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Analysis of Pareto equilibrium in multi-objective games using the semi-tensor product*

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Abstract: Multi-objective games (MOGs) have received much attention in recent years is a class of games with payoff vectors. Based on the semi-tensor product (STP), this paper discusses the MOG, including the existence, finite-step reachability and finite-step controllability of Pareto equilibrium of this model from both static and dynamic perspectives. First, the MOG concept is presented using multi-layer graphs, and the STP is used to convert the payoff function into its algebraic form. Then, from the static perspective, two necessary and sufficient conditions are proposed to verify whether all players can meet their expectations and whether the strategy profile is a Pareto equilibrium, respectively. Thus, from the dynamic perspective, a strategy updating rule is designed to investigate the finite-step reachability of the evolutionary MOG. Finally, the finite-step controllability of the evolutionary MOG is analyzed by adding pseudo-players, and a backward search algorithm is provided to find the shortest evolutionary process and control sequence.

Key words: Multi-objective game; Pareto equilibrium; Semi-tensor product; Finite-step reachability; Finite-step controllability

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1 Introduction

As one of important tools in researching interaction mechanisms and behavior among individuals, game theory has a wide range of applications in areas such as cybernetics, biology, and economics. It has also become a popular interdisciplinary field.

The payoff function is one of the important components of finite games, and can be categorized into scalar and vector forms. Most of the existing references focus on the cases where payoffs are scalar. However, games with payoff vectors, which are commonly known as multi-objective games (MOGs) or multi-criteria games, are ubiquitous in reality. Games with payoff vectors offer a noteworthy study topic when internal motivation is considered (Puerto and Perea, 2017). For instance, consider tobacco consumers as players. As discussed in Ismaili (2018), there are three components of every player's payoff vector: the pleasure of smoking, the cost of cigarettes, and the impact on life expectancy. Consequently, the payoff can be modeled in the form of a vector, the numerical components of which demonstrate commodities (Hamel and Andreas, 2017), and the ultimate payoff depends on each component of the payoff vector. For example, Rădulescu et al. (2020) applied this type of game to MOG multi-agent systems and every component of the payoff vector represented the performance on a different objective.

Equilibrium analysis is an important issues in game theory. For games with scalar payoffs, the existence of equilibria has been investigated, such as pure Nash equilibrium (Shah-Mansouri and Wong, 2018),

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Bayesian Nash equilibrium (He et al., 2024), fuzzy strong Nash equilibrium (Huang and Liu, 2022), and stationary Nash equilibrium (Jaskiewicz and Nowak, 2023). Therefore, it is reasonable and practical to explore the equilibrium of MOGs. Because there is more than one objective for every player in an MOG, an accepted equilibrium is the Pareto equilibrium (Qu et al., 2015). Pareto equilibrium is a strategy profile in which no player can be better without making the others worse (Smith et al., 2014; Yan and Hayakawa, 2024). Based on methods such as Ky Fan's minimax inequality and fixed point theorem (Wang, 1993; Lu, 2008), the stochastic maximum principle with Markov jumps and Poisson jumps and the Lagrangian multiplier technique (Zhang et al., 2018; Hu et al., 2024), and the theorem of quasi-equilibrium existence (Ding, 2000a,b), the existence of Pareto equilibrium has been investigated. To date, there has been no reference to finite-step reachability or finite-step controllability of Pareto equilibria in MOGs.

Specifically, for the Pareto equilibrium problem in MOGs, it is necessary to discuss the individual expectations of players, because cooperation and competition complement each other in the game process. In particular, for MOGs, if we only examine Pareto equilibrium that is cooperation-oriented, then the equilibrium points may be altruistic but not egoistic, which is not true in most real scenarios. Equally important, if we only examine Nash equilibrium, then such strictly competitive games do not reach an equilibrium point (Choobineh and Mohagheghi, 2019; Peng et al., 2022), because it is nearly impossible for all objectives to simultaneously achieve the best equilibrium. Consequently, setting the payoff vector and ultimate payoff expectations for every player in an MOG and then researching the Pareto equilibrium allow more flexible simulation of various problems, where the ultimate payoff can be expressed using the weighted sum, weighted average, polynomial function, and so on. Capătă (2021) provided sufficient conditions for the existence of Pareto equilibrium over set constraints for an MOG with two players. Additionally, several references have explained payoff vector constraints based on reality. For example, Fournier et al. (2021) took investment companies and scholarships of students as examples to explain the practical significance of payoff vector limitations. The payoff vector of an investment company may include the value of the portfolio, the Sharpe ratio, and the conditional value at risk. The company usually requires the value of its portfolio to be higher than the value of other investment companies, requires the Sharpe ratio to be above a specific level, and requires the risk conditionality to be lower than a specific level. Additionally, a student's grade is influenced by several factors, like study progress, extracurricular classes, and spending time with friends and family. To keep receiving the fellowship, the student is expected to meet specific standards to maintain their average grade above a certain level. MOGs have also been used in dynamic networks, where the supplier tried to meet a multidimensional demand (Bertsekas and Rhodes, 1971; Hart, 2005; Bauso et al., 2012). Accordingly, it is meaningful to research MOGs with restricted payoff vectors (Bauso et al., 2008; Fournier et al., 2021).

In recent years, as one of the important mathematical tools, the semi-tensor product (STP) of matrices has been widely used in Boolean control networks (Liu et al., 2023), k-valued logical control networks (Wang et al., 2023), game theoretical control (Kong et al., 2020; Li et al., 2023; Jia et al., 2024), and equilibrium analysis (Cheng and Ji, 2022; Wu et al., 2022). In particular, the STP is an effective tool for solving equilibrium problems with scalar payoffs (Guo and Wang, 2016; Li et al., 2017; Le et al., 2019). Wang et al. (2022) took a class of congestion games with player-specific costs and resource failures as the research target, and provided a necessary and sufficient condition to guarantee the global convergence of the target Nash equilibrium via STP. Wang and Cheng (2016) used STP to obtain a sufficient condition that can ensure the stability of a class of delayed evolutionary potential games at a pure Nash equilibrium. Yu et al. (2024) discussed the existence and convergence of weighted Nash equilibria in games with multiple payoffs via STP. To our knowledge, STP has not been used to study the Pareto equilibrium of MOGs and its existence, finite-step reachability, and finite-step controllability.

In this study, the existence, finite-step reachability, and finite-step controllability of the Pareto equilibrium of MOGs are considered using the STP technique. The main contributions of this paper are as follows: (i) The algebraic expressions of MOGs are obtained using multi-layer graphs and STP. (ii) Two necessary and sufficient conditions are proposed

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to verify whether all players can meet their expectations and verify the existence of Pareto equilibrium, respectively. (iii) A new strategy updating rule is designed and evolutionary MOGs are considered in algebraic form, based on which, some results concerning finite-step reachability and finite-step controllability are presented. (iv) On the basis of (iii), a backward search algorithm is presented to determine the shortest evolutionary process and control sequence.

The rest of this paper is organized as follows. Section 2 gives some preliminaries, including the definitions of STP, the concepts of Pareto-efficient strategy and Pareto equilibrium, and the MOG model description. Section 3 presents the main results and some illustrative examples. In detail, Subsection 3.1 proposes the criteria for the ultimate payoff expectation and Pareto equilibrium from the static perspective. Based on the results of this subsection, a strategy updating rule and a backward search algorithm are provided to guarantee the finite-step reachability and finite-step controllability from the initial strategy profile to the target equilibrium, respectively. Section 4 is the conclusion of this paper.

The important symbols used in this paper are as follows. Let $\mathbf{0}_k$ be $\underbrace{(0,0,\ldots,0)}_{k}^{\mathrm{T}}$, $\mathbf{1}_k$ be $\underbrace{(1,1,\ldots,1)}_{k}^{\mathrm{T}}$, $\mathcal{M}_{m\times n}$ be the set of $m\times n$ real matrices, $\mathcal{M}^B_{m\times n}$ be the set of $m\times n$ matrices, which consist of 0 and 1, $\mathbb{N}_+$ be the set of positive integers, $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^k$ be the set of k-dimension real vectors, and $\mathbb{R}^k_+$ be the set $\{(b_1,\ldots,b_k)\in\mathbb{R}^k | b_j\geq 0,\ j=1,2,\ldots,k\}$. δ^i_k denotes the *i-th* column of the identity matrix I_k, and the set Δ_k consists of δ^i_k, $i=1,2,\ldots,k$. $|B|$ represents the number of elements in the set B. For matrix $A\in\mathcal{M}_{m\times n}$, $Col_j(A)$ is the *j-th* column of A and $(A)_{i,j}$ is an element in row i and column j, where $i\in\{1,2,\ldots,m\}$ and $j\in\{1,2,\ldots,n\}$. $Col(A):=\{Col_j(A)|j=1,2,\ldots,n\}$. For vectors $\alpha=[\alpha_1,\ldots,\alpha_m]$, $\beta=[\beta_1,\ldots,\beta_m]\in\mathbb{R}^m$, α_i is defined as the *i-th* element of α where $i\in\{1,2,\ldots,m\}$, and $\alpha<_\exists\beta$ means that there exists at least one $j\in\{1,2,\ldots,m\}$ such that $\alpha_j<\beta_j$ holds. For m-dimension Boolean vectors $p=[p_1,p_2,\ldots,p_m]$ and $q=[q_1,q_2,\ldots,q_m]$, denote $p\vee q$ as $[p_1\vee q_1,\ldots,p_m\vee q_m]$. In detail, for every $j=1,2,\ldots,m$, $p_j\vee q_j=\begin{cases}0, & p_j=q_j=0;\\ 1, & \text{else.}\end{cases}$

2 Preliminary

2.1 STP

Definition 1 (Cheng et al., 2011) Given $M\in\mathcal{M}_{m\times n}$ and $N\in\mathcal{M}_{p\times q}$, $M\ltimes N$ is the STP of M and N, and can be expressed as

$$M\ltimes N:=(M\otimes I_{t/n})(N\otimes I_{t/p})\in\mathcal{M}_{mt/n\times qt/p},$$

where $t=\mathrm{lcm}\{n,p\}$ is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

For brevity, this paper omits the symbol "$\ltimes$" during the calculation.

Lemma 1 (Cheng, 2014) For any function $v:\Delta_{k^n}\to\mathbb{R}$, the row vector $V\in\mathbb{R}^{k^n}$, which satisfies the following equation

$$v(x_1,\ldots,x_n)=V\ltimes_{i=1}^n x_i,$$

is called the structure vector of v, where $\ltimes_{i=1}^n x_i=x_1\ltimes x_2\cdots\ltimes x_n$ and V is unique for v.

Lemma 2 (Cheng, 2014) Denote

$$D_f^{[p,q]}=\mathbf{1}_p^{\mathrm{T}}\otimes I_q\in\mathcal{M}^B_{q\times pq},$$
$$D_r^{[p,q]}=I_p\otimes\mathbf{1}_q^{\mathrm{T}}\in\mathcal{M}^B_{p\times pq}.$$

Then, for $X\in\Delta_p$ and $Y\in\Delta_q$, the following equations hold

$$D_f^{[p,q]}XY=Y,\quad D_r^{[p,q]}XY=X.$$

Lemma 3 (Cheng and Liu, 2017) For any two column vectors $X\in\mathbb{R}^m$ and $Y\in\mathbb{R}^n$, we have

$$W_{[m,n]}\ltimes X\ltimes Y=Y\ltimes X,$$

where $W_{[m,n]}=(\delta^1_n\ltimes\delta^1_m,\ldots,\delta^n_n\ltimes\delta^1_m,\ldots,\delta^1_n\ltimes\delta^m_m,\ldots,\delta^n_n\ltimes\delta^m_m)\in\mathcal{M}_{mn\times mn}$ is the swap matrix.

2.2 Game model

The MOG is defined in Wang (1993). On that basis, a set of weights is added in the following definition. Particularly, the weight of every objective should be equivalent if all players have no preference for any of the objectives.

Definition 2 An $MOG=(N,S,a,K,C)$ consists of the following factors:

- $N=\{1,2,\ldots,n\}$ is the set of players.
- $S=S_1\times S_2\times\ldots S_n$ is the set of strategy profiles where S_i is the strategy set of player i and "$\times$"

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is the Cartesian product. $s = (s_1, s_2, \ldots, s_n) \in S$ is a strategy profile where player i chooses strategy s_i, $i = 1, 2, \ldots, n$, $s_i \in S_i$.

- $a = (a_1, a_2, \ldots, a_m)$. a_j represents the weight of the j-th layer, $j = 1, 2, \ldots, m$. Different layers represent the different objectives, and $m \in \mathbb{N}_+$ is the number of objectives in this game.

- $K = (K_1, K_2, \ldots, K_n)$. $K_i = \{k_1^i, k_2^i, \ldots, k_{m_i}^i\}$ is the set of layers where player i resides, which means that K_i consists of the objectives with which player i is concerned, $m_i \in \mathbb{N}_+$, $m \geq m_i \geq 1$ and $i = 1, 2, \ldots, n$.

- $C(s) = (c_1(s), c_2(s), \ldots, c_n(s))$. $c_i(s)^{\mathrm{T}} = [c_i^{k_1^i}(s), c_i^{k_2^i}(s), \ldots, c_i^{k_{m_i}^i}(s)]$ is the payoff vector of player i under the strategy profile $s \in S$. Specifically, $c_i^j(s) = \sum\limits_{l \in U_j(i)} c_{i,l}^j(s_i, s_l)$ is the payoff of the j-th objective for player i and $c_{i,l}^j(x_i, x_l)$ is the payoff of player i in a game with player l in terms of the j-th objective, where $U_j(i)$ is the set of players who have games with player i at the j-th layer, $i \in N$, $j \in K_i$.

Remark 1 It is worth noting that the dimensions of vectors $c_i(s)$ and $c_j(s)$ may be different because the size of K_i may be unequal to the size of K_j, $i, j \in N$, $i \neq j$ and $s \in S$.

To describe the game model more intuitively, the following example is provided.

Example 1 Consider the scholarship selection policy for students. In terms of bonus points for scientific research results, the value is usually adjusted according to the level of the published article. There are five students and they are numbered 1 through 5. Students 2 and 4 have published first-level articles, students 1, 2, 3, and 4 have published second-level articles, and students 3, 4, and 5 have published third-level articles.

This situation can be modeled as in Fig. 1: MOG= (N, S, a, K, C). Here, $N = \{1, 2, \ldots, 5\}$, $K = (K_1, \ldots, K_5)$, $K_1 = (2)$, $K_2 = (1, 2)$, $K_3 = (2, 3)$, $K_4 = (1, 2, 3)$, $K_5 = (3)$, $C(s) = (c_1(s), \ldots, c_5(s))$, $c_1(s) = c_1^2(s)$, $c_2(s) = (c_2^1(s), c_2^2(s))^{\mathrm{T}}$, $c_3(s) = (c_3^2(s), c_3^3(s))^{\mathrm{T}}$, $c_4(s) = (c_4^1(s), c_4^2(s), c_4^3(s))^{\mathrm{T}}$, and $c_5(s) = c_5^3(s)$. $a = (a_1, a_2, a_3)$ consists of the weights of articles in different levels. In this example, the following represents the ultimate bonus points for student i: $\sum\limits_{j \in K_i} a_j c_i^j(s)$, $s \in S$.

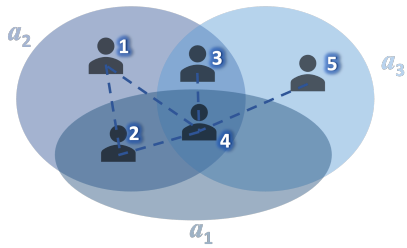

Fig. 1 Three-layer graph representation of the MOG in Example 1.

Definition 3 (Wang, 1993) A strategy $\bar{s}_i \in S_i$ of player i is called a Pareto-efficient strategy against $\bar{s}$ if there is no strategy $s_i \in S_i$ that satisfies

$$c_i(s_i, \bar{s}_{-i}) - c_i(\bar{s}_i, \bar{s}_{-i}) \in \mathbb{R}_+^{k_i} \backslash \{\mathbf{0}_{k_i}\}.$$

Definition 4 (Wang, 1993) A strategy profile $\bar{s} = (\bar{s}_i, \bar{s}_{-i}) \in S$ is called a Pareto equilibrium if, for each player i, $\bar{s}_i \in S_i$ is a Pareto-efficient strategy against $\bar{s}$.

3 Main results

This section describes of the static and dynamic analysis. For convenience of representation, the MOG mentioned in this section satisfies $N = \{1, 2, \ldots, n\}$, $S_i = S_0 = \{1, 2, \ldots, k\}$ and $a = (a_1, a_2, \ldots, a_m)$. Meanwhile, $\delta_k^j \in \Delta_k$ stands for the algebraic form of the strategy $j \in S_0$, and the algebraic form of S_0 is Δ_k. In this way, we can express the strategy profile $s = (s_1, s_2, \ldots, s_n)$ as $x = (x_1, x_2, \ldots, x_n)$, where x_i is the algebraic form of s_i, $s_i \in S_0$, $x_i \in \Delta_k$.

Remark 2 This paper also applies to situations where the number of strategies is different among players because these strategy sets in algebraic form can be unified by Lemma 2.

3.1 Static analysis

In an MOG, the payoff vector of player i is related to the objectives with which player i is concerned and the players who have games with player i. Therefore, assume that $\mathbb{C}_i(s)$ is the ultimate payoff for player i under the strategy profile $s \in S$, and

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can be expressed as

$$\mathbb{C}_i(s) = \sum_{j\in K_i} a_j c_i^j(s), \quad i = 1, 2, \ldots, n.$$

Assume that player i has $d_{i,j}$ neighbors in the j-th layer. Then, it is noticed that

$$\begin{aligned} c_i^j(s) &= \sum_{l\in U_j(i)} c_{i,l}^j(s_i, s_l) \\ &= V_{c_i^1}^j x_i x_{i_1} + V_{c_i^2}^j x_i x_{i_2} + \cdots + V_{c_i^{d_{i,j}}}^j x_i x_{i_{d_{i,j}}}, \end{aligned} \tag{1}$$

where $V_{c_i^l}^j$ is the structure vector of player i when he plays with the l-th neighbor in the j-th layer, $i \in N,\ j \in K_i,\ l \in \{1, 2, \ldots, d_{i,j}\}$.

Next, we analyze the payoff of player i in the j-th layer, $j \in K_i$. Define

$$F_{i,j} = \begin{cases} \mathbf{1}_{k^{i-1}}^{\mathrm{T}} \otimes I_k \otimes \mathbf{1}_{k^{j-i-1}}^{\mathrm{T}} \otimes I_k \otimes \mathbf{1}_{k^{n-j}}^{\mathrm{T}}, & i < j; \\ 0, & i = j; \\ \mathbf{1}_{k^{j-1}}^{\mathrm{T}} \otimes I_k \otimes \mathbf{1}_{k^{i-j-1}}^{\mathrm{T}} \otimes I_k \otimes \mathbf{1}_{k^{n-i}}^{\mathrm{T}}, & i > j. \end{cases}$$

Subsequently, according to Lemma 2, Eq. (1) can be transformed into

$$c_i^j(s) = \mathcal{F}_i^j \ltimes_{p=1}^n x_p,$$

where $\mathcal{F}_i^j = \sum_{l\in U_j(i)} V_{c_i^l}^j F_{i,l}$.

Thus, the ultimate payoff of player i in the MOG can be rewritten as

$$\mathbb{C}_i(s) = \mathcal{G}_i \ltimes_{p=1}^n x_p, \tag{2}$$

where

$$\mathcal{G}_i = \sum_{j\in K_i} a_j \mathcal{F}_i^j. \tag{3}$$

Without loss of generality, we assume that the expectation of player i is r_i, where r_i is a known constant. If the ultimate payoff of player i is in line with r_i, then the following algebraic equation holds

$$\mathbb{C}_i(s) \geq r_i, \quad i \in N.$$

Therefore, combining with Eq. (2) we get

$$\begin{bmatrix} \mathcal{G}_1 \ltimes_{p=1}^n x_p \\ \mathcal{G}_2 \ltimes_{p=1}^n x_p \\ \vdots \\ \mathcal{G}_n \ltimes_{p=1}^n x_p \end{bmatrix} \geq \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix},$$

that is,

$$\mathcal{G} \ltimes_{p=1}^n x_p \geq \mathbf{0}_n,$$

where

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 - r_1 \mathbf{1}_{k^n}^{\mathrm{T}} \\ \mathcal{G}_2 - r_2 \mathbf{1}_{k^n}^{\mathrm{T}} \\ \vdots \\ \mathcal{G}_n - r_n \mathbf{1}_{k^n}^{\mathrm{T}} \end{bmatrix}. \tag{4}$$

In this way, we can obtain the following theorem.

Theorem 1 The individual expectations $(r_1, r_2, \ldots, r_n)$ can be achieved in the $MOG = (N, S, a, K, C)$ if and only if there exists a positive integer j that satisfies

$$Col_j(\mathcal{G}) \geq \mathbf{0}_n,$$

where $\mathcal{G}$ is shown in Eq. (4).

Remark 3 The ultimate payoffs in other forms can also be discussed in a similar way.

Example 2 Consider an $MOG = (N, S, a, K, C)$; the three-layer graph is shown in Fig. 2. Here, we set $N = \{1, 2, 3, 4\}$, $S_1 = S_2 = S_3 = S_4 = S_0 = \{1, 2\}$, $a = (3, 2, 2)$, $K = ((1), (1, 3), (1, 2, 3), (2))$. $C(s) = (c_1(s), \ldots, c_4(s))$ is shown in Table 1, and the payoff expectations of players 1–4 are $r_1 = 21$, $r_2 = 28$, $r_3 = 36$, and $r_4 = 12$, respectively. Then, from Table 1, we can observe

$$\begin{aligned} V_1^c &= [7\ \ 7\ \ 1\ \ 9\ \ 8\ \ 8\ \ 2\ \ 4\ \ 3\ \ 9\ \ 7\ \ 6\ \ 5\ \ 12\ \ 8\ \ 3], \\ V_2^c &= \begin{bmatrix} 5 & 3 & 7 & 10 & 3 & 6 & 9 & 9 & 5 & 6 & 3 & 5 & 1 & 8 & 5 & 3 \\ 4 & 2 & 4 & 4 & 9 & 1 & 2 & 6 & 3 & 2 & 5 & 2 & 3 & 4 & 7 & 6 \end{bmatrix}, \\ V_3^c &= \begin{bmatrix} 6 & 3 & 6 & 7 & 3 & 0 & 5 & 5 & 6 & 3 & 7 & 3 & 4 & 8 & 5 & 3 \\ 1 & 10 & 1 & 4 & 7 & 1 & 2 & 4 & 1 & 10 & 7 & 1 & 6 & 5 & 6 & 3 \\ 2 & 2 & 1 & 5 & 7 & 3 & 3 & 2 & 9 & 4 & 2 & 5 & 3 & 8 & 2 & 4 \end{bmatrix}, \\ V_4^c &= [9\ \ 2\ \ 5\ \ 8\ \ 10\ \ 7\ \ 5\ \ 5\ \ 2\ \ 3\ \ 7\ \ 3\ \ 7\ \ 12\ \ 6\ \ 4]. \end{aligned} \tag{5}$$

Combining with Eqs. (2), (3), and (5) we can obtain

$$\begin{aligned} \mathcal{G}_1 &= \begin{bmatrix} 21\ 21\ 3\ 27\ 24\ 24\ 6\ 12\ 9\ 27\ 21\ 18\ 15\ 36\ 24\ 9 \end{bmatrix}, \\ \mathcal{G}_2 &= \begin{bmatrix} 23\ 13\ 29\ 38\ 27\ 20\ 31\ 39\ 21\ 22\ 19\ 19\ 9\ 32\ 29\ 21 \end{bmatrix}, \\ \mathcal{G}_3 &= \begin{bmatrix} 24\ 33\ 22\ 39\ 37\ 8\ 25\ 27\ 38\ 37\ 39\ 21\ 30\ 50\ 31\ 23 \end{bmatrix}, \\ \mathcal{G}_4 &= \begin{bmatrix} 18\ 4\ 10\ 16\ 20\ 14\ 10\ 10\ 4\ 6\ 14\ 6\ 14\ 24\ 12\ 8 \end{bmatrix}. \end{aligned}$$

Furthermore, based on Eq. (4), $\mathcal{G}$ can be calculated as Eq. (6).Thus, it can be verified that $j = 4$

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Table 1 Payoff matrix of Example 2

c \ s	$s_1 = 1111$	$s_2 = 1112$	$s_3 = 1121$	$s_4 = 1122$	$s_5 = 1211$	$s_6 = 1212$	$s_7 = 1221$	$s_8 = 1222$
c_1	7	7	1	9	8	8	2	4
c_2	$[5,4]^{\mathrm{T}}$	$[3,2]^{\mathrm{T}}$	$[7,4]^{\mathrm{T}}$	$[10,4]^{\mathrm{T}}$	$[3,9]^{\mathrm{T}}$	$[6,1]^{\mathrm{T}}$	$[9,2]^{\mathrm{T}}$	$[9,6]^{\mathrm{T}}$
c_3	$[6,1,2]^{\mathrm{T}}$	$[3,10,2]^{\mathrm{T}}$	$[6,1,1]^{\mathrm{T}}$	$[7,4,5]^{\mathrm{T}}$	$[3,7,7]^{\mathrm{T}}$	$[0,1,3]^{\mathrm{T}}$	$[5,2,3]^{\mathrm{T}}$	$[5,4,2]^{\mathrm{T}}$
c_4	9	2	5	8	10	7	5	5
c \ s	$s_9 = 2111$	$s_{10} = 2112$	$s_{11} = 2121$	$s_{12} = 2122$	$s_{13} = 2211$	$s_{14} = 2212$	$s_{15} = 2221$	$s_{16} = 2222$
c_1	3	9	7	6	5	12	8	3
c_2	$[5,3]^{\mathrm{T}}$	$[6,2]^{\mathrm{T}}$	$[3,5]^{\mathrm{T}}$	$[5,2]^{\mathrm{T}}$	$[1,3]^{\mathrm{T}}$	$[8,4]^{\mathrm{T}}$	$[5,7]^{\mathrm{T}}$	$[3,6]^{\mathrm{T}}$
c_3	$[6,1,9]^{\mathrm{T}}$	$[3,10,4]^{\mathrm{T}}$	$[7,7,2]^{\mathrm{T}}$	$[3,1,5]^{\mathrm{T}}$	$[4,6,3]^{\mathrm{T}}$	$[8,5,8]^{\mathrm{T}}$	$[5,6,2]^{\mathrm{T}}$	$[3,3,4]^{\mathrm{T}}$
c_4	2	3	7	3	7	12	6	4

$$\mathcal{G} = \begin{bmatrix} 14 & 14 & -4 & 20 & 17 & 17 & -1 & 5 & 2 & 20 & 14 & 11 & 8 & 29 & 17 & 2 \\ -5 & -15 & 1 & 10 & -1 & -8 & 3 & 11 & -7 & -6 & -9 & -9 & -19 & 4 & 1 & -7 \\ -12 & -3 & -14 & 3 & 1 & -28 & -11 & -9 & 2 & 1 & 3 & -15 & -6 & 14 & -5 & -13 \\ 12 & -2 & 4 & 10 & 14 & 8 & 4 & 4 & -2 & 0 & 8 & 0 & 8 & 18 & 6 & 2 \end{bmatrix}. \tag{6}$$

and $j = 14$ satisfy Theorem 1, that is, payoff vectors of all players are in line with their expectations under the strategy profiles $\delta_{16}^{4} \sim (1,1,2,2)$ and $\delta_{16}^{14} \sim (2,2,1,2)$.

Next, we discuss the existence of Pareto equilibria in MOGs. Denote

$$V_i^c = \begin{bmatrix} \mathcal{F}_i^{k_1} \\ \mathcal{F}_i^{k_2} \\ \vdots \\ \mathcal{F}_i^{k_{m_i}} \end{bmatrix}.$$

Based on Lemma 3, it can be implied that

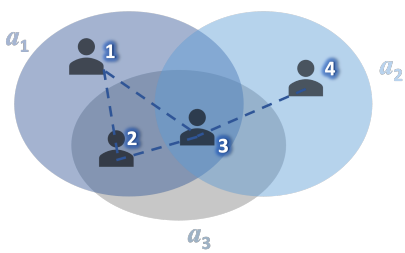

Fig. 2 Three-layer graph representation of the MOG in Example 2.

$$c_i(s) = L_i x_i \ltimes_{p \neq i}^{n} x_p, \tag{7}$$

where $L_i = V_i^c W_{[k,k^{i-1}]} \in \mathcal{M}_{m_i \times k^n}$. Then, divide L_i into k parts:

$$L_i = [L_i^1, L_i^2, \ldots, L_i^k], \tag{8}$$

where $L_i^r \in \mathcal{M}_{m_i \times k^{n-1}}$ and $r = 1, 2, \ldots, k$. Subsequently, arrange the k parts in columns to construct a new matrix, and let H_i^l be the *l-th* column of this matrix, that is,

$$H_i^l = \begin{bmatrix} H_i^{l,1} \\ H_i^{l,2} \\ \vdots \\ H_i^{l,k} \end{bmatrix} = Col_l \begin{bmatrix} L_i^1 \\ L_i^2 \\ \vdots \\ L_i^k \end{bmatrix} \in \mathbb{R}^{km_i}. \tag{9}$$

Based on the above, the following theorem holds.

Theorem 2 For an $MOG = (N, S, a, K, C)$, there exists a Pareto equilibrium $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ if and only if one of the following conclusions holds

a) for every $i \in N$, there exists a vector $\alpha \in \mathbb{R}^{m_i}$ such that $H_i^{-i} = \mathbf{1}_k \otimes \alpha$ holds,

b) for every $i \in N$, there exists a constant $r \in S_0 \backslash \{s_i^*\}$ such that $H_i^{-i,r} <_{\exists} H_i^{-i,s_i^*}$ holds,

where $-i$ satisfies $\ltimes_{p \neq i}^{n} x_p^* = \delta_{k^{n-1}}^{-i}$, $i = 1, 2, \ldots, n$.

Proof. As can be obtained from Eqs. (7) and (8), $Col_l(L_i^q)$ is the payoff vector of player i as i chooses strategy $q \in S_0$ and l satisfies $\ltimes_{p \neq i}^{n} x_p = \delta_{k^{n-1}}^{l}$. Hence, it can be derived that H_i^l as shown in Eq. (9) consists of all possible payoff vectors of player i, where $i \in N$ and $\ltimes_{p \neq i}^{n} x_p = \delta_{k^{n-1}}^{l}$.

(Necessity) According to Definitions 3 and 4, it can be deduced that the payoff vector of player i

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$c_i(s_i, s^*_{-i})$ satisfies

$$\text{either} \quad c_i(s_i, s^*_{-i}) = c_i(s^*_i, s^*_{-i}) \tag{10}$$
$$\text{or} \quad c_i(s_i, s^*_{-i}) <_\exists c_i(s^*_i, s^*_{-i}), \tag{11}$$

where (s^*_i, s^*_{-i}) is the Pareto equilibrium, $s_i \in S_0$. Setting $\ltimes^n_{p\neq i} x^*_p = \delta^{-i}_{k^{n-1}}$ and combining with Eq. (9) we have

$$c_i(s_i, s^*_{-i}) = L_i x_i \ltimes^n_{p\neq i} x^*_p = H_i^{-i,s_i},$$
$$c_i(s^*_i, s^*_{-i}) = L_i x^*_i \ltimes^n_{p\neq i} x^*_p = H_i^{-i,s^*_i}.$$

Furthermore, based on the arbitrariness of s_i, Eq. (10) is equivalent to

$$H_i^{-i,1} = H_i^{-i,2} = \cdots = H_i^{-i,k} \in \mathbb{R}^{m_i}.$$

Therefore, there exists a vector α such that $H_i^{-i} = \mathbf{1}_k \otimes \alpha$ holds.

Similarly, Eq. (11) is equivalent to

$$H_i^{-i,s_i} <_\exists H_i^{-i,s^*_i}.$$

(Sufficiency) From Eq. (9) it can be inferred that $H_i^{-i} = \mathbf{1}_k \otimes \alpha$ means

$$c_i(s_i{=}1, s^*_{-i}) = c_i(s_i{=}2, s^*_{-i}) = \ldots = c_i(s_i{=}k, s^*_{-i}),$$
$$\Longleftrightarrow \quad c_i(s_i, s^*_{-i}) = c_i(s^*_i, s^*_{-i}), \quad \forall s_i \in S_i. \tag{12}$$

Similarly, $H_i^{-i,s_i} <_\exists H_i^{-i,s^*_i}$ means

$$c_i(r, s^*_{-i}) <_\exists c_i(s^*_i, s^*_{-i}), \quad r \in S_0 \backslash \{s^*_i\}. \tag{13}$$

Hence, if Eq. (12) or Eq. (13) holds, when player i changes strategy from s^*_i to s_i and other players do not change strategies, at least one component of the payoff vector of player i decreases, that is,

$$c_i(s_i, s^*_{-i}) - c_i(s^*_i, s^*_{-i}) \notin \mathbb{R}^{k_i}_+ \backslash \{\mathbf{0}_{k_i}\}, \quad s_i \in S_0 \backslash \{s^*_i\}.$$

□

Example 3 Consider the didactic toy example in Ocean Shores city. As shown in Ismaili (2018), there are five players, each player has two strategies, and each player focuses on two objectives. Thus, this example can be described by $MOG = (N, S, a, K, C)$, and the two-layer graph is shown in Fig. 3. Here, $N = \{1, 2, \ldots, 5\}$, $s \in S(= S_1 \times \cdots \times S_5)$, $S_i = S_0 = \{1, 2\}$, $a = \{1, 1\}$, $K = \{K_1, \ldots, K_5\}$ and $K_i = K_0 = \{1, 2\}$. The payoff vector of every player is shown in Fig. 1 in Ismaili (2018). Then, V^c_i can be deduced as Eq. (14), $i = 1, 2, \ldots, 5$. For $j = 1, 2, 3, 4$,

$$W_{[2,2^j]} = \delta_{2^{j+1}}[1, 3, \ldots, 2^{j+1} - 1, 2, 4, \ldots, 2^{j+1}].$$

Then, it can be calculated that

$$c_i(s_i, s_{-i}) = L_i x_i \ltimes^5_{p\neq i} x_p, \quad i = 1, 2, \ldots, 5,$$

where L_i are shown in Eq. (15).

For instance, it can be verified that when $s = (2, 1, 1, 1, 1)$, $\ltimes^5_{j\neq 1} x_j = \delta^1_{16}$ and $\ltimes^5_{j\neq i} x_j = \delta^9_{16}$ hold, and $i = 2, 3, 4, 5$. Furthermore, according to Eq. (9) and Theorem 2, it can be checked that

$$H_1^{1,2} <_\exists H_1^{1,1}, \quad H_2^{9,1} <_\exists H_2^{9,2},$$
$$H_3^{9,1} <_\exists H_3^{9,2}, \quad H_4^{9,1} <_\exists H_4^{9,2},$$
$$H_5^{9,1} <_\exists H_5^{9,2},$$

which reveals that strategy profile (2,1,1,1,1) is a Pareto equilibrium. In a similar way, it can be validated that strategy profiles (2,1,1,1,2), (2,1,1,2,1), (2,1,1,2,2), (2,1,2,1,1), (2,1,2,1,2), (2,2,1,1,1), (2,2,1,1,2), (2,2,1,2,1), (2,2,1,2,2), (2,2,2,2,1), and (2,2,2,2,2) are also Pareto equilibria, which is consistent with the result of Fig. 1 in Ismaili (2018).

Example 4 Recall Example 2. Next, we explore the existence of its Pareto equilibrium. Combining Lemma 3, with Eqs. (5) and (7), we can obtain

$$c_i(s_i, s_{-i}) = L_i x_i \ltimes^4_{p\neq i} x_p, \quad i = 1, 2, 3, 4,$$

where

$$L_1 = \left[\begin{array}{cccccccc|cccccccc} 7 & 7 & 1 & 9 & 8 & 8 & 2 & 4 & 3 & 9 & 7 & 6 & 5 & 12 & 8 & 3 \end{array}\right],$$

$$L_2 = \left[\begin{array}{cccccccc|cccccccc} 5 & 3 & 7 & 10 & 5 & 6 & 3 & 5 & 3 & 6 & 9 & 9 & 1 & 8 & 5 & 3 \\ 4 & 2 & 4 & 4 & 3 & 2 & 5 & 2 & 9 & 1 & 2 & 6 & 3 & 4 & 7 & 6 \end{array}\right],$$

$$L_3 = \left[\begin{array}{cccccccc|cccccccc} 6 & 3 & 3 & 0 & 6 & 3 & 4 & 8 & 6 & 7 & 5 & 5 & 7 & 3 & 5 & 3 \\ 1 & 10 & 7 & 1 & 1 & 10 & 6 & 5 & 1 & 4 & 2 & 4 & 7 & 1 & 6 & 3 \\ 2 & 2 & 7 & 3 & 9 & 4 & 3 & 8 & 1 & 5 & 3 & 2 & 2 & 5 & 2 & 4 \end{array}\right],$$

$$L_4 = \left[\begin{array}{cccccccc|cccccccc} 9 & 5 & 10 & 5 & 2 & 7 & 7 & 6 & 2 & 8 & 7 & 5 & 3 & 3 & 12 & 4 \end{array}\right].$$

Moreover, based on Eq. (9), we have the following results, where $l \in \{1, 2, \ldots, 8\}$.

$$H_1^l = Col_l \begin{bmatrix} 7 & 7 & 1 & 9 & 8 & 8 & 2 & 4 \\ 3 & 9 & 7 & 6 & 5 & 12 & 8 & 3 \end{bmatrix},$$

$$H_2^l = Col_l \begin{bmatrix} 5 & 3 & 7 & 10 & 5 & 6 & 3 & 5 \\ 4 & 2 & 4 & 4 & 3 & 2 & 5 & 2 \\ 3 & 6 & 9 & 9 & 1 & 8 & 5 & 3 \\ 9 & 1 & 2 & 6 & 3 & 4 & 7 & 6 \end{bmatrix},$$

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$$V_1^c = \begin{bmatrix} 14 & 14 & 3 & 3 & 4 & 4 & 10 & 10 & 5 & 5 & 1 & 1 & 11 & 11 & 12 & 12 & 7 & 7 & 10 & 10 & 6 & 6 & 1 & 1 & 6 & 6 & 3 & 3 & 1 & 1 & 15 & 15 \\ 4 & 4 & 15 & 15 & 10 & 10 & 12 & 12 & 10 & 10 & 3 & 3 & 13 & 13 & 15 & 15 & 14 & 14 & 6 & 6 & 13 & 13 & 8 & 8 & 15 & 15 & 11 & 11 & 5 & 5 & 14 & 14 \end{bmatrix},$$

$$V_2^c = \begin{bmatrix} 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 \end{bmatrix},$$

$$V_3^c = \begin{bmatrix} 14 & 14 & 14 & 14 & 1 & 1 & 1 & 1 & 14 & 14 & 14 & 14 & 1 & 1 & 1 & 1 & 5 & 5 & 5 & 5 & 7 & 7 & 7 & 7 & 5 & 5 & 5 & 5 & 7 & 7 & 7 & 7 \\ 18 & 18 & 18 & 18 & 8 & 8 & 8 & 8 & 18 & 18 & 18 & 18 & 8 & 8 & 8 & 8 & 12 & 12 & 12 & 12 & 11 & 11 & 11 & 11 & 12 & 12 & 12 & 12 & 11 & 11 & 11 & 11 \end{bmatrix},$$

$$V_4^c = \begin{bmatrix} 9 & 16 & 12 & 6 & 9 & 16 & 12 & 6 & 9 & 16 & 12 & 6 & 9 & 16 & 12 & 6 & 15 & 11 & 13 & 12 & 15 & 11 & 13 & 12 & 15 & 11 & 13 & 12 & 15 & 11 & 13 & 12 \\ 16 & 14 & 16 & 1 & 16 & 14 & 16 & 1 & 16 & 14 & 16 & 1 & 16 & 14 & 16 & 1 & 8 & 15 & 10 & 5 & 8 & 15 & 10 & 5 & 8 & 15 & 10 & 5 & 8 & 15 & 10 & 5 \end{bmatrix},$$

$$V_5^c = \begin{bmatrix} 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 \\ 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 \end{bmatrix}. \tag{14}$$

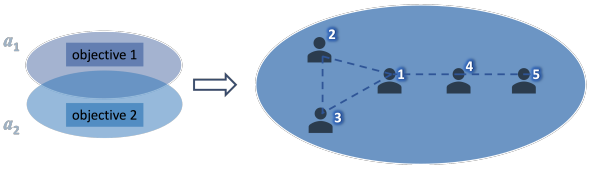

Fig. 3 Two-layer graph representation of the MOG in Example 3.

$$H_3^l = Col_l \begin{bmatrix} 6 & 3 & 3 & 0 & 6 & 3 & 4 & 8 \\ 1 & 10 & 7 & 1 & 1 & 10 & 6 & 5 \\ 2 & 2 & 7 & 3 & 9 & 4 & 3 & 8 \\ 6 & 7 & 5 & 5 & 7 & 3 & 5 & 3 \\ 1 & 4 & 2 & 4 & 7 & 1 & 6 & 3 \\ 1 & 5 & 3 & 2 & 2 & 5 & 2 & 4 \end{bmatrix},$$

$$H_4^l = Col_l \begin{bmatrix} 9 & 5 & 10 & 5 & 2 & 7 & 7 & 6 \\ 2 & 8 & 7 & 5 & 3 & 3 & 12 & 4 \end{bmatrix},$$

Therefore, it can be checked that Theorem 2 holds for every $i \in N$ as $l = 1, 4, 5, 8, 14,$ *and* 15, which means that δ_{16}^1, δ_{16}^4, δ_{16}^5, δ_{16}^8, δ_{16}^{14}, and δ_{16}^{15} are Pareto equilibria. Thus, combining with the result in Example 2, we find that strategy profiles $\delta_{16}^4 \sim (1,1,2,2)$ and $\delta_{16}^{14} \sim (2,2,1,2)$ are Pareto equilibria, and all players can meet their expectations under these strategy profiles.

3.2 Dynamic evolutionary analysis

In this section, a strategy updating rule is proposed and then the finite-step reachability and finite-step controllability of Pareto equilibrium for evolutionary MOGs (EMOGs) are investigated.

Denote by $(s_1(t), s_2(t), \ldots, s_n(t))$ the strategy profile at time t; its algebraic form is described as $(x_1(t), x_2(t), \ldots, x_n(t))$. $x_{-i}(t)$ is $(x_1(t), \ldots, x_{i-1}(t), x_{i+1}(t), \ldots, x_n(t))$. Given $x_i \in \Delta_k \backslash \{x_i(t)\}$ and $X_{-i}(t) = \ltimes_{j \neq i}^n x_j(t) \in \Delta_{k^{n-1}}$, set $f_i(t, x_i) = c_i(x_i, X_{-i}(t)) - c_i(x_i(t), X_{-i}(t))$, $P_i(t) = \{x_i | f_i(t, x_i) \in \mathbb{R}_+^{k_i} \backslash \{\mathbf{0}_{k_i}\}\}$ and $Q_i(t) = \{x_i | -f_i(t, x_i) \in \mathbb{R}_+^{k_i} \backslash \{\mathbf{0}_{k_i}\}\}$. Thus, the strategy updating rule is designed as

$$O_i(t) = \begin{cases} x_i(t), & |Q_i(t)| = k-1; \\ P_i(t), & |Q_i(t)| + |P_i(t)| = k-1 \\ & \text{and } |Q_i(t)| < k-1; \\ \Delta_k \backslash Q_i(t), & \text{else.} \end{cases} \tag{16}$$

Here, $O_i(t)$ is the set of strategies that can be chosen at time $(t+1)$ for player i. Particularly, $O_i(t)$ with $X_{-i}(t) = \delta_{k^{n-1}}^l$ is designated as O_i^l, and it is assumed that player i chooses the strategy in O_i^l with equal probability, $i = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, k^{n-1}$.

Remark 4 The determination of strategy updating rules is an important prerequisite for dynamic analysis of evolutionary games. Several common strategy updating rules, such as myopic best response adjustment, unconditional imitation, and simplified Fermi rule, are all dependent on the player's own payoff at the present moment as comparison, whereas these rules assume that the strategies chosen by other players remain unchanged at the next moment.

Let $\bar{x}_i(t+1)$ represent the feasible strategy of player i at time $(t+1)$, and based on Eq. (16) it can be expressed as

$$\bar{x}_i(t+1) = \vee_{x_i \in O_i(t)} x_i.$$

Using O_i^l, define $R_i = [\vee_{x_i \in O_i^1} x_i, \ldots, \vee_{x_i \in O_i^{k^{n-1}}} x_i]$. Thereby,

$$\bar{x}_i(t+1) = R_i X_{-i}(t), \quad X_{-i}(t) \in \Delta_{k^{n-1}}. \tag{17}$$

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$$
\begin{aligned}
L_1 &= \left[\begin{array}{cccccccccccccccc|cccccccccccccc}
14 & 14 & 3 & 3 & 4 & 4 & 10 & 10 & 5 & 5 & 1 & 1 & 11 & 11 & 12 & 12 & 7 & 7 & 10 & 10 & 6 & 6 & 1 & 1 & 6 & 6 & 3 & 3 & 1 & 1 & 15 & 15 \\
4 & 4 & 15 & 15 & 10 & 10 & 12 & 12 & 10 & 10 & 3 & 3 & 13 & 13 & 15 & 15 & 14 & 14 & 6 & 6 & 13 & 13 & 8 & 8 & 15 & 15 & 11 & 11 & 5 & 5 & 14 & 14
\end{array}\right], \\
L_2 &= \left[\begin{array}{cccccccccccccccc|cccccccccccccc}
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15
\end{array}\right], \\
L_3 &= \left[\begin{array}{cccccccccccccccc|cccccccccccccc}
14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
18 & 18 & 18 & 18 & 18 & 18 & 18 & 18 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11
\end{array}\right], \\
L_4 &= \left[\begin{array}{cccccccccccccccc|cccccccccccccc}
9 & 16 & 9 & 16 & 9 & 16 & 9 & 16 & 15 & 11 & 15 & 11 & 15 & 11 & 15 & 11 & 12 & 6 & 12 & 6 & 12 & 6 & 12 & 6 & 13 & 12 & 13 & 12 & 13 & 12 & 13 & 12 \\
16 & 14 & 16 & 14 & 16 & 14 & 16 & 14 & 8 & 15 & 8 & 15 & 8 & 15 & 8 & 15 & 16 & 1 & 16 & 1 & 16 & 1 & 16 & 1 & 10 & 5 & 10 & 5 & 10 & 5 & 10 & 5
\end{array}\right], \\
L_5 &= \left[\begin{array}{cccccccccccccccc|cccccccccccccc}
10 & 12 & 10 & 12 & 10 & 12 & 10 & 12 & 10 & 12 & 10 & 12 & 10 & 12 & 10 & 12 & 16 & 2 & 16 & 2 & 16 & 2 & 16 & 2 & 16 & 2 & 16 & 2 & 16 & 2 & 16 & 2 \\
8 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 4 & 10 & 4 & 10 & 4 & 10 & 4 & 10 & 4 & 10 & 4 & 10 & 4 & 10 & 4 & 10
\end{array}\right].
\end{aligned}
\tag{15}
$$

Combining with Lemma 2, Eq. (17) is equivalent to

$$\bar{x}_i(t+1) = J_i X(t), \tag{18}$$

where $J_i = R_i\ (I_{k^{i-1}} \otimes \mathbf{1}_k \otimes I_{k^{n-i}}),\ X(t) \in \Delta_{k^n}$.

Based on the above, the one-step evolutionary equation can be expressed as $\bar{X}(t+1)$, and it satisfies

$$\bar{X}(t+1) = JX(t), \tag{19}$$

where $Col_j(J) = Col_j(J_1) \ltimes Col_j(J_2) \ltimes \cdots \ltimes Col_j(J_n)$, $X(t) \in \Delta_{k^n}$. It is worth noting that $J_{p,q} = 1$ and $Col_q(J) \neq \delta_{k^n}^p$ mean that the strategy profile $X(t) = \delta_{k^n}^q$ can evolve to the strategy profile $\delta_{k^n}^p$ with positive probability.

Theorem 3 Consider the EMOG with strategy updating rule Eq. (16) and all players simultaneously update their strategies. The number of Pareto equilibria n_p satisfies

$$n_p = \mathrm{trace}(J).$$

Proof. First, from Eq. (16) it can be verified that $O_i(t)$ contains all Pareto-efficient strategies for player i, $i \in N$. The details are as follows.

i) Case 1: $|Q_i(t)| = k-1$. The elements in $Q_i(t)$ satisfy Definition 3. Consequently, $x_i(t)$ is the Pareto-efficient strategy for player i.

ii) Case 2: $|Q_i(t)| + |P_i(t)| = k-1$, $Q_i(t) < k-1$. Here, $\Delta_k \backslash \{x_i(t)\} = Q_i(t) \cup P_i(t)$ and $P_i(t) \neq \emptyset$. From the definition of $P_i(t)$, it is clear that $P_i(t)$ contains all Pareto-efficient strategies against the strategy profile at time t.

iii) If the two cases above are not satisfied, then $\Delta_k \backslash \{x_i(t)\} = Q_i(t) \cup P_i(t) \cup P_i^{'}(t)$, where $P_i^{'}(t) = \{x_i | f_i(t, x_i) = \mathbf{0}_{k_i},\ or\ f_i(t, x_i)\ has\ both\ positive\ and\ negative\ elements\}$. According to Definition 3, it can be derived that the strategy $x_i \in P_i^{'}(t)$ is a Pareto-efficient strategy against $(x_i, x_{-i}(t))$, and the strategy $x_i(t)$ is also a Pareto-efficient strategy against $(x_i(t), x_{-i}(t))$. Thus, combining with Case 2, we find that the set $\Delta_k \backslash Q_i(t) = P_i(t) \cup P_i^{'}(t) \cup \{x_i(t)\}$ consists of all Pareto-efficient strategies in this case.

In addition, for any $r \in \{1, 2, \ldots, k^n\}$, $J_{r,r} = 1$ demonstrates that if $X(t)$ is $\delta_{k^n}^r$, then one of the feasible strategy profiles at time (t+1) is $\delta_{k^n}^r$ based on Eq. (19). Meanwhile, Eq. (19) is obtained by equivalent transformation of Eq. (17) based on Lemma 2, and Eq. (17) relies on strategy updating rule Eq. (16). Thus, combining Definitions 3 and 4, we find that the strategy profile $\delta_{k^n}^r$, where every player takes the Pareto-efficient strategy, is a Pareto equilibrium, which is equivalent to $J_{r,r} = 1$, $r \in \{1, 2, \ldots, k^n\}$. □

Theorem 4 Consider the EMOG with the one-step evolutionary Eq. (19). There are $(J^s)_{p_*,p_o}$ possible processes to reach the strategy profile $\delta_{k^n}^{p_*}$ from $\delta_{k^n}^{p_o}$ at Step s with positive probability, $s \in \mathbb{N}_+$.

Proof. According to Eq. (19) and $J \in \mathcal{M}_{k^n \times k^n}^B$ we have that $(J)_{p,q} = 1$ holds if and only if the strategy profile $\delta_{k^n}^q$ can evolve to the strategy profile $\delta_{k^n}^p$ at Step 1 with positive probability. Note that the k^n strategy profiles can be regarded as k^n nodes. In this way, the matrix J^{T} represents the adjacency matrix of the directed graph consisting of these k^n nodes. Next, mathematical induction is used to prove this theorem.

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i) Theorem 4 is obviously true for $s = 1$.

ii) $s = 2$. $(J)_{p_*,v}(J)_{v,p_o} = 1$ means that the two-step evolutionary process $\delta_{k^n}^{p_o} \to \delta_{k^n}^{v} \to \delta_{k^n}^{p_*}$ exists with positive probability because $(J)_{p_*,v}(J)_{v,p_o} = 1$ holds if and only if $J_{v,p_o} = J_{p_*,v} = 1$ holds, $v \in \{1, 2, \ldots, k^n\}$. As a result, $(J^2)_{p_*,p_o} = \sum_{v=1}^{k^n}(J)_{p_*,v}(J)_{v,p_o}$ is the number of the evolutionary processes from $\delta_{k^n}^{p_o}$ to $\delta_{k^n}^{p_*}$ at Step 2 with positive probability.

iii) Suppose that Theorem 4 is true for $s = t-1 > 2$. Moreover, when $s = t$, the following equation holds

$$(J^t)_{p_*,p_o} = \left(J^{t-1}(J)\right)_{p_*,p_o} = \sum_{v=1}^{k^n}(J^{t-1})_{p_*,v}(J)_{v,p_o},$$

where J_{v,p_o} represents the number of evolutionary processes from $\delta_{k^n}^{p_o}$ to $\delta_{k^n}^{v}$ at Step 1 with positive probability and $(J^{t-1})_{p_*,v}$ represents the number of evolutionary processes from $\delta_{k^n}^{v}$ to $\delta_{k^n}^{p_*}$ at Step $(t-1)$ with positive probability. Thus, $(J^{t-1})_{p_*,v}(J)_{v,p_o}$ is the number of the paths starting from $\delta_{k^n}^{p_o}$, passing through $\delta_{k^n}^{v}$, and finally reaching $\delta_{k^n}^{p_*}$, $v \in \{1, 2, \ldots, k^n\}$. Therefore, $(J^t)_{p_*,p_o}$ is the number of paths from $\delta_{k^n}^{p_o}$ to $\delta_{k^n}^{p_*}$ at Step t with positive probability.

□

Next, we discuss the fact that the EMOG with the one-step evolutionary Eq. (19) can not naturally evolve to the target Pareto equilibrium at Step s with positive probability, where $s \in \mathbb{N}_+$ is stated. In this case, motivated by control theory, we introduce m pseudo-players as external controls, who participate in the EMOG and update their strategies without being constrained by Eq. (16). Specifically, $N^u = \{1, 2, \ldots, m\}$ and $S_i^u = S_0 = \Delta_k$ represent the set of pseudo-players and the set of strategies for every $i \in N^u$, respectively. $u_j(t) \in \Delta_k$ is the strategy chosen by pseudo-player j at time t, $j \in N^u$. Furthermore, $u(t) = \ltimes_{j=1}^{m} u_j(t) \in \Delta_{k^m}$. Combining with Lemma 2 and Eqs. (16)–(19), the controlled one-step evolutionary equation can be written as

$$\bar{X}_u(t+1) = J_u u(t) X_u(t), \quad X_u(t) \in \Delta_{k^n}, \tag{20}$$

where $J_u \in \mathcal{M}_{k^n \times k^{m+n}}^B$ and $\bar{X}_u(t+1)$ is the feasible strategy profile at time $(t+1)$ after adding m pseudo-players.

For ease of representation, divide J_u into k^m parts:

$$J_u = [J_{u,1}, J_{u,2}, \ldots, J_{u,k^m}], \tag{21}$$

where $J_{u,r} \in \mathcal{M}_{k^n \times k^n}^B$, $r = 1, 2, \ldots, k^m$. Similar to Eq. (19), if there exists a $j \in N^u$ such that $(J_{u,j})_{p,q} = 1$ and $Col_q(J_{u,j}) \neq \delta_{k^n}^{p}$ hold, then strategy profile $\delta_{k^n}^{q}$ can be controlled by $\delta_{k^m}^{j}$ to $\delta_{k^n}^{p}$ at Step 1 with positive probability.

Furthermore, define

$$\bar{J}_u = [\bar{J}_{u,1}, \bar{J}_{u,2}, \ldots, \bar{J}_{u,k^n}] \in \mathcal{M}_{k^n \times k^n}^B, \tag{22}$$

where $\bar{J}_{u,l} = \vee_{r=1}^{k^m} Col_l(J_{u,r})$. Then, the following result holds.

Theorem 5 Consider the EMOG with the controlled one-step evolutionary equation Eq. (20). The initial strategy profile $X(0) = \delta_{k^n}^{p_o}$ can be controlled to the target strategy profile $X(s) = \delta_{k^n}^{p_*}$ at Step s with positive probability if and only if $(\bar{J}_u^s)_{p_*,p_o} > 0$.

Proof. In accordance with Eqs. (21) and (22), $(\bar{J}_u)_{p,q} = 1$ indicates that there is at least one control sequence such that the strategy profile can be controlled from $\delta_{k^n}^{q}$ to $\delta_{k^n}^{p}$ at Step 1 with positive probability. Hence, matrix $\bar{J}_u$ contains all possible evolutionary processes at Step 1. Combined with the proof of Theorem 4 and setting $J = \bar{J}_u$, Theorem 5 holds. □

However, although the initial strategy profile can be controlled at Step s with positive probability, it sometimes does not meet the requirements of the problem. To further demonstrate the impacts of pseudo-players, the shortest evolutionary process from the initial strategy profile to the target strategy profile is studied. Algorithm 1 was designed for this purpose and its effectiveness has been proven.

For ease of representation, Algorithm 1 can be summarized in the following four steps:

Step 1: Identify the initial strategy profile $\delta_{k^n}^{p_o}$ and the target strategy profile $\delta_{k^n}^{p_*}$, initialize the length of the shortest evolutionary process $t = 0$, and define $\Gamma(x) = \{r | x_r \neq 0, x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n, r \in \{1, 2, \ldots, n\}\}$.

Step 2: $t \leftarrow t + 1$. Find the strategy profiles that can be controlled by $\delta_{k^m}^{j}$ to $\delta_{k^n}^{p_*}$ at Step 1 with probability 1 and denote these strategy profiles by set $Z_j(t)$, that is, $Z_j(t) =$

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Algorithm 1 Backward search method to find the shortest evolutionary processes and control sequences

Input: $p_o,\ p_*,\ J_{u,i},\ i = 1, 2, \ldots, k^m$
Output: $t,\ P,\ Q$

1: Initialize $t = 0,\ q = \mathbf{0}^{\mathrm{T}}_{k^n}$;
2: **while** $(q)_{p_o} \neq 1$ **do**
3: $\quad t = t + 1$;
4: $\quad$ **for** $j = 1 : k^m$ **do**
5: $\quad\quad$ **for** $r = 1 : k^n$ **do**
6: $\quad\quad\quad \alpha = \mathbf{0}^{\mathrm{T}}_{\mathbf{k^n}}$;
7: $\quad\quad\quad$ **if** $t = 1$ and $(J_{u,j})_{p_*,r} = 1$ **then**
8: $\quad\quad\quad\quad (\alpha)_r = 1$;
9: $\quad\quad\quad$ **else if** $t \neq 1$ and $\{l|(J_{u,j})_{l,r} = 1\} \subset \{l|(Q)_{t-1,l} = 1\}$ **then**
10: $\quad\quad\quad\quad (\alpha)_r = 1$;
11: $\quad\quad\quad$ **end if**
12: $\quad\quad$ **end for**
13: $\quad\quad$ **if** $j = 1$ **then**
14: $\quad\quad\quad p = \alpha$;
15: $\quad\quad$ **else**
16: $\quad\quad\quad p = [p; \alpha]$;
17: $\quad\quad$ **end if**
18: $\quad$ **end for**
19: $\quad$ **for** $r = 1 : k^n$ **do**
20: $\quad\quad (q)_r = (p)_{1,r} \vee (p)_{2,r} \vee \ldots (p)_{k^m,r}$;
21: $\quad$ **end for**
22: $\quad$ **if** $t = 1$ **then**
23: $\quad\quad P = p$;
24: $\quad\quad Q = q$;
25: $\quad$ **else**
26: $\quad\quad P = [P\ ;\ p]$;
27: $\quad\quad Q = [Q\ ;\ q]$;
28: $\quad$ **end if**
29: **end while**

$\{\delta^r_{k^n}|(J_{u,j})_{p_*,r} = 1, r \in \{1, 2, \ldots, k^n\}\}$, $j = 1, 2, \ldots, k^m$.

Step 3: Determine whether $\delta^{p_o}_{k^n}$ can be controlled to $\delta^{p_*}_{k^n}$ at Step t with probability 1. If $\delta^{p_o}_{k^n} \in Z(t)(= \cup^{k^m}_{j=1} Z_j(t))$ holds, then output t. Otherwise, set $t \leftarrow t + 1$.

Step 4: Find the strategy profiles that can be controlled to the strategy profiles that are in $Z(t)$ at Step 1 with probability 1, and then use these strategy profiles to obtain $Z_j(t)$, that is, $Z_j(t) = \{\delta^r_{k^n}|\Gamma\big(Col_r(J_{u,j})\big) \subset Z(t-1)\}$. Return to Step 3.

Theorem 6 Consider the EMOG with the controlled one-step evolutionary Eq. (20). If the game can be controlled from $X(0) = \delta^{p_o}_{k^n}$ to the target strategy profile $\delta^{p_*}_{k^n}$ in finite steps with probability 1, then Algorithm 1 can be used to find the shortest evolutionary process and the control sequence.

Proof. Assume that $X(0) = \delta^{p_o}_{k^n}$ can be controlled to the strategy profile $\delta^{p_*}_{k^n}$ in finite steps with probability 1. t acts as the output of Algorithm 1, and assume that $(X(0), X(1), \ldots, X(t-1), X(t))$ is the shortest evolutionary process and $(u(0), u(1), \ldots, u(t-1))$ is the control sequence. It is clear that $X(t-1)$ must be able to be controlled to $X(t) = \delta^{p_*}_{k^n}$ at Step 1 with probability 1. Hence, as described in Step 2, we have

$$X(t-1) \in Z(1) \xrightarrow[u(t-1)\in\{\delta^j_{k^m}|Z_j(t)\neq\emptyset\}]{probability\ 1} X(t) = \delta^{p_*}_{k^n}.$$

Case 1: $\delta^{p_o}_{k^n} \in Z(1)$. It is known that $\delta^{p_o}_{k^n}$ can be controlled to $\delta^{p_*}_{k^n}$ at Step 1 with probability 1. In this case, $t = 1$ and $u(0) \in \{\delta^j_{k^m}|Col_{p_o}(J_{u,j}) = \delta^{p_*}_{k^n}\}$ hold.

Case 2: $\delta^{p_o}_{k^n} \notin Z(1)$. This case corresponds to Step 4. On the one hand, if there exists an $s \in \{2, 3, \ldots, t\}$ such that $z_j(s) \in Z_j(s) \subseteq Z(s)$ and $z_j(s) \notin \{\delta^r_{k^n}|\Gamma(Col_r(J_{u,j})) \subset Z(s-1)\}$ hold, the strategy profile $z_j(s)$ can not be controlled to the strategy profile $z_j(s-1) \in Z(s-1)$ at Step 1 with probability 1. Subsequently, $X(t-s) \in Z(s)$ can not be guaranteed to reach $X(t)$ at Step s with probability 1 because $Z(s-1)$ contains all strategy profiles that can be controlled to $X(t)$ at Step $(s-1)$ with probability 1. Therefore, no matter whether $X(t-s)$ is reachable from $X(0)$, it can not ensure that $X(t)$ is reachable from $X(0)$ with probability 1. On the other hand, if there exists an $s \in \{2, 3, \ldots, t\}$ such that $Z_j(s) \subset \{\delta^j_{k^m}|Col_{p_o}(J_{u,j}) = \delta^{p_*}_{k^n}\}$ holds, it can not be guaranteed that $Z(s)$ contains all strategy profiles that can be controlled to $X(t)$ at Step s with probability 1. As a result, the evolutionary process and control sequence are incomplete.

In addition, Algorithm 1 starts from the target strategy profile $\delta^{p_*}_{k^n}$ and relies on a backward search method to find the set $Z(t)$ that contains of all the strategy profiles that can be controlled to $\delta^{p_*}_{k^n}$ at Step t with probability 1. Hence, if the initial strategy profile satisfies $\delta^{p_o}_{k^n} \in Z(t)$ and $\delta^{p_o}_{k^n} \notin Z(s)$ for $s = 1, 2, \ldots, t-1$, then t is the length of the shortest evolutionary process. □

Remark 5 (Complexity analysis) According to Algorithm 1 it is obvious that the complexity mainly depends on Lines 2, 4, and 5. Now, the time and space complexity of Algorithm 1 are both $O(k^{2n+m})$.

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Example 5 Recall Example 2. We explore its finite-step reachability to the target Pareto equilibrium. According to Eqs. (16) and (17), it can be determined that

$$\bar{x}_i(t+1) = R_i X_{-i}(t), \quad i = 1,2,3,4,$$

where

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, R_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, R_4 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Furthermore, Eqs. (18) and (19) hold with

$$J_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

$$J_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$J_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

$$J_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix},$$

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Based on Theorem 3 we find that the number of Pareto equilibria is trace(J)=6, which is in agreement with Example 4. In this example, we set the length of the evolution process is 3 and $X(0) = \delta_{16}^2 \sim (1,1,1,2)$. By calculating we find that there are $(J^3)_{4,2} = 4$ and $(J^3)_{14,2} = 0$ paths to the strategy profiles $\delta_{16}^4 \sim (1,1,2,2)$ and $\delta_{16}^{14} \sim (2,2,1,2)$ with positive probability, respectively.

Example 6 Recall Example 2 and assume that player 3 is a pseudo-player. Hence, the set of players is $N = \{1,2,4\}$ and the payoff matrix $C(s) = (c_1(s), c_2(s), c_4(s))$ under the strategy profile $s \in (s_1, s_2, \ldots, s_8)$ is shown in Table 2. Based on Table 1 we can deduce the payoff matrices of players 1, 2, and 4 under control δ_2^1 and δ_2^2, which are shown in Tables 3 and 4, respectively. Combining with Table 2 and Theorem 2 it can be calculated that $\delta_8^2 \sim (1,1,2)$ and $\delta_8^8 \sim (2,2,2)$ are Pareto equilibria. In this example, the length of the evolution process is still 3, the target strategy profile is $\delta_8^2 \sim (1,1,2)$ and $X(0) = \delta_8^4 \sim (1,2,2)$. Based on Eq. (19), it can be computed that

$$\bar{X}(t+1) = JX(t),$$

where

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then, it can be calculated that $(J^3)_{2,4} = 0$, which means that the EMOG can not naturally evolve to $\delta_8^2 \sim (1,1,2)$ at Step 3. Thus, taking player 3 in Example 2 as a pseudo player, the controlled one-step evolutionary equation is

$$\bar{X}_u(t+1) = J_u u(t) X_u(t), \tag{23}$$

where

$$J_u = \left[\begin{array}{cccccccc|cccccccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Furthermore, from Eq. (22) we have

$$\bar{J}_u = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

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Table 2 Payoff matrix of players 1, 2, and 4

c \ s	$s_1 = 111$	$s_2 = 112$	$s_3 = 121$	$s_4 = 122$	$s_5 = 211$	$s_6 = 212$	$s_7 = 221$	$s_8 = 222$
c_1	5	8	2	3	1	6	4	5
c_2	$[4, 8]^{\mathrm{T}}$	$[6, 5]^{\mathrm{T}}$	$[2, 7]^{\mathrm{T}}$	$[4, 3]^{\mathrm{T}}$	$[2, 1]^{\mathrm{T}}$	$[5, 3]^{\mathrm{T}}$	$[5, 4]^{\mathrm{T}}$	$[6, 6]^{\mathrm{T}}$
c_4	3	6	7	7	10	4	8	9

Table 3 Payoff matrix of players 1, 2, and 4 under control δ_2^1

c \ s	$s_1 = 111$	$s_2 = 112$	$s_3 = 121$	$s_4 = 122$	$s_5 = 211$	$s_6 = 212$	$s_7 = 221$	$s_8 = 222$
c_1	7	7	8	8	3	9	5	2
c_2	$[5, 4]^{\mathrm{T}}$	$[3, 2]^{\mathrm{T}}$	$[3, 9]^{\mathrm{T}}$	$[6, 1]^{\mathrm{T}}$	$[5, 3]^{\mathrm{T}}$	$[6, 2]^{\mathrm{T}}$	$[1, 3]^{\mathrm{T}}$	$[8, 4]^{\mathrm{T}}$
c_4	9	2	10	7	2	3	7	12

Table 4 Payoff matrix of players 1, 2, and 4 under control δ_2^2

c \ s	$s_1 = 111$	$s_2 = 112$	$s_3 = 121$	$s_4 = 122$	$s_5 = 211$	$s_6 = 212$	$s_7 = 221$	$s_8 = 222$
c_1	1	9	2	4	7	6	8	3
c_2	$[7, 4]^{\mathrm{T}}$	$[10, 4]^{\mathrm{T}}$	$[9, 2]^{\mathrm{T}}$	$[9, 6]^{\mathrm{T}}$	$[3, 5]^{\mathrm{T}}$	$[5, 2]^{\mathrm{T}}$	$[5, 7]^{\mathrm{T}}$	$[3, 6]^{\mathrm{T}}$
c_4	5	8	5	5	7	3	6	4

Based on Theorem 5 it can be verified that $(\bar{J}_u^3)_{2,4} = 4$, which means that there are 4 evolutionary processes that can be controlled from δ_8^4 to δ_8^2 at Step 3 with positive probability.

Next we find the shortest evolutionary process and control sequence from δ_8^4 to δ_8^2 with probability 1. According to Algorithm 1, the backward search process is shown in Fig. 4. Therefore, the evolutionary processes and control sequences are $\delta_8^4 \xrightarrow{u(0)=\delta_2^1} \delta_8^5 \xrightarrow{u(1)=\delta_2^1} \delta_8^2$ and $\delta_8^4 \xrightarrow{u(0)=\delta_2^1} \delta_8^7 \xrightarrow{u(1)=\delta_2^1} \delta_8^2$.

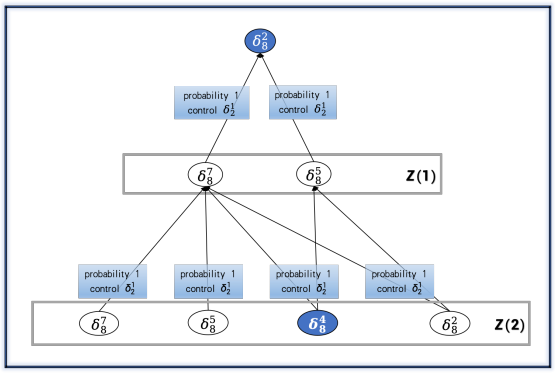

Fig. 4 The backward search process of Example 6.

4 Conclusion

This paper has investigated the existence, finite-step reachability, and finite-step controllability of Pareto equilibria for MOGs. STP was used to convert the payoff function into its algebraic form, which is more concise and facilitates computations. The main results have been presented through static and dynamic analyses. In the static analysis, it is assumed that the expectations of all players in an MOG are satisfied for ultimate payoffs; then, a necessary and sufficient condition is given to determine whether a strategy profile is a Pareto equilibrium. In the dynamic analysis, a strategy updating rule was introduced, and based on that rule, the one-step evolutionary equation was obtained, which as used to find the next feasible strategy profiles. Meanwhile, two necessary and sufficient conditions are proposed to determine the number of Pareto equilibria and guarantee the finite-step reachability, respectively. Particularly, motivated by control theory, finite pseudo-players were used as external controls, and the finite-step controllability of the EMOG was researched. To find the shortest evolutionary process and control sequence, a backward search algorithm was designed and its effectiveness was proved.

MOGs with network topology structure and unknown disturbances can describe more complex situations in reality. Thus, further work on the equilibrium analysis for this type of game is promising.

Contributors

Fanyueyang ZHANG and Jun'e FENG designed the research. Fanyueyang ZHANG processed the data and drafted the paper. Jun'e FENG helped organize the paper. Fanyueyang ZHANG and Jun-e FENG revised and finalized the

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paper.

Conflict of interest

Fanyueyang ZHANG and Jun'e FENG declare that they have no conflict of interest.

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