Frontiers of Information Technology & Electronic Engineering
www.zju.edu.cn/jzus; engineering.cae.cn; www.springerlink.com
ISSN 2095-9184 (print); ISSN 2095-9230 (online)
E-mail: jzus@zju.edu.cn

Boundedness of Marcinkiewicz integral with rough kernel on Triebel-Lizorkin spaces*

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Received Mar. 18, 2015; Revision accepted June 1, 2015; Crosschecked July 8, 2015

Abstract: This paper is a continuation of our previous work (Zhang and Chen, 2010b). Following the same general steps of the proof there, we make essential improvement on our previous theorem by recalculating a key inequality. Our result shows that the Marcinkiewicz integral, with a bounded radial function in its kernel, is still bounded on the Triebel-Lizorkin space.

Key words: Marcinkiewicz integral, Triebel-Lizorkin spaces
doi:10.1631/FITEE.1500082 **Document code:** A **CLC number:** O174.5

1 Introduction

Let $\psi \in C_c^\infty(\mathbb{R}^n)$, with $\operatorname{supp}(\psi) \subset \{\xi : 1/2 < |\xi| < 2\}$ and $\psi(\xi) > 1$ if $3/5 < |\xi| < 5/3$. We call $f \in F_p^{\alpha,q}$ for $\alpha > 0$ and $1 < p, q < +\infty$ if

$$\|f\|_{F_p^{\alpha,q}} = \|f\|_{L^p} + \left\| \left(\sum_k |2^{-k\alpha} \varPsi_k * f|^q \right)^{1/q} \right\|_{L^p} < +\infty.$$

Here $\varPsi_k(x)$ is defined by $\hat{\varPsi}_k(\xi) = \psi(2^k \xi)$. Since $F_p^{\alpha,q}$ is a generalization of the classical Lebesgue space L^p, the boundedness of singular integrals with rough kernels in the Triebel-Lizorkin space has been well studied in recent years. Readers are referred to Chen *et al.* (2002; 2003), Chen and Zhang (2004), Chen *et al.* (2005), Chen and Ding (2008), Chen and Zhang (2008), Zhang and Chen (2010a), Al-Qassem *et al.* (2012), and the references therein for the history of this topic. See also Chen *et al.* (2005), Zhang and Zhang (2013), and Chen and Zhu (2014) for the boundedness of oscillatory singular integrals and their commutators on Triebel-Lizorkin spaces. The Triebel-Lizorkin boundedness for some nonlinear operators, e.g., the g-function, the maximal singular integral, and the Marcinkiewicz integral, is a little bit tricky. To the best of our knowledge, all existing results (Zhang and Chen, 2009; 2010a; 2010b) were proved by reducing these nonlinear operators to some vector-valued singular integrals.

In this paper, we continue the study of Marcinkiewicz integral with rough kernel

$$\mu_{\Omega,b}(f)(x) = \left(\int_{-\infty}^{+\infty} |\sigma_{2^t} * f(x)|^2 \, \mathrm{d}t \right)^{1/2},$$

where

$$\sigma(x) = \Omega(x)|x|^{1-n} b(|x|) \chi_{|x|<1}(x)$$

and

$$\sigma_{2^t}(x) = 2^{-tn} \sigma\left(\frac{x}{2^t}\right)$$

for some radial function b, Ω being homogeneous of degree 0 and satisfying

$$\int_{\mathbb{S}^{n-1}} \Omega(x') \mathrm{d}\sigma(x') = 0. \tag{1}$$

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Denote $\mu_{\Omega,b} = \mu_\Omega$ when $b \equiv 1$. In our previous work (Zhang and Chen, 2009), we proved the following theorem:

Theorem 1 Let $\Omega \in H^1(\mathbb{S}^{n-1})$ be homogeneous of degree 0 and satisfy Eq. (1). Then μ_Ω is bounded on $F_p^{\alpha,q}, 0 < \alpha < 1, 1 < p, q < \infty$.

However, it is difficult to obtain Triebel-Lizorkin boundedness once radial function b is introduced. We have only the following unsatisfactory result in Zhang and Chen (2010b):

Theorem 2 Let $b \in L^\infty(\mathbb{R}_+)$ and $\Omega \in L^r(\mathbb{S}^{n-1})$ with $r > 1$ satisfy Eq. (1). Then $\mu_{\Omega,b}$ is bounded on $F_p^{\alpha,q}$ for $0 < \alpha < 1$ and $1 + \dfrac{n+1}{n+2-1/r} < p, q < 2 + \dfrac{1-1/r}{n+1}$.

Note that there are strong restrictions on the ranges of p and q in Theorem 2. The aim of this paper is to remove this restriction for $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Below is the main theorem of this paper:

Theorem 3 Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy Eq. (1) and $b \in L^\infty$. Then $\mu_{\Omega,b}$ is bounded on $F_p^{\alpha,q}$ for $0 < \alpha < 1$ and $1 < p, q < +\infty$.

Here and below, notation $A \preceq B$ means that there is a constant $C > 0$ independent of all essential variables such that $A \leq CB$.

In addition, for general $\Omega \in L^r(\mathbb{S}^{n-1})$ with $r > 1$, we can enlarge the ranges of p and q compared with Theorem 2. The proof follows the same framework as in Zhang and Chen (2010b). The improvement comes from the better decaying rate of a key inequality (inequality (4) below) in our proof. So, this paper might be regarded as a continuation of Zhang and Chen (2010b).

Remark 1 We learned that YABUTA K. has recently submitted his work on the same Triebel-Lizorkin boundedness for a more general form of Marcinkiewicz integral using a different method.

2 Proof of Theorem 3

First we need the following characterization of $F_p^{\alpha,q}$. Denote B_n the unit ball in $\mathbb{R}^n$. For $r \geq 1$ and $f \in \mathcal{S}(\mathbb{R}^n)$, define

$$S_{q,r}(f)(x) = \left(\int_0^{+\infty} \left(\int_{B_n} \left|\frac{f(x+l\zeta)-f(x)}{l^\alpha}\right|^r \mathrm{d}\zeta\right)^{q/r} \frac{\mathrm{d}l}{l}\right)^{1/q}.$$

Then by Lemma 2.1 of Zhang and Chen (2010b), we have

$$\|f\|_{F_p^{\alpha,q}} \sim \|f\|_{L^p} + \|S_{q,r}(f)\|_{L^p},$$

whenever $0 < \alpha < 1$, $1 < p, q < \infty$, and $1 \leq r < \min\{q, p\}$. Since the L^p boundedness for $\mu_{\Omega,b}$ is well known, we only have to prove

$$\|S_{q,r}(\mu_{\Omega,b}f)\|_{L^p} \preceq \|S_{q,r}(f)\|_{L^p}. \tag{2}$$

Notation $L^{\boldsymbol{p}}(E)$ will also be used for $\boldsymbol{p} = (p_1, p_2, p_3)$, $1 \leq p_i < +\infty$ and $E = \mathbb{R}^n \times (0, +\infty) \times B_n$. Letting $H = L^2(\mathbb{R}, \mathrm{d}t)$, we call an H-valued function $F(x, l, \zeta) \in L^{\boldsymbol{p}}(E)$, if

$$\|F\|_{L^{\boldsymbol{p}}(E)} = \left(\int_{\mathbb{R}^n} \left(\int_0^{+\infty} \left(\int_{B_n} |F(x,l,\zeta)|_H^{p_1} \mathrm{d}\zeta\right)^{p_2/p_1} \cdot \frac{1}{l}\mathrm{d}l\right)^{p_3/p_2} \mathrm{d}x\right)^{1/p_3} < \infty.$$

To prove inequality (2), we first rewrite $\mu_{\Omega,b}$ into some appropriate form. Take a C^∞ radial function ϕ such that $\mathrm{supp}(\phi) \subset \{y : 1/2 < |y| < 2\}$ and $\int_0^\infty \phi(t)/t\mathrm{d}t = 1$. Let $\widehat{\Phi}(\xi) = \phi(\xi)$ and $\Phi_t(x) = t^{-n}\Phi(\frac{x}{t})$. Then by the Fourier transform, there is

$$f(x) = \int_0^{+\infty} \frac{\Phi_t * f(x)}{t}\mathrm{d}t = \int_{-\infty}^{+\infty} \Phi_{2^t} * f(x)\mathrm{d}t$$

for each $f \in \mathcal{S}(\mathbb{R}^n)$. Similar to calculations in Zhang and Chen (2010b), we have

$$\begin{aligned}
\mu_{\Omega,b}f(x) &= \left\|\sigma_{2^t} * \int_{-\infty}^{+\infty} \Phi_{2^{s+t}} * f(x)\mathrm{d}s\right\|_{L^2(\mathbb{R},\mathrm{d}t)} \\
&= \left\|\int_{-\infty}^{+\infty} \Phi_{2^{s+t}} * \sigma_{2^t} * f(x)\mathrm{d}s\right\|_{L^2(\mathbb{R},\mathrm{d}t)} \\
&\leq \int_{-\infty}^{+\infty} \|\Phi_{2^{s+t}} * \sigma_{2^t} * f(x)\|_{L^2(\mathbb{R},\mathrm{d}t)}\mathrm{d}s \\
&= \int_{-\infty}^{+\infty} \|\Phi_{2^t} * \sigma_{2^{t-s}} * f(x)\|_{L^2(\mathbb{R},\mathrm{d}t)}\mathrm{d}s \\
&:= \int_{-\infty}^{+\infty} H_s f(x)\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|S_{q,r}(\mu_{\Omega,b}f)(x)\|_{L^p} \\
&= \left\|\frac{\mu_{\Omega,b}f(x+l\zeta) - \mu_{\Omega,b}f(x)}{l^\alpha}\right\|_{L^{\boldsymbol{p}}(E)} \\
&\leq \left\|\mu_{\Omega,b}\left(\frac{f(x+l\zeta)-f(x)}{l^\alpha}\right)\right\|_{L^{\boldsymbol{p}}(E)} \\
&\leq \int_{-\infty}^{+\infty} \|H_s F(\cdot, l, \zeta)(x)\|_{L^{\boldsymbol{p}}(E)}\mathrm{d}s.
\end{aligned}$$

Here we have set $\boldsymbol{p} = (r, q, p)$ and

$$F(x, l, \zeta) = \frac{f(x + l\zeta) - f(x)}{l^\alpha}.$$

To prove inequality (2), we need to show that for some $\gamma > 0$, there is

$$\|H_s F(\cdot, l, \zeta)(x)\|_{L^{\boldsymbol{p}}(E)} \preceq 2^{-\gamma|s|}\|F(x, l, \zeta)\|_{L^{\boldsymbol{p}}(E)},$$

which is further obtained by interpolation between

$$\|H_s F(\cdot, l, \zeta)\|_{L^{\boldsymbol{2}}(E)} \preceq 2^{-\beta|s|}\|F(x, l, \zeta)\|_{L^{\boldsymbol{2}}(E)} \quad (3)$$

and

$$\|H_s F(\cdot, l, \zeta)\|_{L^{\boldsymbol{q}}(E)} \preceq 2^{\epsilon|s|}\|F(x, l, \zeta)\|_{L^{\boldsymbol{q}}(E)}. \quad (4)$$

Here $\boldsymbol{2} = (2, 2, 2)$, $\boldsymbol{q} = (q_1, q_2, q_3)$ for any $1 < q_i < +\infty$, $\beta > 0$ is some fixed real number, and $\epsilon > 0$ could be arbitrarily small. Inequality (3) was proved in Zhang and Chen (2010b) for some fixed $\beta > 0$. Inequality (4) is the key inequality we mentioned in the abstract. It represents better decay than inequality (3.3) in Zhang and Chen (2010b), which was applied to prove Theorem 2.

By applying Lemma 2.2 of Zhang and Chen (2010b), inequality (4) will hold if we can prove the following vector-valued Hörmander's condition:

$$\begin{aligned}\int_{|x|>2|y|} &|\varPhi_{2^t} * \sigma_{2^{t-s}}(x + y) \\ &- \varPhi_{2^t} * \sigma_{2^{t-s}}(x)|_{L^2(\mathrm{d}t)}\mathrm{d}x \\ &\preceq 2^{\epsilon|s|}, \quad \forall\, y \neq 0.\end{aligned}$$

To be clear we set $\rho_s(x) = \varPhi * \sigma_{2^{-s}}(x)$, noting that

$$\varPhi_{2^t} * \sigma_{2^{t-s}}(x) = 2^{-tn}(\varPhi * \sigma_{2^{-s}})\left(\frac{x}{2^t}\right).$$

Assume that we have already proved

$$|\rho_s(x)| \preceq \min\left\{1, \frac{2^{\epsilon|s|}}{|x|^{n+\epsilon}}\right\} \quad \forall\, \epsilon > 0, \quad (5)$$

and the same estimate for $|\nabla\rho_s(x)|$. Now check whether the kernel $2^{-tn}\rho_s(\frac{x}{2^t})$ satisfies a certain Lipshitz condition, from which Hörmander's condition follows. Since

$$\begin{aligned}&\int_{\mathbb{R}} \left|2^{-tn}\rho_s\left(\frac{x+y}{2^t}\right) - 2^{-tn}\rho_s\left(\frac{x}{2^t}\right)\right|^2 \mathrm{d}t \\ =&\left(\int_{-\infty}^{\log_2|x|} + \int_{\log_2|x|}^{+\infty}\right)|2^{-tn}\rho_s((x+y)/2^t) \\ &- 2^{-tn}\rho_s\left(x/2^t\right)|^2\mathrm{d}t \\ =&I_1 + I_2,\end{aligned}$$

we next estimate these two terms, respectively. By our assumption on $\nabla\rho_s(x)$ mentioned above, it is clear that for some $0 < \theta < 1$, there is

$$\begin{aligned}I_2^{1/2} &= \left(\int_{\log_2|x|}^{+\infty}\left|2^{-t(n+1)}\nabla\rho_s\left(\frac{x+\theta y}{2^t}\right)y\right|^2\mathrm{d}t\right)^{1/2} \\ &\preceq |y|\left(\int_{\log_2|x|}^{+\infty}2^{-2t(n+1)}\mathrm{d}t\right)^{1/2} \\ &\preceq \frac{|y|}{|x|^{n+1}}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\left|2^{-t(n+1)}\nabla\rho_s\left(\frac{x+\theta y}{2^t}\right)\cdot y\right| \\ \preceq&|y|\cdot\frac{2^{\epsilon|s|}2^{-t(n+1)}}{[(x+\theta y)/2^t]^{n+\epsilon}} \preceq 2^{\epsilon|s|}\frac{2^{-t(1-\epsilon)}|y|}{|x|^{n+\epsilon}},\end{aligned}$$

whenever $|x| > 2|y|$ and

$$\begin{aligned}&2^{-tn}\left|\rho_s\left(\frac{x+y}{2^t}\right) - \rho_s\left(\frac{x}{2^t}\right)\right| \\ \preceq&2^{-tn+\epsilon|s|}\left(\frac{2^{t(n+\epsilon)}}{|x+y|^{n+\epsilon}} + \frac{2^{t(n+\epsilon)}}{|x|^{n+\epsilon}}\right) \\ \preceq&2^{\epsilon|s|}\frac{2^{t\epsilon}}{|x|^{n+\epsilon}}.\end{aligned}$$

Combining the two we obtained, for any $0 < \theta < \epsilon$, there is

$$2^{-tn}\left(\rho_s\left(\frac{x+y}{2^t}\right) - \rho_s\left(\frac{x}{2^t}\right)\right) \preceq 2^{\epsilon|s|}\frac{|y|^\theta}{|x|^{n+\epsilon}}2^{t(\epsilon-\theta)}.$$

So, for I_1, we obtain

$$\begin{aligned}I_1^{1/2} &\preceq 2^{\epsilon|s|}\frac{|y|^\theta}{|x|^{n+\epsilon}}\left(\int_{-\infty}^{\log_2|x|}2^{2t(\epsilon-\theta)}\mathrm{d}t\right)^{1/2} \\ &= 2^{\epsilon|s|}\frac{|y|^\theta}{|x|^{n+\theta}}.\end{aligned}$$

Note that when $|x| > 2|y|$, we also have

$$I_2^{1/2} \preceq \frac{|y|}{|x|^{n+1}} \preceq 2^{\epsilon|s|}\frac{|y|^\theta}{|x|^{n+\theta}},$$

if ϵ is set to be less than 1. The Hörmander condition then follows by a standard argument using the estimates of I_1 and I_2.

Finally we turn to inequality (5). It is obvious that

$$|\rho_s(x)| = \left|2^s\int_{|z|<2^{-s}}\frac{\varOmega(z)b(|z|)}{|z|^{n-1}}\varPhi(x-z)\mathrm{d}z\right| \preceq \|\varOmega\|_{L^1},$$

since Φ is a Schwarz function. When $|x| > 2|z|$ or $|z| > 2|x|$, we always have

$$|\Phi(x-z)| \preceq \frac{1}{|x|^{n+1}}.$$

Consequently,

$$\left|2^s \int_{|z|<2^{-s},|x|\geq 2|z|} \frac{\Omega(z)b(|z|)}{|z|^{n-1}}\Phi(x-z)\mathrm{d}z\right| \preceq \frac{1}{|x|^{n+1}},$$

$$\left|2^s \int_{|z|<2^{-s},|x|\leq 2|z|} \frac{\Omega(z)b(|z|)}{|z|^{n-1}}\Phi(x-z)\mathrm{d}z\right| \preceq \frac{1}{|x|^{n+1}}.$$

When $|x|/2 < |z| < 2|x|$ and $|z| < 2^{-s}$, we have $|x|/2 < 2^{-s}$. Therefore,

$$\begin{aligned}&\left|\int_{|z|<2^{-s},\frac{|x|}{2}<|z|<2|x|} 2^s \frac{\Omega(z)b(|z|)}{|z|^{n-1}}\Phi(x-z)\mathrm{d}z\right|\\ \preceq&\frac{1}{|x|^{n-1}}\left|\int_{|z|<2^{-s},\frac{|x|}{2}<|z|<2|x|} 2^s\Phi(x-z)\mathrm{d}z\right|\\ \preceq&\frac{2^{-\epsilon s}}{|x|^{n+\epsilon}}\int_{\mathbb{R}^n}|\Phi(x-z)|\mathrm{d}z\\ \preceq&\frac{2^{\epsilon|s|}}{|x|^{n+\epsilon}}.\end{aligned}$$

The computation for $\nabla\rho_s(x) = (\nabla\Phi) * \sigma_{2^{-s}}(x)$ is exactly the same. Theorem 3 thus has been proved.

Remark 2 When $\Omega \in L^r(\mathbb{S}^{n-1}), r > 1$, we modify the last estimate as

$$\begin{aligned}&\left|\int_{|z|<2^{-s},\frac{|x|}{2}<|z|<2|x|} 2^s \frac{\Omega(z)b(|z|)}{|z|^{n-1}}\Phi(x-z)\mathrm{d}z\right|\\ \preceq&2^s\left(\int_{|z|<2^{-s},\frac{|x|}{2}<|z|<2|x|}\frac{|\Omega(z)|^r}{|z|^{(n-1)r}}\mathrm{d}z\right)^{1/r}\\ &\cdot\left(\int_{|z|<2^{-s},\frac{|x|}{2}<|z|<2|x|}|\Phi(x-z)|^{r'}\mathrm{d}z\right)^{1/r'}.\end{aligned}$$

The second term on the right side of '$\preceq$' is bounded, while the first term is less than

$$\begin{aligned}&\frac{2^{\epsilon|s|}}{|x|^{1+\epsilon}}\left(\int_{|x|/2}^{|x|}\frac{t^{n-1}}{t^{(n-1)r}}\int_{\mathbb{S}^{n-1}}|\Omega(z')|^r\mathrm{d}z'\mathrm{d}t\right)^{1/r}\\ \preceq&\frac{2^{(\epsilon+n/r)|s|}}{|x|^{n+\epsilon}}.\end{aligned}$$

Plugging this into the calculation of Hörmander's condition, we finally reach inequality (6) instead of inequality (4):

$$\|H_sF\|_{L^q(E)} \preceq 2^{(\epsilon+n/r)|s|}\|F\|_{L^q(E)}. \qquad (6)$$

By the remark in Zhang and Chen (2010b), we can take β in inequality (3) to be close enough to $1/2(1-1/r)$ whenever $\Omega \in L^r(\mathbb{S}^{n-1})$. Thus, by interpolation we obtain the Triebel-Lizorkin boundedness of $\mu_{\Omega,b}$ for

$$2-\frac{r-1}{n+r-1} < p,q < 2+\frac{r-1}{n},$$

which improves the ranges of p and q in Theorem 2.

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