Electronic Supplementary Materials

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A novel stress-based formulation of finite element analysis

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Data S1 Proposed finite element formulation

Consider the three-dimensional body in Fig. S1, with the given boundary conditions u_i=0 on surface S_0. The body has a traction f_i^S acting on surface S_f. The body undergoes internal displacements u_{ij} because of the internal stresses τ_{ij}.

For the elastic body, to be in static equilibrium for the given combination of loading condition (Timoshenko and Goodier, 1934),

$$\sigma_{ij,j}=0, \tag{S1}$$

with the natural boundary conditions

$$\sigma_{ij}n_j=f_i^S,\ \text{ on } S_f, \tag{S2}$$

and essential boundary conditions,

$$u_i=0, \qquad \text{on } S_0, \tag{S3}$$

where, 'n_j' is the unit normal vector anywhere at surface 'S' of the body. Let us say that the body is displaced by the virtual displacement $\bar{u}_i$. It is evident that following the boundary condition satisfies,

$$\bar{u}_i=0, \qquad \text{on } S_0 \tag{S4}$$

hence,

$$\sigma_{ij,j}\bar{u}_i=0, \tag{S5}$$

integrating,

$$\int_V \sigma_{ij,j}\bar{u}_i\,\mathrm{d}V=0. \tag{S6}$$

Since

$$\begin{aligned}&\left(\sigma_{ij}\bar{u}_i\right)_{,j}=\sigma_{ij,j}\bar{u}_i+\sigma_{ij}\bar{u}_{i,j},\\&\int_V\left(\left(\sigma_{ij}\bar{u}_i\right)_{,j}-\sigma_{ij}\bar{u}_{i,j}\right)\mathrm{d}V=0.\end{aligned} \tag{S7}$$

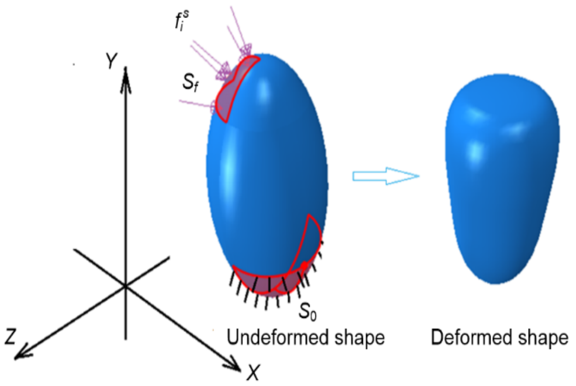

Fig. S1 A three-dimensional body subjected to typical load

We wish to emphasize here in the continuum for a structural body, either it is linear or a linear range of the loading condition, the divergence theorem is satisfied at each point of the loading condition (Gauss and Friedrich, 1867). Hence, from the divergence theorem,

$$\int_V \left(\sigma_{ij}\bar{u}_i\right)_{,j} \mathrm{d}V = \int_S \left(\sigma_{ij}\bar{u}_i\right) n_j \,\mathrm{d}S, \tag{S8}$$

substituting it in Eq. (S7),

$$-\int_V \sigma_{ij}\bar{u}_{i,j}\,\mathrm{d}V + \int_S \left(\sigma_{ij}\bar{u}_i\right) n_j \,\mathrm{d}S = 0. \tag{S9}$$

From the natural boundary condition of (S2) and Eq. (S4),

$$-\int_V \sigma_{ij}\bar{u}_{i,j}\,\mathrm{d}V + \int_{S_f} f_i^S \bar{u}_i \,\mathrm{d}S = 0.$$

Substituting $\bar{u}_{i,j} = \bar{\varepsilon}_{ij}$ and rearranging,

$$\int_V \sigma_{ij}\bar{\varepsilon}_{ij}\,\mathrm{d}V = \int_{S_f} f_i^S \bar{u}_i \,\mathrm{d}S. \tag{S10}$$

In this equation, virtual displacements ($\overline{u_i}$) and virtual strains $\overline{\varepsilon_{ij}}$ can be replaced by real displacement/strain and the equation can be applied to solve a typical structural problem. Thus, this equation can further be implied in general form as,

$$\int_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V = \int_{S_f} f_i^S u_i \,\mathrm{d}S. \tag{S10.a}$$

This equation is used for our nonlinear analysis finite element formulation. Stress, strain and displacement can be directly substituted in the equation in order to obtain the desired results. Consider the three-dimensional element in the Cartesian coordinate as shown in Fig. 3b. Consider that only the axial stress in x-direction 'σ_x' is experienced by the element. The total axial force on the element can be evaluated by,

$$F_x = \sigma_x \, \mathrm{d} z \, \mathrm{d} y. \tag{S11}$$

If the longitudinal strain in the element is ε_x, its deformation can be evaluated by,

$$u_x = \varepsilon_x \mathrm{d}x. \tag{S12}$$

The strain energy stored in the element can be evalauted by,

$$\begin{aligned} &\text{Strain energy} = F_x u_x = \sigma_x \varepsilon_x \mathrm{d}x\mathrm{d}z\mathrm{d}y, \\ &\text{or, Strain energy} = \sigma_x \varepsilon_x \mathrm{d}V. \end{aligned} \tag{S13}$$

Here *'V'* stands for volume. Thus, the total strain energy stored in it can be evaluated by simple integration.

$$\text{Strain energy} = \int_V \sigma_x \varepsilon_x \mathrm{d}V. \tag{S14}$$

Consider now that the discrete element of Fig. 3b experiences only axial stress σ_x. Eq. (S10.a) can be modified for this element as,

$$\int_V \sigma_x \varepsilon_x \, \mathrm{d} V = F_x u_x. \tag{S15}$$

Here, $\int_{S_f} f_i^S \mathrm{d}S = F_x$, as stress at any *YZ*-plane is constant. Eq. (S15) can be written as,

$$\int_A \mathrm{d}\, y \, \mathrm{d}\, z \int_L \sigma_x \varepsilon_x \, \mathrm{d}\, x = F_x u_x. \tag{S16}$$

Substituting stress, strain, and displacement functions from Eqs. (1)–(3),

$$\begin{aligned} &A \int_L \left(473.9\mathrm{e}^{0.06458r} - \frac{0.03438}{\mathrm{e}^{9.5r}} \right) \left(0.01545r + 0.01546 \right) \mathrm{d}x \\ &= F_x \left(0.00773r^2 + 0.01546r + 0.00773 \right) \end{aligned}$$

Here '*A*' is the total area of the element cross section in the *YZ*-plane and *'J'* is the Jacobian which transfers Cartesian coordinates to the reference coordinate system. Differentiating the equation,

$$\begin{aligned} &A \left(473.9\mathrm{e}^{0.06458r} - \frac{0.03438}{\mathrm{e}^{9.5r}} \right) \left(0.01545r + 0.01546 \right) \\ &= F_x \left(0.01545r + 0.01546 \right), \\ &\text{or, } 473.9\mathrm{e}^{0.06458r} - \frac{0.03438}{\mathrm{e}^{9.5r}} = \frac{F_x}{A}. \end{aligned} \tag{S17}$$

This force balance equation is used in order to find the reference coordinate system (r) for the structural body subjected to loading.

Data S2 Derivation of shear stress and shear strain functions from normal stress and normal strain functions

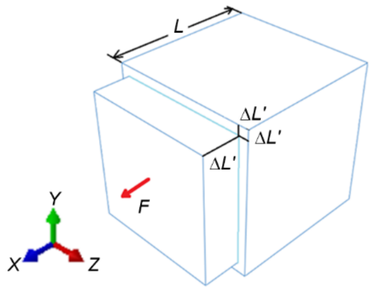

Fig. S2 A 3-D element with uniaxial loading

With the definition of Poisson's ratio (Fig. S2), as it is the ratio of lateral strain to longitudinal strain (Timoshenko and Goodier, 1934),

$$\nu = \frac{\text{Strain}_{\text{lateral}}}{\text{Strain}_{\text{longitudinal}}} = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x},$$

(S18)

$$\text{or,} \quad \varepsilon_y = \varepsilon_z = -\nu\varepsilon_x \qquad \text{(S18.a)}$$

Here,

$$\varepsilon_x = \frac{\Delta L}{L}; \varepsilon_y = \frac{\Delta L^{'}}{L}; \varepsilon_z = \frac{\Delta L^{'}}{L}. \qquad \text{(S19)}$$

With the definition of Poisson's ratio, the strain components in three dimensions can be written as,

$$\begin{aligned} &\varepsilon_x = \varepsilon_x - \nu\left(\varepsilon_y + \varepsilon_z\right), \varepsilon_y = \varepsilon_y - \nu\left(\varepsilon_x + \varepsilon_z\right), \\ &\varepsilon_z = \varepsilon_z - \nu\left(\varepsilon_x + \varepsilon_y\right). \end{aligned} \qquad \text{(S20)}$$

Consider the special load case of a 2D plane body subjected to stresses σ_x and σ_y as shown in Fig. S3.

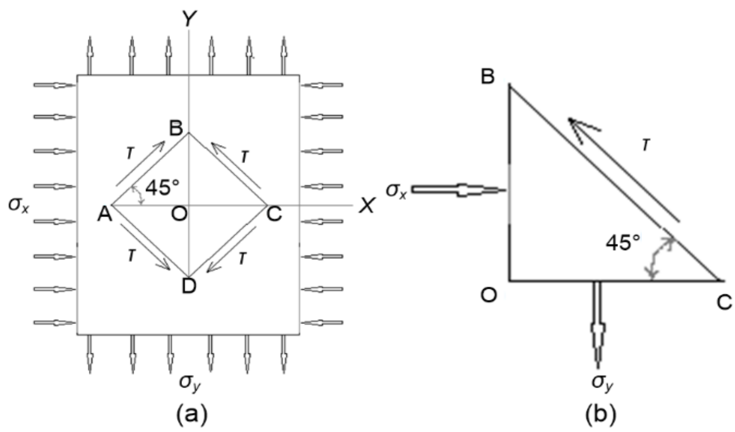

Fig. S3 The two-dimensional plane body

(a) A two-dimensional sheet subjected to normal stresses; (b) The free body diagram of element OBC

From Fig. S3 (b), the force equilibrium:

$$\tau = \frac{1}{2} = \left(\sigma_x + \sigma_y\right) = \sigma_x . \tag{S21}$$

After deformation, shear strain in the element can be represented as,

$$\gamma = \frac{\mathrm{OC}'}{\mathrm{OB}'} = \tan\left(45^0 - \frac{\gamma}{2}\right). \tag{S22}$$

Here OC' and OB' are the lengths after deformation.

$$\mathrm{OB}' = \mathrm{OB} + \varepsilon_x \mathrm{OB},\ \mathrm{OC}' = \mathrm{OC} - \varepsilon_y \mathrm{OC}, \tag{S23}$$

substituting,

$$\frac{\mathrm{OB} + \varepsilon_x \mathrm{OB}}{\mathrm{OC} - \varepsilon_y \mathrm{OC}} = \tan\left(45^0 - \frac{\gamma}{2}\right) = \frac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}},$$

considering initial lengths OC and OB as a unity,

$$\frac{1 - \varepsilon_x}{1 + \varepsilon_y} = \frac{2 - \gamma}{2 + \gamma}.$$

For this two-dimensional case as ε_z=0 and ε_x and ε_y from Eq. (S20) can be written as,

$$\varepsilon_x = \varepsilon_x - \nu\varepsilon_y, \varepsilon_y = \varepsilon_y - \nu\varepsilon_x.$$

Substituting ε_x and ε_x,

$$\frac{1 - \varepsilon_x + \nu\varepsilon_x}{1 + \varepsilon_x - \nu\varepsilon_x} = \frac{2 - \gamma}{2 + \gamma}$$

solving,

$$\gamma = 2\left(1 + \nu\right)\varepsilon. \tag{S24}$$

From Eqs. (S21) and (S24), the shear stress and shear strain functions can be derived as follows,

$$\tau_{xy} = \tau_{xz} = \tau_{yz} = 473.9\mathrm{e}^{0.06458r} - \frac{0.03438}{\mathrm{e}^{9.5r}}, \tag{S25}$$

$$\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 2\left(1 + \nu\right)\left(0.01545r + 0.01546\right). \tag{S26}$$

References

Gauss CF, 1867. Werke. Cambridge Library Collection– Mathematics. Cambridge University Press, Cambridge, UK.

Timoshenko SP, Goodier JN, 1934. Theory of Elasticity. 3rd Edition. Mcgraw Hill, New York, USA.