(x) = 0$, $\forall t > T$.

2 Theorem 5.3 in Bhat and Bernstein (2000)

Theorem 5.3 Consider the system $\dot{x} = f(x) + g(x)$, where $x \in \mathbb{R}^n$ and g is the perturbation term. Let $V : \mathcal{D} \to \mathbb{R}$ be a positive definite continuous function that satisfies $\dot{V}(x) + c(V(x))^\alpha \leq 0$, $x \in \mathcal{V}\backslash\{0\}$, where $\mathcal{V} \subseteq \mathcal{D}$ is an open neighborhood of the origin, $c > 0$, and $\alpha \in (0, \frac{1}{2})$. Then, for every $L \geq 0$, there exists an open neighborhood $\mathcal{U}$ of the origin and $T > 0$ such that, for every continuous function $g : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}^n$ satisfying

$$\|g(t, x)\| \leq L\|x\|,\ (t, x) \in \mathbb{R}_+ \times \mathcal{D},$$

every right maximally defined solution x with $x(0) \in \mathcal{U}$ is defined on $\mathbb{R}_+$ and satisfies $x(t) \in \mathcal{U}$, for all $t \in \mathbb{R}_+$, and $x(t) = 0$ for all $t > T$.

3 Proof of Eq. (10)

Because the arc-lengths are given by

$$s_i\left(\lambda_i, \phi_i\right) \triangleq \int_{\phi_i^*}^{\phi_i} \frac{\partial s_i\left(\lambda_i, \tau\right)}{\partial \tau} \mathrm{d}\tau,$$

where ϕ_i^* is the parameter associated with the starting point of the arc of s_i, the total variation of arc-length s_i is given by

$$\dot{s}_i = \frac{\partial s_i(\lambda_i, \phi_i)}{\partial \phi_i}\frac{\partial \phi_i}{\partial t} + \frac{\partial s_i(\lambda_i, \phi_i)}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial t} = \frac{\mathrm{d}s_i}{\mathrm{d}t}\Big|_{\lambda_i=\mathrm{const}} + \frac{\partial s_i(\lambda_i, \phi_i)}{\partial \lambda_i}\dot{\lambda}_i = T_i^{\mathrm{T}} v_i + \frac{\partial s_i(\lambda_i, \phi_i)}{\partial \lambda_i}\left\|\nabla \lambda_i\right\| v_{N_i}.$$

Note that $\dot{\lambda}_i = \left\|\nabla \lambda_i\right\| v_{N_i}$. The derivative of generalized arc-length ξ_i is

$$\dot{\xi}_i = \frac{\partial \xi_i}{\partial s_i} T_i^{\mathrm{T}} v_i + \frac{\partial \xi_i}{\partial s_i}\frac{\partial s_i}{\partial \lambda_i}\left\|\nabla \lambda_i\right\| v_{N_i}.$$

Then the dynamics of ξ_i is given by

$$\dot{\xi}_i = \frac{\partial \xi_i}{\partial s_i} v_{T_i} + \varDelta_{\xi_i},$$

where $v_{T_i} = T_i^{\mathrm{T}} v_i$ and $\varDelta_{\xi_i} = \frac{\partial \xi_i}{\partial s_i}\frac{\partial s_i}{\partial \lambda_i}\left\|\nabla \lambda_i\right\| v_{N_i}$.

4 Proof of inequality (15)

Let $\bar{v}_{N_i} = v_{N_i}^{\alpha} - \hat{v}_{N_i}^{\alpha}$, $\tilde{v}_{N_i} = v_{N_i} - \hat{v}_{N_i}$, and $\hat{v}_{N_i} = -k_1 \left(\nabla \Psi_i\right)^{\frac{1}{\alpha}}$. Differentiating Eq. (14) along Eqs. (8), (9), and (13) yields

$$\begin{aligned}
\dot{V}_{\mathrm{P}} =& \sum_{i=1}^{n} \nabla \Psi_i \|\nabla \lambda_i\| \left(\tilde{v}_{N_i} + \hat{v}_{N_i}\right) + \gamma_1 \sum_{i=1}^{n} \left(v_{N_i}^{\alpha} - \hat{v}_{N_i}^{\alpha}\right)^{2-\frac{1}{\alpha}} \left(u_{N_i} + \Delta_{N_i}\right) \\
&+ \sum_{i=1}^{n} \left(\frac{-1}{k_1^{1+\alpha}} \frac{\partial \hat{v}_{N_i}^{\alpha}}{\partial \lambda_i} \int_{\hat{v}_{N_i}}^{v_{N_i}} \left(\tau^{\alpha} - \hat{v}_{N_i}^{\alpha}\right)^{1-\frac{1}{\alpha}} \mathrm{d}\tau \right) \dot{\lambda}_i \\
=& \sum_{i=1}^{n} \nabla \Psi_i \left\|\nabla \lambda_i\right\| \tilde{v}_{N_i} - \sum_{i=1}^{n} k_1 \left\|\nabla \lambda_i\right\| \left(\nabla \Psi_i\right)^{1+\frac{1}{\alpha}} + \gamma_1 \sum_{i=1}^{n} \left(v_{N_i}^{\alpha} - \hat{v}_{N_i}^{\alpha}\right)^{2-\frac{1}{\alpha}} \left(u_{N_i} + \Delta_{N_i}\right) \\
&+ \sum_{i=1}^{n} \left(k_1^{-1} \nabla^2 \Psi_i \int_{\hat{v}_{N_i}}^{v_{N_i}} \left(\tau^{\alpha} - \hat{v}_{N_i}^{\alpha}\right)^{1-\frac{1}{\alpha}} \mathrm{d}\tau \right) \left\|\nabla \lambda_i\right\| v_{N_i} \\
\leq& \sum_{i=1}^{n} \left\|\nabla \lambda_i\right\| \left|\nabla \Psi_i \tilde{v}_{N_i}\right| - \sum_{i=1}^{n} k_1 \left\|\nabla \lambda_i\right\| \left(\nabla \Psi_i\right)^{1+\frac{1}{\alpha}} + \gamma_1 \sum_{i=1}^{n} \bar{v}_{N_i}^{2-\frac{1}{\alpha}} \left(u_{N_i} + \Delta_{N_i}\right) + f_{\mathrm{P}},
\end{aligned}$$

where

$$f_{\mathrm{P}} = \sum_{i=1}^{n} k_1^{-1} \left\|\nabla \lambda_i\right\| \left|\nabla^2 \Psi_i\right| \left|\bar{v}_{N_i}\right|^{1-\frac{1}{\alpha}} \left|v_{N_i}\right| \left|\tilde{v}_{N_i}\right|.$$

5 Proof of inequality (16)

Because $\left|\tilde{v}_{N_i}\right| = \left|\left(v_{N_i}^{\alpha}\right)^{\frac{1}{\alpha}} - \left(\hat{v}_{N_i}^{\alpha}\right)^{\frac{1}{\alpha}}\right|$, according to Lemmas A.1 and A.2 in Qian and Lin (2001), we have

$$\begin{aligned}
\left|\tilde{v}_{N_i}\right| &\leq 2^{1-\frac{1}{\alpha}} \left|v_{N_i}^{\alpha} - \hat{v}_{N_i}^{\alpha}\right|^{\frac{1}{\alpha}}, \\
\left|\nabla \Psi_i \tilde{v}_{N_i}\right| &\leq \frac{1}{1+\frac{1}{\alpha}} \phi_1 \left|2 \nabla \Psi_i\right|^{1+\frac{1}{\alpha}} + \frac{\frac{1}{\alpha}}{1+\frac{1}{\alpha}} \phi_1^{-\alpha} \left|2^{-1} \bar{v}_{N_i}\right|^{1+\frac{1}{\alpha}} = \left|\nabla \Psi_i\right|^{1+\frac{1}{\alpha}} + c_{\phi_1} \left|\bar{v}_{N_i}\right|^{1+\frac{1}{\alpha}}, \\
\left|\bar{v}_{N_i}\right| \left|\nabla \Psi_i\right|^{\frac{1}{\alpha}} &\leq \frac{\frac{1}{\alpha}}{1+\frac{1}{\alpha}} \phi_2 \left|\nabla \Psi_i\right|^{1+\frac{1}{\alpha}} + \frac{1}{1+\frac{1}{\alpha}} \phi_2^{-\frac{1}{\alpha}} \left|\bar{v}_{N_i}\right|^{1+\frac{1}{\alpha}} = \left|\nabla \Psi_i\right|^{1+\frac{1}{\alpha}} + c_{\phi_2} \left|\bar{v}_{N_i}\right|^{1+\frac{1}{\alpha}},
\end{aligned}$$

where $\phi_1 = 2^{-1-\frac{1}{\alpha}}(1+\alpha)/\alpha$, $\phi_2 = 1+\alpha$, $c_{\phi_1} = 2^{-1-\frac{1}{\alpha}} \phi_1^{-\alpha}/(1+\alpha)$, and $c_{\phi_2} = \phi_2^{-\frac{1}{\alpha}} \alpha/(1+\alpha)$.

6 Proof of inequality (18)

Exploiting $\left|\nabla \Psi_i \tilde{v}_{N_i}\right| \leq \left|\nabla \Psi_i\right|^{1+\frac{1}{\alpha}} + c_{\phi_1} \left|\bar{v}_{N_i}\right|^{1+\frac{1}{\alpha}}$ and inequality (17) yields

$$\begin{aligned}
\dot{V}_{\mathrm{P}} \leq& \sum_{i=1}^{n} \left\|\nabla \lambda_i\right\| \left(\left|\nabla \Psi_i\right|^{1+\frac{1}{\alpha}} + c_{\phi_1} \left|\bar{v}_{N_i}\right|^{1+\frac{1}{\alpha}}\right) - \sum_{i=1}^{n} k_1 \left\|\nabla \lambda_i\right\| \left(\nabla \Psi_i\right)^{1+\frac{1}{\alpha}} + \gamma_1 \sum_{i=1}^{n} \bar{v}_{N_i}^{2-\frac{1}{\alpha}} \left(u_{N_i} + \Delta_{N_i}\right) \\
&+ \sum_{i=1}^{n} k_1^{-1} \left\|\nabla \lambda_i\right\| \left|\nabla^2 \Psi_i\right| \left[2^{2-\frac{2}{\alpha}} \bar{v}_{N_i}^{1+\frac{1}{\alpha}} + k_1 2^{1-\frac{1}{\alpha}} \left(\left|\nabla \Psi_i\right|^{1+\frac{1}{\alpha}} + c_{\phi_2} \left|\bar{v}_{N_i}\right|^{1+\frac{1}{\alpha}}\right)\right] \\
=& \sum_{i=1}^{n} \left\|\nabla \lambda_i\right\| \left|\nabla \Psi_i\right|^{1+\frac{1}{\alpha}} - \sum_{i=1}^{n} k_1 \left\|\nabla \lambda_i\right\| \left(\nabla \Psi_i\right)^{1+\frac{1}{\alpha}} + \sum_{i=1}^{n} 2^{1-\frac{1}{\alpha}} \left\|\nabla \lambda_i\right\| \left|\nabla^2 \Psi_i\right| \left|\nabla \Psi_i\right|^{1+\frac{1}{\alpha}} \\
&+ \gamma_1 \sum_{i=1}^{n} \bar{v}_{N_i}^{2-\frac{1}{\alpha}} \left(u_{N_i} + \Delta_{N_i}\right) + \sum_{i=1}^{n} k_1^{-1} 2^{2-\frac{2}{\alpha}} \left\|\nabla \lambda_i\right\| \left|\nabla^2 \Psi_i\right| \left|\bar{v}_{N_i}\right|^{1+\frac{1}{\alpha}} \\
&+ \sum_{i=1}^{n} \left\|\nabla \lambda_i\right\| c_{\phi_1} \left|\bar{v}_{N_i}\right|^{1+\frac{1}{\alpha}} + \sum_{i=1}^{n} 2^{1-\frac{1}{\alpha}} \left\|\nabla \lambda_i\right\| \left|\nabla^2 \Psi_i\right| c_{\phi_2} \left|\bar{v}_{N_i}\right|^{1+\frac{1}{\alpha}}.
\end{aligned}$$

Note that $c_{\underline{\lambda}} \le \|\nabla \lambda_i\| \le c_{\bar{\lambda}}$ and $\left|\nabla^2 \Psi_i\right| \le c_{\Psi^2}$. We have

$$\dot{V}_{\mathrm{P}} \le -\sum_{i=1}^{n}\left(k_1 c_{\underline{\lambda}} - c_{\mathrm{N}1}\right)(\nabla \Psi_i)^{1+\frac{1}{\alpha}} + \gamma_1 \sum_{i=1}^{n} \bar{v}_{N_i}^{2-\frac{1}{\alpha}}\left(u_{N_i} + \Delta_{N_i}\right) + \sum_{i=1}^{n} c_{\mathrm{N}2} \bar{v}_{N_i}^{1+\frac{1}{\alpha}},$$

where $c_{\mathrm{N}1} = c_{\bar{\lambda}} + c_{\bar{\lambda}} 2^{1-\frac{1}{\alpha}} c_{\Psi^2}$ and $c_{\mathrm{N}2} = c_{\bar{\lambda}}\left(c_{\phi_1} + k_1^{-1} 2^{2-\frac{2}{\alpha}} c_{\Psi^2} + 2^{1-\frac{1}{\alpha}} c_{\phi_2} c_{\Psi^2}\right)$.

7 Proof of Eq. (27)

Note that $\hat{v}_{T_i} = -\left(\frac{\partial \xi_i}{\partial s_i}\right)^{-1} k_3 \varsigma_i^{\frac{1}{\alpha}}$. Differentiating Eq. (26) along Eqs. (11), (12), and (25) yields

$$\begin{aligned}
\dot{V}_{\mathrm{F}} =& \sum_{i=1}^{n} \varsigma_i \dot{\varsigma}_i + \gamma_2 \sum_{i=1}^{n}\left(v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha}\right)^{2-\frac{1}{\alpha}} \dot{v}_{T_i} + \gamma_2 \sum_{i=1}^{n} \int_{\hat{v}_{T_i}}^{v_{T_i}} -(2-\frac{1}{\alpha}) \dot{\hat{v}}_{T_i}^{\alpha}\left(\tau^{\alpha} - \hat{v}_{T_i}^{\alpha}\right)^{1-\frac{1}{\alpha}} \mathrm{d}\tau \\
=& \sum_{i=1}^{n} \varsigma_i \sum_{j=0}^{n} a_{ij}\left(\frac{\partial \xi_i}{\partial s_i} v_{T_i} + \Delta_{\xi_i} - \frac{\partial \xi_j}{\partial s_j} v_{T_j} - \Delta_{\xi_j}\right) + \gamma_2 \sum_{i=1}^{n}\left(v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha}\right)^{2-\frac{1}{\alpha}}\left(u_{T_i} + \Delta_{T_i}\right) \\
&+ \frac{1}{k_3} \sum_{i=1}^{n}\left(\frac{\partial \xi_i}{\partial s_i}\right)^{-\alpha} \int_{\hat{v}_{T_i}}^{v_{T_i}}\left(\tau^{\alpha} - \hat{v}_{T_i}^{\alpha}\right)^{1-\frac{1}{\alpha}} \mathrm{d}\tau \sum_{j=0}^{n} a_{ij}\left(\frac{\partial \xi_i}{\partial s_i} v_{T_i} + \Delta_{\xi_i} - \frac{\partial \xi_j}{\partial s_j} v_{T_j} - \Delta_{\xi_j}\right) \\
=& -k_3 \sum_{i=1}^{n} \varsigma_i \sum_{j=0}^{n} a_{ij}\left(\varsigma_i^{\frac{1}{\alpha}} - \varsigma_j^{\frac{1}{\alpha}}\right) + f_{\mathrm{F}} + \gamma_2 \sum_{i=1}^{n}\left(v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha}\right)^{2-\frac{1}{\alpha}}\left(u_{T_i} + \Delta_{T_i}\right) + g_{\mathrm{F}1} + g_{\mathrm{F}2},
\end{aligned}$$

where

$$f_{\mathrm{F}} = \sum_{i=1}^{n} \varsigma_i \sum_{j=0}^{n} a_{ij}\left(\frac{\partial \xi_i}{\partial s_i} \tilde{v}_{T_i} - \frac{\partial \xi_j}{\partial s_j} \tilde{v}_{T_j}\right)$$

$$g_{\mathrm{F}1} = \sum_{i=1}^{n} \varsigma_i \sum_{j=0}^{n} a_{ij}\left(\Delta_{\xi_i} - \Delta_{\xi_j}\right),$$

$$g_{\mathrm{F}2} = \frac{1}{k_3} \sum_{i=1}^{n}\left(\frac{\partial \xi_i}{\partial s_i}\right)^{-\alpha} \int_{\hat{v}_{T_i}}^{v_{T_i}}\left(\tau^{\alpha} - \hat{v}_{T_i}^{\alpha}\right)^{1-\frac{1}{\alpha}} \mathrm{d}\tau \sum_{j=0}^{n} a_{ij}\left(\frac{\partial \xi_i}{\partial s_i} v_{T_i} + \Delta_{\xi_i} - \frac{\partial \xi_j}{\partial s_j} v_{T_j} - \Delta_{\xi_j}\right).$$

8 Proof of inequality (37)

From $|\tilde{v}_{T_i}| \le 2^{1-\frac{1}{\alpha}}\left|v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha}\right|^{\frac{1}{\alpha}}$ and $\hat{v}_{T_i} = -\left(\frac{\partial \xi_i}{\partial s_i}\right)^{-1} k_3 \varsigma_i^{\frac{1}{\alpha}}$, we have

$$\begin{cases}
\left|\hat{v}_{T_i}\right|\left|v_{T_i} - \hat{v}_{T_i}\right| \le k_3 c_{\underline{\xi}}^{-1} 2^{1-\frac{1}{\alpha}} |\varsigma_i|^{\frac{1}{\alpha}}\left|v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha}\right|^{\frac{1}{\alpha}}, \\
\left|\hat{v}_{T_j}\right|\left|v_{T_i} - \hat{v}_{T_i}\right| \le k_3 c_{\underline{\xi}}^{-1} 2^{1-\frac{1}{\alpha}} |\varsigma_j|^{\frac{1}{\alpha}}\left|v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha}\right|^{\frac{1}{\alpha}}.
\end{cases}$$

Due to the fact that

$$\begin{aligned}
|v_{T_i}|\left|v_{T_i} - \hat{v}_{T_i}\right| &\le \left|v_{T_i} - \hat{v}_{T_i}\right|^2 + \left|\hat{v}_{T_i}\right|\left|v_{T_i} - \hat{v}_{T_i}\right| \\
&\le 2^{2-\frac{2}{\alpha}}\left(v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha}\right)^{\frac{2}{\alpha}} + k_3 c_{\underline{\xi}}^{-1} 2^{1-\frac{1}{\alpha}} |\varsigma_i|^{\frac{1}{\alpha}}\left|v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha}\right|^{\frac{1}{\alpha}}, \\
|v_{T_j}|\left|v_{T_i} - \hat{v}_{T_i}\right| &\le \left|v_{T_j} - \hat{v}_{T_j}\right|\left|v_{T_i} - \hat{v}_{T_i}\right| + \left|\hat{v}_{T_j}\right|\left|v_{T_i} - \hat{v}_{T_i}\right| \\
&\le 2^{2-\frac{2}{\alpha}}\left|v_{T_j}^{\alpha} - \hat{v}_{T_j}^{\alpha}\right|^{\frac{1}{\alpha}}\left|v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha}\right|^{\frac{1}{\alpha}} + k_3 c_{\underline{\xi}}^{-1} 2^{1-\frac{1}{\alpha}} |\varsigma_j|^{\frac{1}{\alpha}}\left|v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha}\right|^{\frac{1}{\alpha}},
\end{aligned}$$

we have

$$g_{\mathrm{F}21} \leq \sum_{i=1}^{n} k_3^{-1} c_{\underline{\xi}}^{-\alpha} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right|^{1-\frac{1}{\alpha}} g_{\mathrm{F}23},$$

where

$$\begin{aligned} g_{\mathrm{F}23} =& \gamma_3 c_{\bar{\xi}} \left[2^{2-\frac{2}{\alpha}} \left(v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right)^{\frac{2}{\alpha}} + k_3 c_{\underline{\xi}}^{-1} 2^{1-\frac{1}{\alpha}} |\varsigma_i|^{\frac{1}{\alpha}} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right|^{\frac{1}{\alpha}} \right] \\ &+ \sum_{j=0}^{n} \gamma_4 c_{\bar{\xi}} \left(2^{2-\frac{2}{\alpha}} \left| v_{T_j}^{\alpha} - \hat{v}_{T_j}^{\alpha} \right|^{\frac{1}{\alpha}} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right|^{\frac{1}{\alpha}} + k_3 c_{\underline{\xi}}^{-1} 2^{1-\frac{1}{\alpha}} \left| \varsigma_j \right|^{\frac{1}{\alpha}} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right|^{\frac{1}{\alpha}} \right). \end{aligned}$$

Simplifying the above equation, we obtain

$$\begin{aligned} g_{\mathrm{F}21} \leq& \sum_{i=1}^{n} k_3^{-1} c_{\underline{\xi}}^{-\alpha} \gamma_3 c_{\bar{\xi}} 2^{2-\frac{2}{\alpha}} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right|^{1+\frac{1}{\alpha}} + \sum_{i=1}^{n} c_{\underline{\xi}}^{-\alpha-1} \gamma_3 c_{\bar{\xi}} 2^{1-\frac{1}{\alpha}} |\varsigma_i|^{\frac{1}{\alpha}} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right| \\ &+ \sum_{i=1}^{n} k_3^{-1} c_{\underline{\xi}}^{-\alpha} \sum_{j=0}^{n} \gamma_4 c_{\bar{\xi}} 2^{2-\frac{2}{\alpha}} \left| v_{T_j}^{\alpha} - \hat{v}_{T_j}^{\alpha} \right|^{\frac{1}{\alpha}} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right| + \sum_{i=1}^{n} c_{\underline{\xi}}^{-\alpha-1} \sum_{j=0}^{n} \gamma_4 c_{\bar{\xi}} 2^{1-\frac{1}{\alpha}} \left| \varsigma_j \right|^{\frac{1}{\alpha}} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right| \\ =& \sum_{i=1}^{n} c_{\mathrm{g}1} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right|^{1+\frac{1}{\alpha}} + \sum_{i=1}^{n} c_{\mathrm{g}2} |\varsigma_i|^{\frac{1}{\alpha}} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right| + \sum_{i=1}^{n} \sum_{j=0}^{n} c_{\mathrm{g}3} \left| v_{T_j}^{\alpha} - \hat{v}_{T_j}^{\alpha} \right|^{\frac{1}{\alpha}} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right| \\ &+ \sum_{i=1}^{n} \sum_{j=0}^{n} c_{\mathrm{g}4} \left| \varsigma_j \right|^{\frac{1}{\alpha}} \left| v_{T_i}^{\alpha} - \hat{v}_{T_i}^{\alpha} \right|, \end{aligned}$$

where $c_{\mathrm{g}1} = k_3^{-1} c_{\underline{\xi}}^{-\alpha} \gamma_3 c_{\bar{\xi}} 2^{2-\frac{2}{\alpha}}$, $c_{\mathrm{g}2} = c_{\underline{\xi}}^{-\alpha-1} \gamma_3 c_{\bar{\xi}} 2^{1-\frac{1}{\alpha}}$, $c_{\mathrm{g}3} = k_3^{-1} c_{\underline{\xi}}^{-\alpha} \gamma_4 c_{\bar{\xi}} 2^{2-\frac{2}{\alpha}}$, and $c_{\mathrm{g}4} = c_{\underline{\xi}}^{-\alpha-1} \gamma_4 c_{\bar{\xi}} 2^{1-\frac{1}{\alpha}}$.