Frontiers of Information Technology & Electronic Engineering
www.jzus.zju.edu.cn; engineering.cae.cn; www.springerlink.com
ISSN 2095-9184 (print); ISSN 2095-9230 (online)
E-mail: jzus@zju.edu.cn

Bifurcation-based fractional-order $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller design approach for nonlinear chaotic systems*

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Received Sept. 11, 2016; Revision accepted Jan. 4, 2017; Crosschecked Feb. 15, 2018

Abstract: We propose a novel approach called the robust fractional-order proportional-integral-derivative (FOPID) controller, to stabilize a perturbed nonlinear chaotic system on one of its unstable fixed points. The stability analysis of the nonlinear chaotic system is made based on the proportional-integral-derivative actions using the bifurcation diagram. We extract an initial set of controller parameters, which are subsequently optimized using a quadratic criterion. The integral and derivative fractional orders are also identified by this quadratic criterion. By applying numerical simulations on two nonlinear systems, namely the multi-scroll Chen system and the Genesio-Tesi system, we show that the fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller provides the best closed-loop system performance in stabilizing the unstable fixed points, even in the presence of random perturbation.

Key words: Fractional order system; Bifurcation diagram; Fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller; Multi-scroll Chen chaotic system; Genesio-Tesi chaotic system

https://doi.org/10.1631/FITEE.1601543 **CLC number:** TP311

1 Introduction

In recent years, the emergence of effective methods for solving differentiation and integration of non-integer-order equations has made fractional-order systems more attractive for the control system community (Machado, 1997; Machado and Galhano, 2009; Ladaci and Bensafia, 2016). To improve the performance of linear feedback systems, Podlubny (1999b) proposed a generalization of the classical proportional-integral-derivative (PID) controller to the $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ form called the 'fractional PID', which has recently become very popular due to its additional flexibility to meet design specifications. Since then, fractional order $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ (FOPID) controllers have found application in several power systems. For example, Pan and Das (2012) designed a fractional-order PID controller to take care of various contradictory objective functions for an automatic voltage regulator (AVR) system. Another application of the fractional controller was given by Bouafoura and Braiek (2010), dealing with the design of a fractional-order PID controller for integer and fractional plants.

Chen Z et al. (2014) designed an FOPID for a hydraulic turbine regulating system (HTRS) with the consideration of contradictory performance objectives. They showed the superiority of the fractional-order controllers over the integer controllers by means of a comparative study between the optimum of PID and FOPID controllers. Faieghi

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* Project supported by the Ministry of Higher Education and Scientific Research, Algeria (CNEPRU No. A10N01UN210120150002)

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et al. (2011) proposed a fractional-order PID design for ship roll motion control using an embedded chaos embedded particle swarm optimization (PSO) algorithm. Tang et al. (2012) studied an optimum design of a fractional-order PID controller for an AVR system using chaotic ant swarm (CAS). Based on the fractional high-gain adaptive control approach, Charef et al. (2013) introduced new tuning parameters for the performance improvement of the behavior of the controlled plant. Auto-tuning algorithms for fractional PID controllers have also been proposed in the literature (Ladaci and Charef, 2006a; Delavari et al., 2010; Neçaibia and Ladaci, 2014).

Chaotic behavior of dynamical systems can be used in many real-world applications such as circuits (Liu Y, 2012), mathematics (Liu and Yang, 2010), power systems (Harb and Abdel-Jabbar, 2003), medicine (Ditto, 1996), biology (Ma et al., 2009), and chemical reactors (Lamba and Hudson, 1987). The importance of these application fields for chaotic systems makes chaos and related topics very popular for researchers in various scientific domains (Chen et al., 2012).

Chaotic systems are also affected by the fractional order, in modeling or control, or in both. The work of Rabah et al. (2015a) is an example of stabilization of the fractional Chen chaotic system by linear feedback control, and the work of Rabah et al. (2015b) is an example of Genesio-Tesi chaos stabilization using a fractional-order $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller. Chen et al. (2013) proposed fractional-order chaotic synchronization and anti-synchronization with stochastic parameters using a controller composed of a compensation control action and a fuzzy controller. Tavazoei and Haeri (2008) performed stabilization of unstable fixed points of chaotic fractional-order systems by a fractional proportional integral (PI) controller. The stability analysis of such fractional-order nonlinear systems was thoroughly investigated in Chen D et al. (2014), where a mathematical description of a fractional-order Lyapunov stability theorem was presented.

In this work, we propose a design approach for a fractional-order $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller, based on a graphical bifurcation diagram coupled with an optimization algorithm for adjustment of the regulator parameters. We investigate the problem of chaos stabilization for two nonlinear systems: multi-scroll Chen system (Liu X et al., 2012) and Genesio-Tesi system (Genesio and Tesi, 1992). The stability analysis of the nonlinear chaotic system is made for the fractional-order integral and derivative actions using the bifurcation diagram. The fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller is implemented by means of the fractional Gründwald-Leitnikov numerical method. The results show the effectiveness of fractional-order integral action in achieving the desired stability even in the presence of perturbation.

2 Elements of fractional calculus theory

Fractional calculus is an old mathematical research topic, but it is currently enjoying increased popularity. The fractional-order derivative theory was developed mainly in the 19$^{\text{th}}$ century. Hentenryck et al. (1993) and Podlubny (1999a) provided a good source of references on fractional calculus. However, the application of fractional-order calculus to dynamic system control is a recent focus of interest, as presented by Oustaloup and Mathieu (1991), Podlubny (1999b), and Ladaci and Charef (2006b).

2.1 Basic definitions

There are many mathematical definitions for fractional integration and derivation. Here, we present two currently used definitions.

2.1.1 Riemann-Liouville (R-L) definition

The R-L definition is one of the commonly used definitions of fractional-order integrals and the derivative procedure.

The R-L fractional-order integral of order $\lambda > 0$ is defined by (Podlubny, 1999a)

$$\mathrm{I}_{\mathrm{RL}}^{\lambda} f(t) = \mathrm{D}^{-\lambda} f(t) = \frac{1}{\Gamma(\lambda)} \int_0^t (t-\tau)^{\lambda-1} f(\tau)\mathrm{d}\tau, \quad (1)$$

and the expression of the R-L fractional-order derivative of order μ is (Podlubny, 1999a)

$$\mathrm{D}_{\mathrm{RL}}^{\mu} f(t) = \frac{1}{\Gamma(n-\mu)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-\tau)^{n-\mu-1} f(\tau)\mathrm{d}\tau, \quad (2)$$

where $\Gamma(\cdot)$ is the Euler function and the integer n satisfies $n-1 < \mu < n$. This fractional-order derivative of Eq. (2) may also be defined from Eq. (1) as follows (Hentenryck et al., 1993):

$$\mathrm{D}_{\mathrm{RL}}^{\mu} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left\{ \mathrm{I}^{(n-\mu)} f(t) \right\}. \quad (3)$$

2.1.2 Grünwald-Leitnikov (G-L) definition

The G-L fractional-order integral of order $\lambda > 0$ is given by (Hentenryck et al., 1993)

$$\begin{aligned}\mathrm{I}_{\mathrm{GL}}^{\lambda} f(t) &= \mathrm{D}^{-\lambda} f(t)\\ &= \lim_{h\to 0} h^{\lambda}\sum_{j=0}^{k}(-1)^j\binom{-\lambda}{j}f(kh-jh),\end{aligned} \tag{4}$$

where h is the sampling period and $\omega_j^{(-\lambda)}$ are the coefficients of the following polynomial:

$$(1-z)^{-\lambda} = \sum_{j=0}^{\infty}(-1)^j\binom{-\lambda}{j}z^j = \sum_{j=0}^{\infty}\omega_j^{(-\lambda)}z^j, \tag{5}$$

with $\omega_0^{(-\lambda)} = \begin{pmatrix} -\lambda \\ 0 \end{pmatrix} = 1$.

The G-L fractional-order derivative of order $\mu > 0$ is also given by (Hentenryck et al., 1993)

$$\begin{aligned}\mathrm{D}_{\mathrm{GL}}^{\mu} f(t) &= \frac{\mathrm{d}^{\mu}}{\mathrm{d}t^{\mu}} f(t)\\ &= \lim_{h\to 0} h^{-\mu}\sum_{j=0}^{k}(-1)^j\binom{\mu}{j}f(kh-jh),\end{aligned} \tag{6}$$

where h is the sampling period and $\omega_j^{(\mu)}$ are the coefficients of the polynomial

$$(1-z)^{\mu} = \sum_{j=0}^{\infty}(-1)^j\binom{\mu}{j}z^j = \sum_{j=0}^{\infty}\omega_j^{(\mu)}z^j. \tag{7}$$

The coefficients can be expressed as

$$\omega_j^{(\mu)} = \binom{\mu}{j} = \frac{\Gamma(\mu+1)}{\Gamma(j+1)\Gamma(\mu-j+1)}, \tag{8}$$

with $\omega_0^{(\mu)} = \begin{pmatrix} \mu \\ 0 \end{pmatrix} = 1$.

2.2 Implementation of a fractional operator

Generally, in industrial control processes, the data is sampled, so a numerical approximation of the fractional operator is necessary. There exist several approximation approaches based on the temporal or frequency domain. In the literature, the currently used approaches in the frequency domain are those of Oustaloup and Mathieu (1991), Ladaci and Charef (2006b), and Ladaci and Bensafia (2016). In the temporal domain, there is a lot of work on the numerical solution of fractional differential equations. Diethlem (2003) proposed an efficient method based on the predictor-corrector Adams algorithm. The definitions cited above also have numerical approximations, as given below.

2.2.1 Riemann-Liouville (R-L) approximation

Numerical approximation of the fractional R-L integral is based on the rectangular method (Ladaci and Charef, 2006b). By putting

$$t = k\varDelta, \tag{9}$$

where t is the current time, k is an integer, and $\varDelta$ is the sampling period, we obtain

$$\begin{aligned}\mathrm{I}_{\mathrm{RL}}^{\lambda} f(k\varDelta) &= \frac{\varDelta}{\Gamma(\lambda)}\sum_{\tau=0}^{k-1}(k\varDelta-\tau\varDelta)^{\lambda-1}f(\tau\varDelta)\\ &= \frac{\varDelta^{\lambda}}{\Gamma(\lambda)}\sum_{\tau=0}^{k-1}(k-\tau)^{\lambda-1}f(\tau\varDelta).\end{aligned} \tag{10}$$

2.2.2 Grünwald-Leitnikov (G-L) approximation

For numerical calculus of fractional-order integrals and derivatives, we can use the G-L definitions and Eqs. (5) and (6), respectively.

Thus, for a causal function $f(t)$ and for $t = kh$, the fractional-order derivative is given as follows (Podlubny, 1999a):

$$\mathrm{D}^{\mu} f(kh) = \frac{\mathrm{d}^{\mu}}{\mathrm{d}t^{\mu}} f(t) \cong h^{-\mu}\sum_{j=0}^{k}\omega_j^{(\mu)}f(kh-jh), \tag{11}$$

where the coefficients $\omega_j^{(\mu)}$ are those of Eq. (7), which can be computed by using the following recursive formula:

$$\omega_j^{(\mu)} = \begin{cases} 1, & j=0,\\ \left(1-\frac{1+\mu}{j}\right)\omega_{j-1}^{(\mu)}, & j=1,2,\ldots,k.\end{cases} \tag{12}$$

3 Control strategy

To improve the performance of linear feedback systems, Podlubny (1999b) proposed a generalization of the classical PID controller to the $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ form, the so-called fractional-order PID, which has recently become very popular due to its additional flexibility to meet design specifications. Since that time, FOPID controllers have found application in several power systems.

3.1 Fractional-order proportional-integral-derivative (FOPID) controller

FOPID is a feedback mechanism from the control loop. It has been widely used in industrial control systems. The $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller calculates an error value and tries to minimize it by adjusting the process using a manipulated variable (Rabah et al., 2016).

We assume a system with the following classic dynamics:

$$\dot{\boldsymbol{X}} = f(\boldsymbol{X}) + \boldsymbol{u}(t), \tag{13}$$

where $\boldsymbol{X} \in \mathbb{R}^n$ and $\boldsymbol{u}(t)$ are the control signals.

The $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ control law is given by the following function:

$$\boldsymbol{u}(t) = k_{\mathrm{P}}\boldsymbol{X} + k_{\mathrm{I}}\mathrm{I}^{\lambda}\boldsymbol{X} + k_{\mathrm{D}}\mathrm{D}^{\mu}\boldsymbol{X}, \tag{14}$$

where λ and μ are the fractional integral and derivative order, respectively.

A schematic diagram of the fractional-order PID controller applied in the multi-scroll Chen system (Liu X et al., 2012) is shown in Fig. 1.

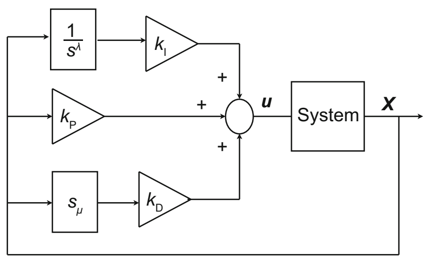

Fig. 1 Diagram of the fractional-order proportional-integral-derivative control system

3.2 Design construction

3.2.1 Bifurcation diagram

The bifurcation diagram is an effective tool to quickly evaluate all possible solutions of the system based on the variation of one of its parameters. It enables identification of the particular values that introduce the stability area (Chen et al., 2016), which is shown in Fig. 2. Consequently, bifurcation control has attracted much research effort in recent years (Berns et al., 1998; Colonius and Grne, 2002).

In this work, we will use this tool to identify k_{P}, k_{I}, and k_{D} gains.

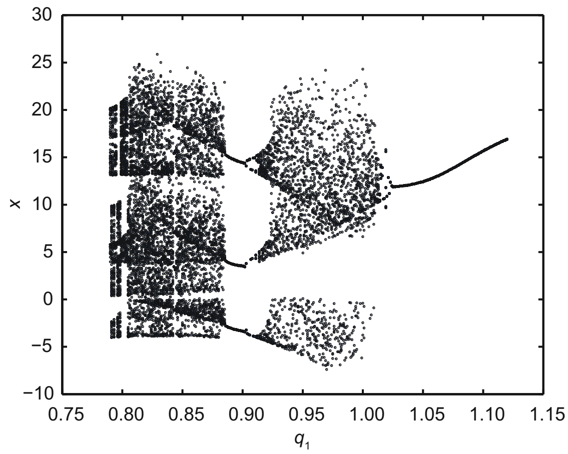

Fig. 2 Bifurcation diagram of a fractional Chen system $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{q}_1)$

3.2.2 Optimization

To choose the proper values of the fractional order and optimize the controller gains, we use a quadratic criterion formulated as

$$J = \sqrt{\sum (x - x_f)^2 + (y - y_f)^2 + (z - z_f)^2}, \tag{15}$$

where x, y, z are the state variables and $F = (x_f, y_f, z_f)$ is the desired fixed point.

The proposed design methodology can be represented by Algorithm 1.

Algorithm 1 Design methodology

Input: initialize parameters k_{P}, k_{I}, and k_{D} by k_1, k_2, and k_3, respectively; initialize k_1^0, k_2^0, and k_3^0.

Step 1: obtain classical PID controller parameters

for $i = 1$ to 3 **do**
- Trace the bifurcation diagram of state x vs. k_i
- Find interval I_i where the system stabilizes the desired fixed point for k_i
- Search for the optimal value $k_{i\text{-opt}}$ on interval I_i using the quadratic criterion J given by Eq. (15)
- Update the k_i value by $k_i = k_{i\text{-opt}}$

end for

Step 2: obtain fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller parameters

Fix the parameters $(k_{\mathrm{P}}, k_{\mathrm{I}}, k_{\mathrm{D}})$ to the optimal values obtained in **Step 1**

$\mu = 1$

Search for the optimal value λ_{opt} on interval $[0, 2]$ using the quadratic criterion J given by Eq. (15)

$\lambda = \lambda_{\mathrm{opt}}$

Search for the optimal value μ_{opt} on interval $[0, 2]$ using the quadratic criterion J given by Eq. (15)

Output: parameters k_{P}, k_{I}, k_{D}, λ, and μ.

4 Application examples

In this part, we apply the control strategy presented previously to control the multi-scroll Chen chaotic system as a first step, and then control the Genesio-Tesi chaotic system as a second step. The fractional Gründwald-Leitnikov method is used for numerical approximation of the control system using Matlab/Simulink.

4.1 Multi-scroll Chen chaotic system

The multi-scroll system was derived from the Chen system in Liu X et al. (2012). It is described by the following mathematical model:

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = (c-a-z+d\sin z)x + cy, \\ \dot{z} = xy - bz, \end{cases} \tag{16}$$

where x, y, z are state variables and a, b, c are the system parameters. As indicated in Hadef and Boukabou (2014), when $a=35$, $b=3$, $c=28$, and $d=8$, a six-scroll attractor is produced as shown in Fig. 3.

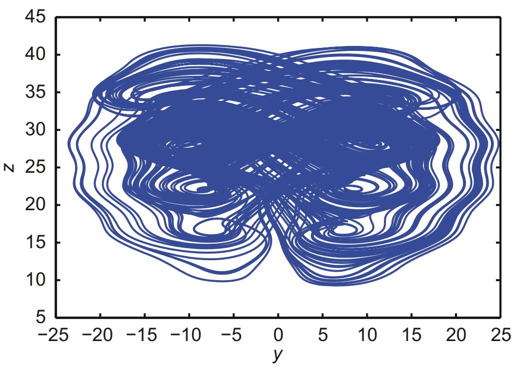

Fig. 3 Phase plane of the multi-scroll Chen system

Let us compute the fixed points of the multi-scroll chaotic Chen system. By solving the equation $(\dot{x},\dot{y},\dot{z})=(0,0,0)$, we have

$$\dot{x} = a(y-x) = 0, \tag{17}$$

$$\dot{y} = (c-a-z+d\sin z)x + cy = 0, \tag{18}$$

$$\dot{z} = xy - bz = 0. \tag{19}$$

From Eq. (17) we obtain

$$x = y. \tag{20}$$

By replacing Eq. (20) into Eq. (19) we have

$$z = x^2/b, \tag{21}$$

and from Eqs. (18), (20), and (21), we obtain the following equation:

$$(2c-a)x - x^3/b + dx\sin x^2/b = 0. \tag{22}$$

Eq. (22) can then be resolved by a graphical method as shown in Fig. 4.

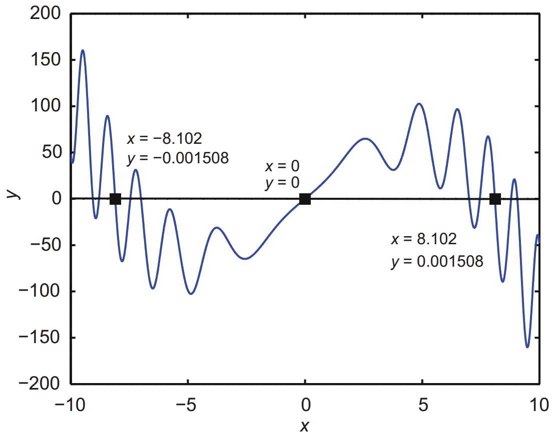

Fig. 4 Fixed points of the multi-scroll chaotic Chen system

We obtain the original fixed points $F_0=(0,0,0)$ and 10 other fixed points:

$$\begin{cases} F_{\pm1} = (\pm6.999,\ \pm6.999,\ 16.331), \\ F_{\pm2} = (\pm7.457,\ \pm7.457,\ 18.536), \\ F_{\pm3} = (\pm8.102,\ \pm8.102,\ 21.880), \\ F_{\pm4} = (\pm9.059,\ \pm9.059,\ 27.356), \\ F_{\pm5} = (\pm8.792,\ \pm8.792,\ 25.771). \end{cases} \tag{23}$$

To control this system on its fixed point $F_{+3}=(8.102, 8.102, 21.880)$, we simultaneously use the classical PID controller and the fractional $\mathrm{PI}^\lambda\mathrm{D}^\mu$ controller.

4.1.1 Using a classical PID controller

In this subsection, $(k_\mathrm{P}, k_\mathrm{I}, k_\mathrm{D})$ parameters are identified using bifurcation diagrams and a quadratic criterion J. First, we fix parameters $(k_\mathrm{I}, k_\mathrm{D})=(0.08, 0.02)$ and trace the bifurcation diagram, which presents the evolution of the x state with the k_P gain shown in Fig. 5. According to this diagram, we determine that the system stabilizes at the desired fixed point for $k_\mathrm{P}\in[0.670, 0.745]$.

So, the search for the optimal value $k_{\mathrm{P\text{-}opt}}$ using the quadratic criterion J is limited on this interval as presented in Fig. 6.

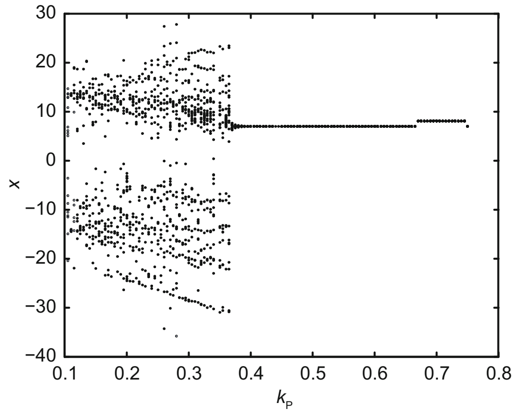

Fig. 5 Bifurcation diagram $x = f(k_P)$

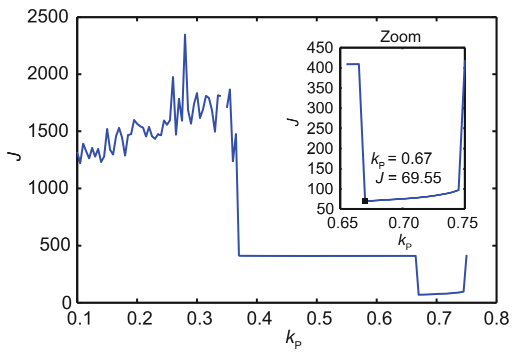

Fig. 6 Quadratic criterion $J = f(k_P)$

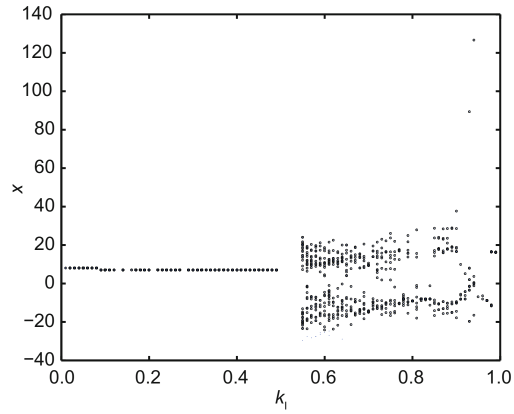

Fig. 7 Bifurcation diagram $x = f(k_I)$

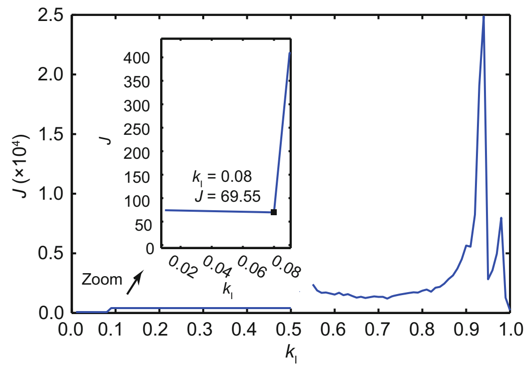

Fig. 8 Quadratic criterion $J = f(k_I)$

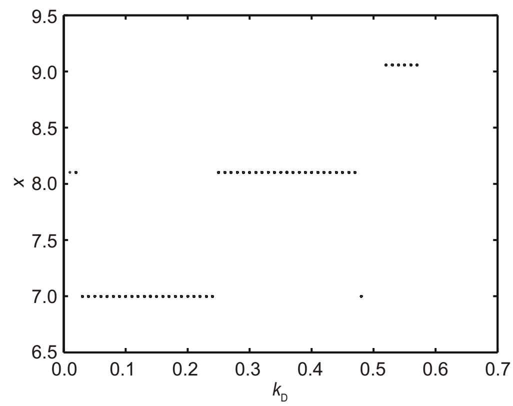

Fig. 9 Bifurcation diagram $x = f(k_D)$

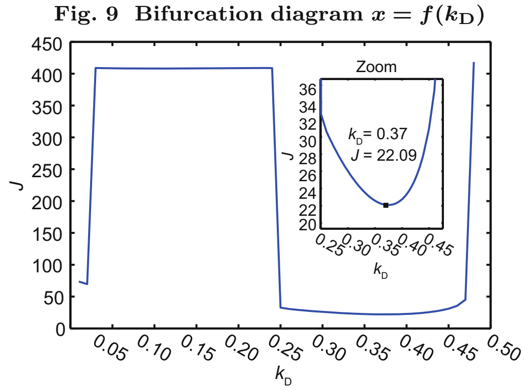

Fig. 10 Quadratic criterion $J = f(k_D)$

Once we have the optimal value $k_{\text{P-opt}} = 0.67$, we fix it by keeping $k_D = 0.02$, and then trace the evolution of x with k_I. This last configuration is presented in Fig. 7. In this case, the convergence is guaranteed for $k_I \in [0.01, 0.08]$, and the optimal value $k_{\text{I-opt}}$ is pulled from Fig. 8, which shows the evolution of the quadratic criterion J versus k_I.

Now we have two optimal values $(k_{\text{P-opt}}, k_{\text{I-opt}}) = (0.67, 0.08)$, and will look for the third parameter by the same procedure. In this case, the evolution of the x state and the quadratic criterion J versus k_D is presented in Figs. 9 and 10, respectively. The system reaches the desired fixed point for $k_D = 0.01$, $k_D = 0.02$, and also $k_D \in [0.25, 0.47]$.

The values of J corresponding to three parameters that ensure system stability on the desired fixed point $F_{+3} = (8.102, 8.102, 21.880)$ are presented in Tables 1–3. Therefore, we can easily determine $k_{\text{D-opt}}$.

At this stage, we have the three optimal control parameters $(k_{\text{P-opt}}, k_{\text{I-opt}}, k_{\text{D-opt}}) = (0.67, 0.08, 0.37)$, and can control the multi-scroll Chen system. The resulting system control responses

Table 1 Quadratic criterion $J = f(k_P)$

k_P	J	k_P	J
0.670	69.55	0.710	77.77
0.675	70.59	0.715	79.26
0.680	71.49	0.720	80.95
0.685	72.37	0.725	82.93
0.690	73.27	0.730	85.28
0.695	74.24	0.735	88.13
0.700	75.30	0.740	91.75
0.705	76.47		

Table 2 Quadratic criterion $J = f(k_I)$

k_I	J	k_I	J
0.01	74.17	0.05	71.58
0.02	73.49	0.06	70.95
0.03	72.83	0.07	70.27
0.04	72.20	0.08	69.55

Table 3 Quadratic criterion $J = f(k_D)$

k_D	J	k_D	J
0.01	73.69	0.35	22.58
0.02	69.55	0.37	22.09
0.25	32.62	0.39	22.32
0.27	29.22	0.41	23.50
0.29	26.97	0.43	26.04
0.31	25.12	0.45	30.96
0.33	23.63	0.47	45.15

are exhibited in Figs. 11 and 12. The control is triggered at $t = 5$ s.

Remark 1 For $k_D = 0.02$, we have $J = 69.65$ and the control signal $u = (0.17, 0.17, 0.45)$, but for $k_D = 0.37$, J is smaller ($J = 22.09$) with a more important control $u = (3.25, 3.25, 8.35)$.

4.1.2 Using a fractional PI$^\lambda$D$^\mu$ controller

In this step, we have three optimal parameters $(k_{\text{P-opt}}, k_{\text{I-opt}}, k_{\text{D-opt}}) = (0.67, 0.08, 0.37)$ and need to obtain the optimal values λ_{opt} and μ_{opt} for the system to reach the desired fixed point. For this case, we use the quadratic criterion defined previously.

Putting $\mu = 1$ and varying λ between 0.7 and 3, the J values obtained are gathered in Table 4 beyond 2 ($\lambda = 2$), and the values vary around 22, but the system stabilizes slightly away from the desired fixed point. From the results obtained (Fig. 13), J is optimal for $\lambda_{\text{opt}} = 1.05$.

We set $\lambda_{\text{opt}} = 1.05$ and seek the optimal value μ_{opt} by calculating the J criterion for different values of μ (Table 5 and Fig. 14).

We have finally defined five optimal values. To test the robustness of the fractional controller, we

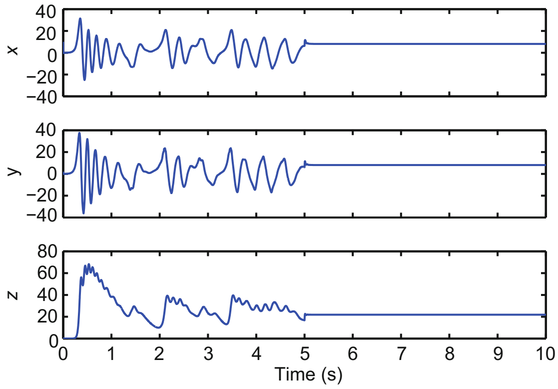

Fig. 11 State variables of the controlled multi-scroll Chen system

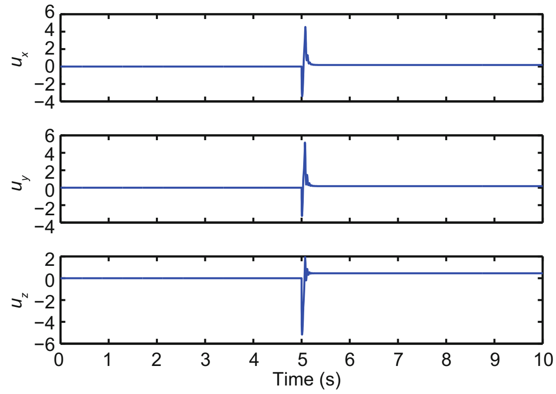

Fig. 12 Proportional-integral-derivative control signal

introduce a random perturbation into the input as follows:

$$u = u + A \cdot \text{rand}(). \tag{24}$$

In our case, $A = 0.02$. The controller results obtained are presented in Figs. 15 and 16.

One can remark that for the multi-scroll Chen system example, the fractional PI$^\lambda$D$^\mu$ controller provides better closed-loop system performance in stabilizing the fixed point when compared with the predictive control studied in Boukabou et al. (2007) and Hadef and Boukabou (2014).

4.2 Chaotic Genesio-Tesi system

Many researchers have been attracted by the problem of control and synchronization of the Genesio-Tesi nonlinear system (Hosseinia et al., 2010; Faieghi and Delavari, 2012). They have proposed different strategies for control and synchronization such as adaptive control (Park, 2007; Fayazi and Rafsanjani, 2011), linear matrix inequality (LMI) optimization approach (Park et al., 2008),

Table 4 Quadratic criterion $J = f(\lambda)$

λ	J	x	y	z
0.70	Nan	Nan	Nan	Nan
0.75	30.9131	8.101	8.101	21.881
0.80	24.6470	8.101	8.101	21.881
0.85	23.1903	8.102	8.101	21.881
0.90	22.5495	8.102	8.101	21.881
0.95	22.2310	8.102	8.102	21.881
0.97	22.1573	8.102	8.102	21.881
0.99	22.1060	8.102	8.102	21.881
1.00	22.0874	8.102	8.102	21.881
1.01	22.0731	8.102	8.102	21.881
1.03	22.0557	8.102	8.102	21.881
1.05	22.0511	8.102	8.102	21.881
1.07	22.0571	8.102	8.102	21.881
1.09	22.0720	8.102	8.103	21.881
1.10	22.0821	8.102	8.103	21.881
1.15	22.1536	8.103	8.103	21.881
1.20	22.2459	8.103	8.103	21.881
1.25	22.3446	8.103	8.104	21.881
1.30	22.4390	8.104	8.104	21.881
1.40	22.5889	8.104	8.106	21.881
1.50	22.6680	8.105	8.107	21.881
1.60	22.6774	8.106	8.109	21.881
1.70	22.6332	8.108	8.111	21.880
1.80	22.5565	8.109	8.114	21.880
1.90	22.4671	8.111	8.116	21.880
2.00	22.3805	8.113	8.120	21.880
2.10	22.3077	8.115	8.123	21.880
2.20	22.2553	8.117	8.127	21.980
2.50	22.2446	8.127	8.142	21.879
3.00	22.6732	8.149	8.178	21.878

Nan: not a number

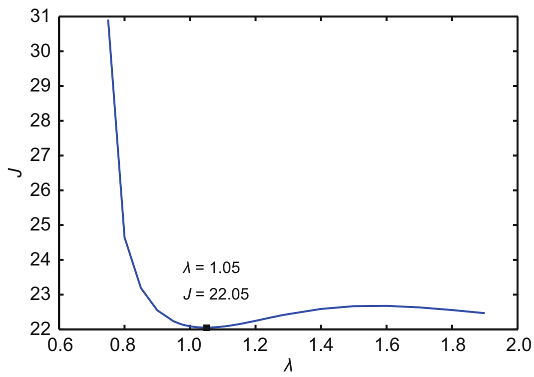

Fig. 13 Quadratic criterion $J = f(\lambda)$

sliding mode control (Dadras and Momeni, 2009; Ghamati and Balochian, 2015), single variable feedback control (Wang, 2010), and back stepping control (Gholipour et al., 2012).

The Genesio-Tesi system is defined by the following mathematical model:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -cx - by - az + x^2. \end{cases} \tag{25}$$

Table 5 Quadratic criterion $J = f(\mu)$

μ	J	x	y	z
0.1	22.0540	8.102	8.102	21.881
0.2	22.0540	8.102	8.102	21.881
0.3	22.0540	8.102	8.102	21.881
0.4	22.0540	8.102	8.102	21.881
0.5	22.0540	8.102	8.102	21.881
0.6	22.0540	8.102	8.102	21.881
0.7	22.0539	8.102	8.102	21.881
0.8	22.0535	8.102	8.102	21.881
0.9	22.0528	8.102	8.102	21.881
1.0	22.0511	8.102	8.102	21.881
1.1	22.0475	8.102	8.102	21.881
1.2	22.0403	8.102	8.102	21.881
1.3	22.0263	8.102	8.102	21.881
1.4	22.0017	8.102	8.102	21.881
1.5	21.9723	8.102	8.102	21.881
1.6	149.783	Variable around fixed point		
1.7	Nan	Divergence		

Nan: not a number

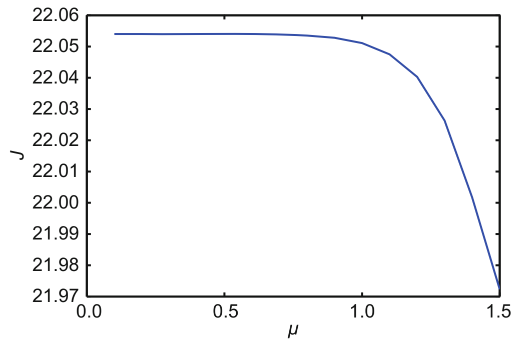

Fig. 14 Quadratic criterion $J = f(\mu)$

When $(a, b, c) = (1.2, 2.992, 6)$, as presented in Hosseinia et al. (2010), the Genesio-Tesi system presents a chaotic behavior as shown in Fig. 17. In this numerical simulation, initial conditions are the same as those used in Hosseinia et al. (2010): $x(0) = -1.0032$, $y(0) = 2.3445$, and $z(0) = -0.087$.

By a simple analysis, we obtain two unstable fixed points: $F_1 = (0, 0, 0)$ and $F_2 = (6, 6, 0)$.

To stabilize the system on the original fixed point $F_1 = (0, 0, 0)$, we use the same procedure as what is applied to the multi-scroll Chen system.

4.2.1 Using a classical PID controller

Here, the bifurcation diagrams are traced to obtain the interval values for $(k_{\rm P}, k_{\rm I}, k_{\rm D})$ parameters, which will be optimized using a quadratic criterion.

We begin by varying the x state depending on $k_{\rm P}$ as presented in Fig. 18. With $(k_{\rm I}, k_{\rm D}) = (0.1, 0.2)$, the system stabilizes on the desired fixed point for $k_{\rm P} \in [0.8, 0.998]$.

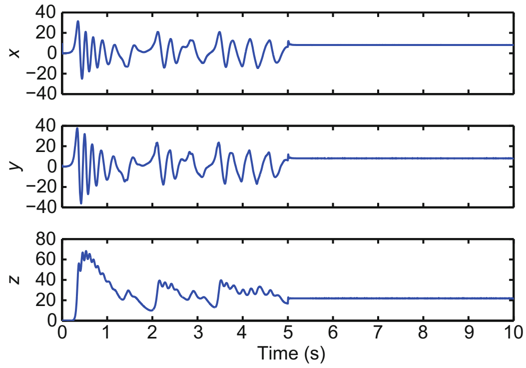

Fig. 15 State variables of the controlled multi-scroll Chen perturbed system

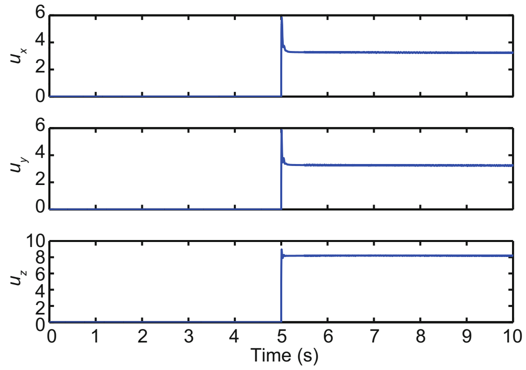

Fig. 16 PI$^\lambda$D$^\mu$ control signal for the perturbed system

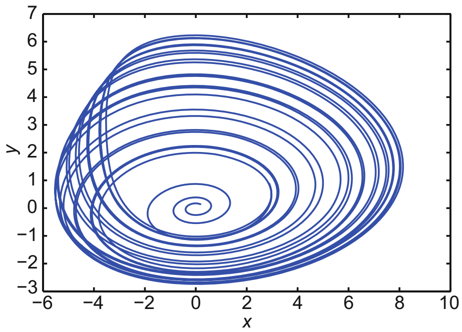

Fig. 17 Phase plane of the chaotic Genesio-Tesi system

The same procedure is repeated to obtain the intervals for k_D and k_I (Figs. 19 and 20), respectively. Then, using the quadratic criterion J given by Eq. (15), we search for the optimal values of the parameters k_P, k_I, and k_D on the corresponding stability intervals (Figs. 21, 22, and 23).

4.2.2 Using a fractional PI$^\lambda$D$^\mu$ controller

After identifying the parameters of the classic controller, we search for the fractional order of the integral and derivative actions to create an optimal fractional controller. The choice of the fractional order μ is based on the quadratic criterion. We vary μ between 0.1 and 1.1. The results obtained are shown in Table 6 and Fig. 24.

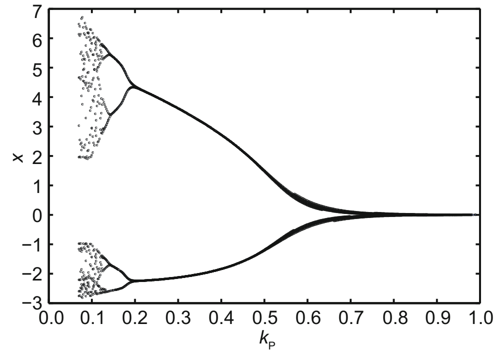

Fig. 18 Bifurcation diagram $x = f(k_\mathrm{P})$

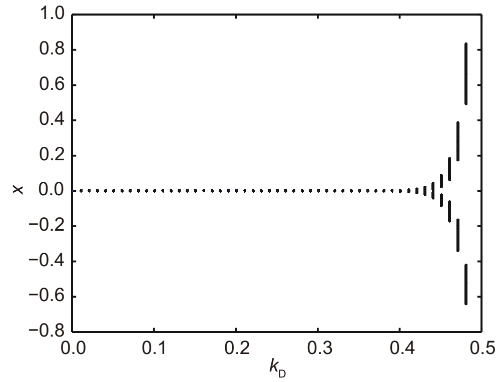

Fig. 19 Bifurcation diagram $x = f(k_\mathrm{D})$

By applying the optimal controller on the Genesio-Tesi chaotic system, this controller stabilizes the system on the fixed point $F_1 = (0, 0, 0)$ as shown in Figs. 25 and 26.

We have finally described the optimal controller parameters. Now we proceed to test the robustness of the controller. The system is disturbed with added input noise as defined in Eq. (24). The results obtained are presented in Figs. 27 and 28.

Table 6 Quadratic criterion $J = f(\mu)$

μ	J	μ	J
0.100	125.8101	0.993	123.7468
0.200	125.8036	0.995	123.7435
0.300	125.7925	0.997	123.7424
0.400	125.7731	0.999	123.7438
0.500	125.7381	1.000	123.7456
0.600	125.6719	1.001	123.7480
0.700	125.5387	1.003	123.7551
0.800	125.2510	1.005	123.7655
0.900	124.6098	1.010	123.8078
0.990	123.7556	1.050	126.0309
0.991	123.7522	1.100	168.1654

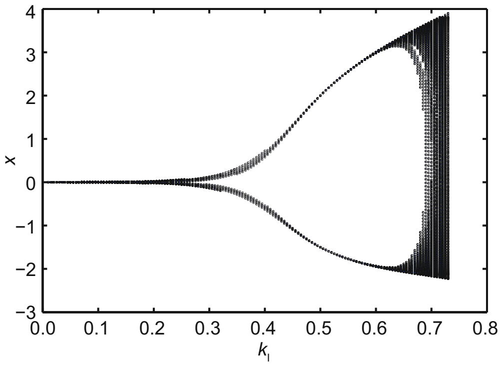

Fig. 20 Bifurcation diagram $x = f(k_I)$

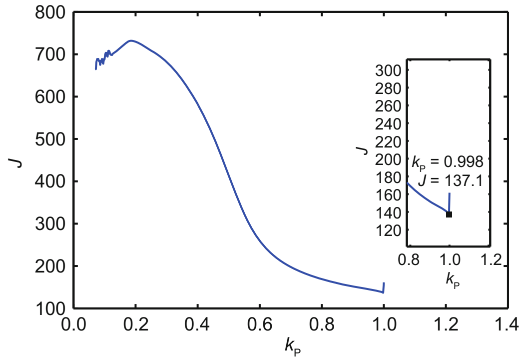

Fig. 21 Quadratic criterion $J = f(k_P)$

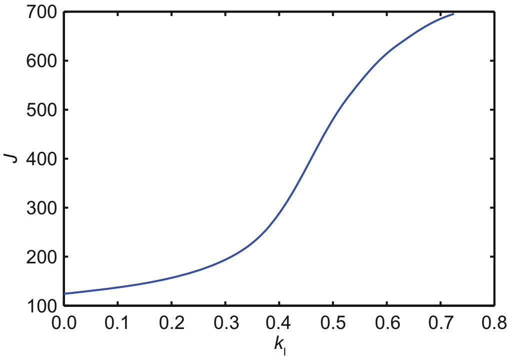

Fig. 22 Quadratic criterion $J = f(k_I)$

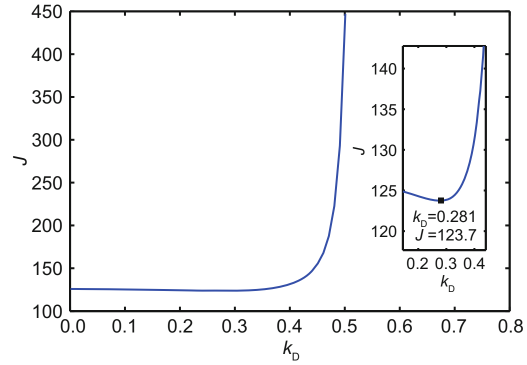

Fig. 23 Quadratic criterion $J = f(k_D)$

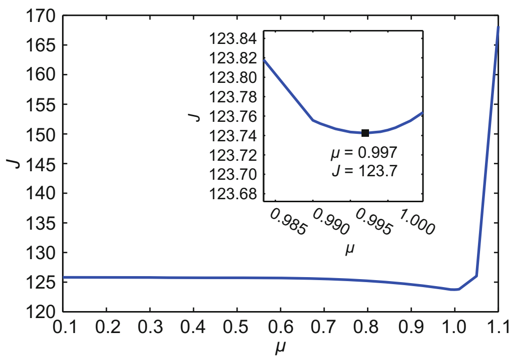

Fig. 24 Quadratic criterion $J = f(\mu)$

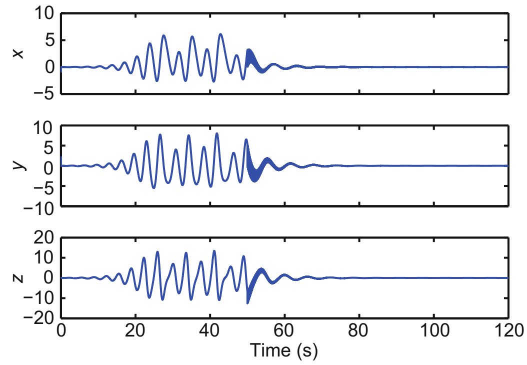

Fig. 25 State variables of the controlled Genesio-Tesi system

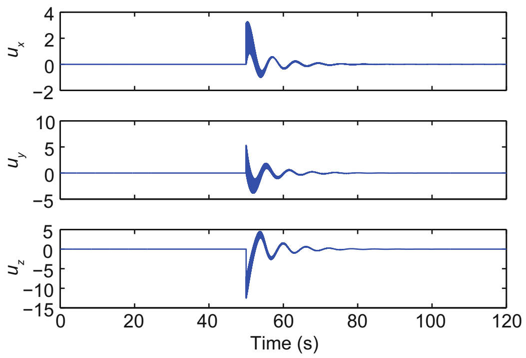

Fig. 26 PD^{μ} control signal

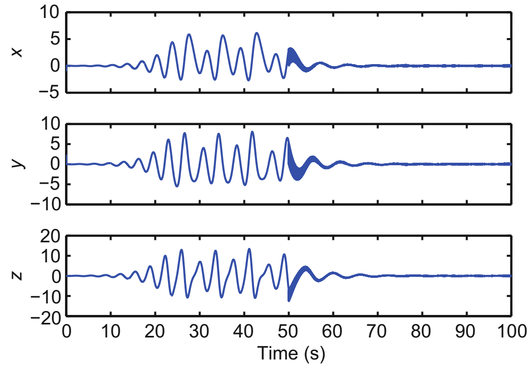

Fig. 27 State variables of the controlled Genesio-Tesi perturbed system

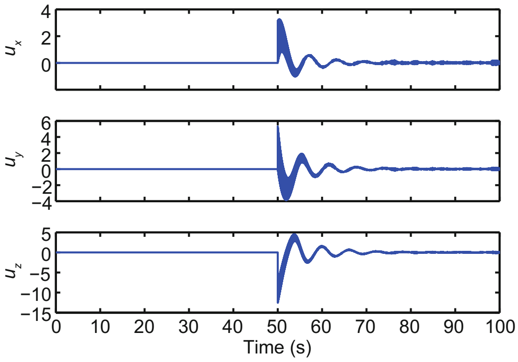

Fig. 28 PD$^{\mu}$ control signal for the perturbed system

5 Conclusions

A robust FOPID controller is investigated to stabilize a chaotic system on one of its fixed points. The three parameters $k_{\rm P}$, $k_{\rm I}$, and $k_{\rm D}$ are initially defined through the bifurcation diagram, and then optimized by means of the quadratic error criterion, which includes the fractional orders. Note that the state variables of the controlled system are asymptotically stabilized on the desired fixed point even in the presence of a random perturbation. Using numerical simulations on the proposed design approach in the first example of the multi-scroll Chen system, we find that the fractional PI$^{\lambda}$D$^{\mu}$ controller provides the best closed-loop system performance in stabilizing an unstable fixed point compared to previous studies. In the second example of the Genesio-Tesi system, the fractional proportional derivative controller presents some improvement in terms of the system behavior compared to the integer-order controller.

References

Berns D, Moiola J, Chen G, 1998. Feedback control of limit cycle amplitude from a frequency domain approach. *Automatica*, 34(12):1567-1573. https://doi.org/10.1016/S0005-1098(98)80010-1

Bouafoura M, Braiek N, 2010. PI$^{\lambda}$D$^{\mu}$ controller design for integer and fractional plants using piecewise orthogonal functions. *Commun Nonl Sci Numer Simul*, 15:1267-1278. https://doi.org/10.1016/j.cnsns.2009.05.047

Boukabou A, Chebbah A, Belmahboul A, 2007. Stabilizing unstable periodic orbits of the multi-scroll Chua's attractor. *Nonl Anal Model Contr*, 12(4):469-477.

Charef A, Assabaa M, Ladaci S, et al., 2013. Fractional order adaptive controller for stabilized systems via high-gain feedback. *IET Contr Theory Appl*, 7(6):822-828. https://doi.org/10.1049/iet-cta.2012.0309

Chen D, Liu Y, Ma X, et al., 2012. Control of a class of fractional-order chaotic systems via sliding mode. *Nonl Dynam*, 67:893-901. https://doi.org/10.1007/s11071-011-0002-x

Chen D, Zhao W, Sprott J, et al., 2013. Application of Takagi-Sugeno fuzzy model to a class of chaotic synchronization and anti-synchronization. *Nonl Dynam*, 73(3):1495-1505. https://doi.org/10.1007/s11071-013-0880-1

Chen D, Zhang R, Liu X, et al., 2014. Fractional order Lyapunov stability theorem and its applications in synchronization of complex dynamical networks. *Commun Nonl Sci Numer Simul*, 19:4105-4121. https://doi.org/10.1016/j.cnsns.2014.05.005

Chen L, Zhao T, Li W, et al., 2016. Bifurcation control of bounded noise excited Duffing oscillator by a weakly fractional-order pI$^{\lambda}$D$^{\mu}$ feedback controller. *Nonl Dynam*, 83(1):529-539. https://doi.org/10.1007/s11071-015-2345-1

Chen Z, Yuan X, Ji B, et al., 2014. Design of a fractional order PID controller for hydraulic turbine regulating system using chaotic non-dominated sorting genetic algorithm II. *Energy Conv Manag*, 84:390-404. https://doi.org/10.1016/j.enconman.2014.04.052

Colonius F, Grne L, 2002. Dynamics, Bifurcations, and Control. Springer Science & Business Media, New York, USA.

Dadras S, Momeni H, 2009. Control uncertain Genesio-Tesi chaotic system: adaptive sliding mode approach. *Chaos Sol Fract*, 42:3140-3146. https://doi.org/10.1016/j.chaos.2009.04.018

Delavari H, Ghaderi R, Ranjbar N, et al., 2010. Adaptive fractional PID controller for robot manipulator. 4$^{\rm th}$ Int Workshop on Fractional Differentiation and its Applications, p.1-7.

Diethlem K, 2003. Efficient solution of multi-term fractional differential equations using P(EC)mE methods. *Computing*, 71:305-319. https://doi.org/10.1007/s00607-003-0033-3

Ditto W, 1996. Applications of chaos in biology and medicine. *Chaos Chang Nat Sci Med*, 376:175-202. https://doi.org/10.1063/1.51060

Faieghi MR, Delavari H, 2012. Chaos in fractional-order Genesio-Tesi system and its synchronization. *Commun Nonl Sci Numer Simul*, 17:731-741. https://doi.org/10.1016/j.cnsns.2011.05.038

Faieghi MR, Naderi M, Jalali AA, 2011. Design of fractional-order PID for ship roll motion control using chaos embedded PSO algorithm. 2$^{\rm nd}$ Int Conf on Control, Instrumentation, and Automation, p.606-610. https://doi.org/10.1109/ICCIAutom.2011.6356727

Fayazi A, Rafsanjani H, 2011. Synchronization in the Genesio-Tesi and Coullet system using a fractional-order adaptive controller. 9$^{\rm th}$ IEEE Int Conf on Control and Automation, p.889-894. https://doi.org/10.1109/ICCA.2011.6137988

Genesio R, Tesi A, 1992. Harmonic balance methods for the analysis of chaotic dynamics in nonlinear systems. *Automatica*, 28(3):531-548. https://doi.org/10.1016/0005-1098(92)90177-H

Ghamati M, Balochian S, 2015. Design of adaptive sliding mode control for synchronization Genesio-Tesi chaotic system. *Chaos Sol Fract*, 75:111-117. https://doi.org/10.1016/j.chaos.2015.02.010

Gholipour R, Khosravi A, Mojallali H, 2012. Intelligent back stepping control for Genesio-Tesi chaotic system using a chaotic particle swarm optimization algorithm. *Int J Comput Electr Eng*, 4(5):618-625. https://doi.org/10.7763/IJCEE.2012.v4.570

Hadef S, Boukabou A, 2014. Control of multi-scroll Chen system. *J Frankl Inst*, 351:2728-2741. https://doi.org/10.1016/j.jfranklin.2014.01.015

Harb A, Abdel-Jabbar N, 2003. Controlling Hopf bifurcation and chaos in a small power system. *Chaos Sol Fract*, 18:1055-1063. https://doi.org/10.1016/S0960-0779(03)00073-0

Hentenryck P, Bent R, Upfal E, 1993. An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley.

Hosseinia S, Ghadri R, Ranjbar A, et al., 2010. Control of chaos via fractional order state feedback controller. In: New Trend in Nanotechnology and Fractional Calculus Applications. Springer, Netherlands, p.511-519. https://doi.org/10.1007/978-90-481-3293-5_46

Ladaci S, Bensafia Y, 2016. Indirect fractional order pole assignment based adaptive control. *Int J Eng Sci Technol*, 19:518-530. https://doi.org/10.1016/j.jestch.2015.09.004

Ladaci S, Charef A, 2006a. An adaptive fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller. 6th Int Symp on Tools and Methods of Competitive Engineering, p.1533-1540.

Ladaci S, Charef A, 2006b. On fractional adaptive control. *Nonl Dynam*, 43(4):365-378. https://doi.org/10.1007/s11071-006-0159-x

Lamba P, Hudson J, 1987. Experiments on bifurcations to chaos in a forced chemical reactor. *Chem Eng Sci*, 42:1-8. https://doi.org/10.1016/0009-2509(87)80203-8

Liu X, Shen X, Zhang H, 2012. Multi-scroll chatic and hyperchaotic attractors generated from Chen system. *Int J Bifurc Chaos*, 22(2):1250033. https://doi.org/10.1142/S0218127412500332

Liu Y, 2012. Circuit implementation and finite-time synchronization of the 4D Rabinovich hyperchaotic system. *Nonl Dynam*, 67:89-96. https://doi.org/10.1007/s11071-011-9960-2

Liu Y, Yang Q, 2010. Dynamics of a new Lorenz-like chaotic system. *Nonl Anal Real World Appl*, 11(4):2563-2572. https://doi.org/10.1016/j.nonrwa.2009.09.001

Ma J, Wang C, Tang J, et al., 2009. Suppression of the spiral wave and turbulence in the excitability-modulated media. *Int J Theor Phys*, 48:150-157. https://doi.org/10.1007/s10773-008-9790-2

Machado J, 1997. Analysis and design of fractional-order digital control systems. *J Syst Anal Model Simul*, 27(2-3):107-122.

Machado J, Galhano A, 2009. Approximating fractional derivatives in the perspective of system control. *Nonl Dynam*, 56:401-407. https://doi.org/10.1007/s11071-008-9409-4

Neçaibia A, Ladaci S, 2014. Self-tuning fractional order $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller based on extremum seeking approach. *Int J Autom Contr*, 8(2):99-121. https://doi.org/10.1504/IJAAC.2014.063361

Oustaloup A, Mathieu B, 1991. La commande CRONE (in French). HERMES Science Publ., Paris.

Pan I, Das S, 2012. Chaotic multi-objective optimization based design of fractional order $\mathrm{PI}^{k}\mathrm{D}^{l}$ controller in AVR system. *Electr Power Energy Syst*, 43:393-407. https://doi.org/10.1016/j.ijepes.2012.06.034

Park J, 2007. Adaptive controller design for modified projective synchronization of Genesio-Tesi chaotic system with uncertain parameters. *Chaos Sol Fract*, 34(4):1154-1159. https://doi.org/10.1016/j.chaos.2006.04.053

Park J, Kwon O, Lee S, 2008. LMI optimization approach to stabilization of Genesio-Tesi chaotic system via dynamic controller. *Appl Math Comput*, 196:200-206. https://doi.org/10.1016/j.amc.2007.05.045

Podlubny I, 1999a. Fractional differential equations. Mathematics in Science & Engineering, San Diego, USA.

Podlubny I, 1999b. Fractional-order systems and $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controllers. *IEEE Trans Autom Contr*, 44(1):208-214. https://doi.org/10.1109/9.739144

Rabah K, Ladaci S, Lashab M, 2015a. Stabilization of fractional Chen chaotic system by linear feedback control. 3rd IEEE Int Conf on Control, Engineering & Information Technology, p.1-5. https://doi.org/10.1109/CEIT.2015.7232990

Rabah K, Ladaci S, Lashab M, 2015b. State feedback with fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ control structure for Genesio-Tesi chaos stabilization. 16th IEEE Int Conf on Sciences and Techniques of Automatic Control & Computer Engineering, p.328-333. https://doi.org/10.1109/STA.2015.7505112

Rabah K, Ladaci S, Lashab M, 2016. Stabilization of a Genesio-Tesi chaotic system using a fractional order $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ regulator. *Int J Sci Tech Autom Contr Comput Eng*, 10(1):2085-2090.

Tang Y, Cui M, Hua C, et al., 2012. Optimum design of fractional order $\mathrm{PI}^{k}\mathrm{D}^{l}$ controller for AVR system using chaotic ant swarm. *Expert Syst Appl*, 39:6887-6896. https://doi.org/10.1016/j.eswa.2012.01.007

Tavazoei M, Haeri M, 2008. Stabilization of unstable fixed points of chaotic fractional order systems by a state fractional PI controller. *Eur J Contr*, 3:247-257. https://doi.org/10.3166/ejc.14.247-257

Wang G, 2010. Stabilization and synchronization of Genesio-Tesi system via single variable feedback controller. *Phys Lett A*, 374:2831-2834. https://doi.org/10.1016/j.physleta.2010.05.007