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A fractional-order multifunctional *n*-step honeycomb RLC circuit network*

Ling ZHOU[†1], Zhi-zhong TAN[†‡1], Qing-hua ZHANG[2]
([1]*Department of Physics, Nantong University, Nantong 226019, China*)
([2]*Department of Mathematics, Nantong University, Nantong 226019, China*)
[†]E-mail: zl7103@163.com; tanz@ntu.edu.cn

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Abstract: We investigate a multifunctional *n*-step honeycomb network which has not been studied before. By adjusting the circuit parameters, such a network can be transformed into several different networks with a variety of functions, such as a regular ladder network and a triangular network. We derive two new formulae for equivalent resistance in the resistor network and equivalent impedance in the LC network, which are in the fractional-order domain. First, we simplify the complex network into a simple equivalent model. Second, using Kirchhoff's laws, we establish a fractional difference equation. Third, we construct an equivalent transformation method to obtain a general solution for the nonlinear differential equation. In practical applications, several interesting special results are obtained. In particular, an *n*-step impedance LC network is discussed and many new characteristics of complex impedance have been found.

Key words: Honeycomb network; Equivalent transformation; Fractional differential equation; Impedance characteristics
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1 Introduction

Resistor networks are important models in the fields of natural science and engineering technology. How to calculate the equivalent resistance between two nodes has been a classical problem of circuit theory and graph theory. Since the German physicist Kirchhoff described the current law for a node and the circuit voltage law in 1845, many abstract and complex scientific and engineering problems have been solved by establishing resistor network models (Klein and Randi, 1993; Cserti, 2000; Xiao and Gutman, 2003; Wu, 2004; Tzeng and Wu, 2006; Izmailian and Huang, 2010; Tan, 2011; 2015a; 2015b; 2015c; 2015d; Asad, 2013a; Tan *et al.*, 2013; Essam *et al.*, 2014; 2015; Izmailian and Kenna, 2014; Tan and Fang, 2015; Tan and Zhang, 2015). Since resistor network models are intuitive and easy to analyze, they have become a basic method for studying a series of scientific problems. Many practical problems can also be simulated by resistor network models. Therefore, research on resistor network models has important theoretical and practical significance.

In the process of studying resistor networks, new techniques and methods are constantly being developed. Before 2004, researchers focused mainly on infinite lattices using Green's function technique (Klein and Randi, 1993; Cserti, 2000; Xiao and Gutman, 2003). Later, many resistor network problems were solved using Green's function technique (Asad, 2013b). Wu (2004) established a theorem by calculating the equivalent resistance between any two nodes in a resistor network using the Laplacian approach. Soon afterwards, the Laplacian approach was further developed (Izmailian and Huang, 2010; Izmailian and Kenna, 2014; Izmailian *et al.*, 2014; Essam *et al.*, 2015). Tan (2011) established a new recursion-transform (RT) method, which could

‡ Corresponding author
* Project supported by the Jiangsu Provincial Science Foundation (No. BK20161278)
ORCID: Zhi-zhong TAN, http://orcid.org/0000-0001-6068-3112

calculate the equivalent resistance of an $m\times n$ resistor network with different boundaries such as in a cobweb model (Tan, 2011; 2015a; 2015b; Tan *et al.*, 2013; 2015), hammock network (Essam *et al.*, 2015), globe network (Tan *et al.*, 2014), fan network (Essam *et al.*, 2015), and cobweb network with a $2r$ boundary (Tan and Fang, 2015). The RT method was further improved by Tan (2015a; 2015b; 2015c) and it could be applied to a resistor network with arbitrary boundaries. With the development of various methods, many complex impedance networks have been studied (Whan and Lobb, 1996; Gabelli *et al.*, 2006; Tan, 2012; 2016; Zhuang *et al.*, 2014; Tan *et al.*, 2015).

A precise expression of equivalent resistance with an equivalent complex impedance has been a difficult problem, especially when the network model is complex. According to various conclusions in the literature (Cserti, 2000; Wu, 2004; Tzeng and Wu, 2006; Izmailian and Huang, 2010; Tan, 2011; 2015a; 2015b; 2015c; Asad *et al.*, 2013; Tan *et al.*, 2013; Essam *et al.*, 2014; 2015; Izmailian and Kenna, 2014), we find that they are expressed by a fractional order. Research shows that the equivalent resistance of the resistor network expressed by a fractional order is a more effective method. The fractional-order mathematical models developed for inductors and capacitors can describe the electrical characteristics more accurately. Indeed, the inductors and capacitors themselves may be of a fractional order in nature (Biswas *et al.*, 2006). At present, some researchers are also devoted to the design and realization of fractional electronic components (Radwan and Salama, 2011; Machado and Galhano, 2012; Elshurafa *et al.*, 2013), and some researchers have concentrated on the study of fractional-order circuit theory (Radwan and Salama, 2012; Chen and He, 2013; Jia *et al.*, 2013; Wang and Ma, 2013; Zhou and Huang, 2014). However, few researchers are studying the electrical characteristics of n-step honeycomb circuit networks in a fractional-order sense. Therefore, we focus on the resistance and impedance of fractional-order n-step honeycomb resistors and an LC circuit network, which may lay a foundation for future research on graphene networks and the structures of some non-metallic crystals.

Though resistor network models have been studied since the 1840s and multiple types of resistor network problems have been solved, exact equivalent resistance equations are still difficult to obtain due to the complexity of resistor networks in real life.

Therefore, more innovations in circuit theory knowledge and mathematical methods are needed. In this study, we investigate a multifunctional n-step honeycomb resistor network as shown in Fig. 1. In the model, the network unit number is n, each resistance element on the upper and lower boundaries is r, and the one in the vertical direction is r_0. Resistance elements in the vertical and horizontal directions on the inside rectangular grid in each network unit are r_1 and r_3, respectively. The resistances that connect the inside and outside rectangular grids are all r_2 and the right boundary is an arbitrary load resistance R_0. Clearly, there are six different resistance elements in the model. This is a complex network, which has never been investigated before. In particular, the multi-parameter network contains a variety of network functions. Since the six different resistors are arbitrary elements, this network model can be transformed into several special resistor networks, such as a regular ladder network, a triangular network, and some new networks, by adjusting the circuit parameters. Here, we study mainly the equivalent resistance formula between the two nodes A_n and B_n.

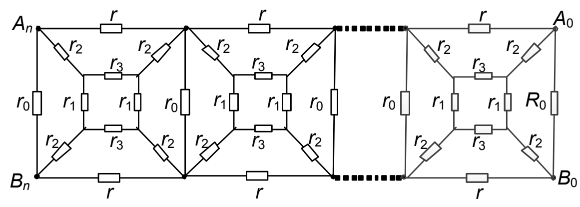

Fig. 1 A type of multifunctional n-step honeycomb resistor-network model, which contains six different resistance elements and an arbitrary load resistance R_0

2 General equivalent resistance formula

We consider a multifunctional n-step honeycomb resistor network as shown in Fig. 1. In this network, there are six different resistors. The right boundary is a load resistor R_0. The general equivalent resistance between the two nodes A_n and B_n can be expressed in a fractional form:

$$R_{A_nB_n}(n)=\lambda r_0-\alpha\beta\frac{F_n+(R_0-\lambda r_0)F_{n-1}}{F_{n+1}+(R_0-\lambda r_0)F_n}, \quad (1)$$

where

$$\begin{cases} F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}, \\ \lambda = \dfrac{b}{d}, \end{cases} \tag{2}$$

$$\begin{cases} \alpha = \dfrac{1}{d}\left(r + br_0 + \sqrt{r^2 + 2dr_0 r}\right), \\ \beta = \dfrac{1}{d}\left(r + br_0 - \sqrt{r^2 + 2dr_0 r}\right), \end{cases} \tag{3}$$

$$\begin{cases} b = 1 + \dfrac{2r}{2r_2 + r_1} + \dfrac{rr_1^2}{u(2r_2 + r_1)}, \\ u = 2r_2(r_1 + r_3) + r_1 r_3, \\ d = 1 + \dfrac{2(r_0 + r)}{2r_2 + r_1} + \dfrac{r_1^2 + 2r_0(r_3 + r_1)}{u(2r_2 + r_1)} r. \end{cases} \tag{4}$$

The general equivalent resistance Eq. (1) can be derived in two steps: first, the model for the recursion equation is set up by Kirchhoff's law; second, the nonlinear differential equation is solved by constructing the transformation.

3 Equivalent model and recursion equation

According to the structural features in Fig. 1, we set the equivalent resistance between A_n and B_n on the nth network as R_n, and that between A_{n-1} and B_{n-1} as R_{n-1}. Therefore, the model in Fig. 1 can be reduced to a simpler model as shown in Fig. 2.

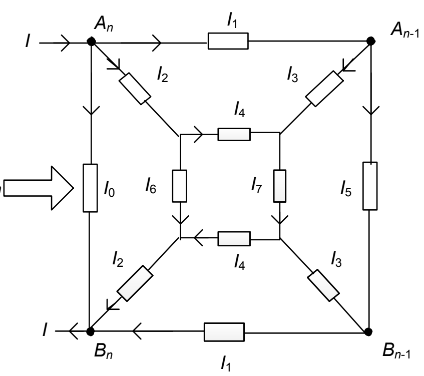

Fig. 2 The equivalent model marked with current parameters

Using Kirchhoff's law, we can establish a connection between R_n and R_{n-1}. In Fig. 2, the constant electricity I is fed into the network at node A_n, and flows out at node B_n, and the other branch currents are defined as I_1–I_7.

According to Kirchhoff's second law, the loop voltage equations can be written as

$$\begin{cases} 2I_2 r_2 + I_6 r_1 - I_0 r_0 = 0, \\ I_1 r + I_3 r_2 - I_4 r_3 - I_2 r_2 = 0, \\ 2I_3 r_2 + I_7 r_1 - I_5 R_{n-1} = 0, \\ 2I_4 r_3 + (I_7 - I_6) r_1 = 0. \end{cases} \tag{5}$$

From Kirchhoff's first law, the nodal current equations can be expressed as

$$\begin{cases} I_1 + I_2 + I_0 = I, \\ I_4 + I_6 = I_2, \\ I_3 + I_5 = I_1, \\ I_3 + I_4 = I_7. \end{cases} \tag{6}$$

The solution for the above equations gives an important recursive relationship as follows:

$$\frac{I_0}{I} = \frac{2r + bR_{n-1}}{(2r + r_0 b) + dR_{n-1}}, \tag{7}$$

where b and d are given by Eq. (4). Putting Eq. (7) into the relation $R_n = U/I = I_0 r_0 / I$, we obtain

$$R_n = \frac{2rr_0 + br_0 R_{n-1}}{(2r + r_0 b) + dR_{n-1}}. \tag{8}$$

Eq. (8) is a key recursive formula in the form of a nonlinear differential equation. In the next section the general and special solutions to Eq. (8) are studied.

4 Equivalent transformation and formula derivation

Using the variable substitution method, Eq. (8) can be solved. We assume that there exists a sequence $\{x_n\}$ with $x_0=1$, which satisfies

$$R_n = \frac{x_{n+1}}{x_n} - \frac{2r + r_0 b}{d}. \tag{9}$$

Taking $n=0$, we have

$$\begin{cases} x_0 = 1, \\ x_1 = R_0 + \dfrac{2r + r_0 b}{d}. \end{cases} \tag{10}$$

Substituting Eq. (9) and the recursive formula R_{n-1} into Eq. (8) gives

$$x_{n+1} = 2\frac{r + br_0}{d}x_n - \frac{r_0[b(2r + r_0 b) - 2rd]}{d^2}x_{n-1}. \tag{11}$$

Assuming that α and β are the two roots of the characteristic equation x_n, we can obtain the parameters in Eq. (3). From Eq. (11), we have

$$x_{n+1} = (\alpha + \beta)x_n - \alpha\beta x_{n-1}. \tag{12}$$

Using the method proposed by Tan (2011; 2015a; 2015b; 2015c; 2015d), to solve the differential Eq. (12), one has

$$x_n = \frac{1}{\alpha - \beta}[(x_1 - \beta x_0)\alpha^n - (x_1 - \alpha x_0)\beta^n]. \tag{13}$$

Putting the initial item (10) into Eq. (13), we obtain

$$\begin{aligned} x_n = \frac{1}{\alpha - \beta}\Bigg[&\left(R_0 + \frac{2r + r_0 b}{d} - \beta\right)\alpha^n \\ &- \left(R_0 + \frac{2r + r_0 b}{d} - \alpha\right)\beta^n\Bigg]. \end{aligned} \tag{14}$$

To further simplify the above formula, using Eqs. (11) and (12), we obtain

$$\alpha + \beta - \frac{2r + r_0 b}{d} = \frac{b}{d}r_0 = \lambda r_0. \tag{15}$$

Substituting Eq. (15) into Eq. (14), one has

$$x_n = \frac{1}{\alpha - \beta}[(R_0 + \alpha - \lambda r_0)\alpha^n - (R_0 + \beta - \lambda r_0)\beta^n]. \tag{16}$$

Substituting Eq. (16) and the recursive formula x_{n+1} into Eq. (9) and using Eq. (15) for simplification, we have

$$R_n = \lambda - \alpha\beta\frac{(R_0 + \alpha - \lambda r_0)\alpha^{n-1} - (R_0 + \beta - \lambda r_0)\beta^{n-1}}{(R_0 + \alpha - \lambda r_0)\alpha^n - (R_0 + \beta - \lambda r_0)\beta^n}. \tag{17}$$

Here, we define $F_n=(\alpha^n-\beta^n)/(\alpha-\beta)$. Then we can obtain Eq. (1) from Eq. (17). So, Eq. (1) is proven.

5 Special cases and comparisons

Case 1: $r_3=r_1$. The equivalent resistance formula in this case is still expressed by Eqs. (1) and (5). The parameters b and d can be reduced as

$$\begin{cases} b = 1 + \dfrac{2r}{2r_2 + r_1} + \dfrac{rr_1}{(4r_2 + r_1)(2r_2 + r_1)}, \\ d = 1 + \dfrac{2(r_0 + r)}{2r_2 + r_1} + \dfrac{(r_1 + 4r_0)r}{(4r_2 + r_1)(2r_2 + r_1)}. \end{cases} \tag{18}$$

Case 2: $r_3=r_2=r_1=r$. From Eq. (4) we have

$$\begin{cases} b = \dfrac{26}{15}, \\ d = \dfrac{26r + 14r_0}{15r}. \end{cases} \tag{19}$$

Substituting Eq. (19) into Eq. (3) gives

$$\begin{cases} \alpha = \dfrac{15r}{26r + 14r_0}\left(r + \dfrac{26}{15}r_0 + \sqrt{r^2 + \dfrac{4r_0}{15}(13r + 7r_0)}\right), \\ \beta = \dfrac{15r}{26r + 14r_0}\left(r + \dfrac{26}{15}r_0 - \sqrt{r^2 + \dfrac{4r_0}{15}(13r + 7r_0)}\right), \\ \lambda = \dfrac{b}{d} = \dfrac{13r}{13r + 7r_0}. \end{cases} \tag{20}$$

Then, substituting the above parameters into Eq. (1), the equivalent resistance can be written as

$$R_{A_nB_n} = \frac{13r_0 r}{13r + 7r_0} - \left(\frac{8rr_0}{13r + 7r_0}\right)^2 \frac{F_n + (R_0 - \lambda r_0)F_{n-1}}{F_{n+1} + (R_0 - \lambda r_0)F_n}. \tag{21}$$

Case 3: $r_3=0$. The n-step honeycomb network in Fig. 1 can be reduced to an n-step resistor network with a vertical resistance bridge (Fig. 3). The resistance r_1 in Fig. 3 has been expanded to twice the original resistance value ($r_1 \rightarrow 2r_1$). The parameters in Eq. (4) are given by

$$\begin{cases} b = 1 + \dfrac{r}{r_2 + r_1} + \dfrac{rr_1}{2r_2(r_2 + r_1)}, \\ d = 1 + \dfrac{r_0 + r}{r_2 + r_1} + \dfrac{(r_1 + r_0)r}{2r_2(r_2 + r_1)}. \end{cases} \tag{22}$$

Therefore, the equivalent resistance formula is still expressed by Eqs. (1) and (5). However, the parameters are degenerated into Eq. (22).

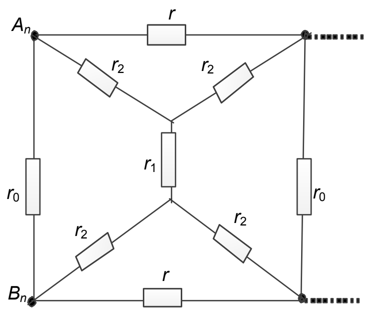

Fig. 3 Sub-network model with a vertical resistance bridge

Case 4: r_1=0. In this case, the n-step honeycomb network in Fig. 1 can be reduced to an n-step resistor network with a horizontal resistance bridge (Fig. 4). Here the resistor r_3 in Fig. 4 has been expanded to twice the original resistance value ($r_3 \rightarrow 2r_3$). The parameters in Eq. (4) are simplified as

$$\begin{cases} b = 1 + \dfrac{r}{r_2}, \\ d = 1 + \dfrac{r_0 + r}{r_2} + \dfrac{r_0 r}{2r_2^2}. \end{cases} \tag{23}$$

Putting Eq. (23) into Eq. (3), one has

$$\begin{cases} \alpha = \dfrac{1}{d}\left(r + r_0 + \dfrac{r_0 r}{r_2} + \sqrt{r^2 + 2dr_0 r} \right), \\ \beta = \dfrac{1}{d}\left(r + r_0 + \dfrac{r_0 r}{r_2} - \sqrt{r^2 + 2dr_0 r} \right), \end{cases} \tag{24}$$

$$\lambda = \frac{2(r_2 + r)r_2}{2r_2^2 + 2(r_0 + r)r_2 + r_0 r}. \tag{25}$$

Therefore, putting the above parameters into Eq. (1), we obtain

$$R_{A_n B_n} = \lambda r_0 - \left(\frac{r_0}{d} \right)^2 \frac{F_n + (R_0 - \lambda r_0)F_{n-1}}{F_{n+1} + (R_0 - \lambda r_0)F_n}, \tag{26}$$

where F_n is defined by Eq. (2) and λ is given by Eq. (25).

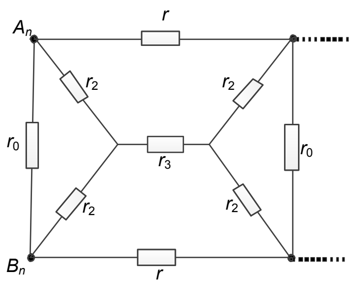

Fig. 4 Sub-network model with a horizontal resistance bridge

Case 5: r_3=r_1=0. In this case, the n-step honeycomb network in Fig. 1 is reduced to an n-step resistor network with embedded cross resistors (Fig. 5).

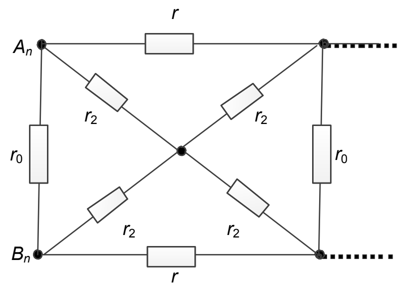

Fig. 5 Sub-network model with embedded cross resistors

According to Eq. (4), we have

$$\begin{cases} b = 1 + \dfrac{r}{r_2}, \\ d = 1 + \dfrac{r_0 + r}{r_2} + \dfrac{r_0 r}{2r_2^2}. \end{cases} \tag{27}$$

Obviously, Eq. (27) is the same as Eq. (23). Therefore, their equivalent resistances are completely the same as given by Eq. (26). Such results can be explained by the symmetry structure in Fig. 4. The network in Fig. 4 shows that no current passes through resistor r_3 when r_1=0. That is to say, there is no voltage on resistor r_3. Therefore, it can be regarded as a short circuit. The result also shows that the general formula obtained in this study is correct.

Case 6: r_2=0. In this case, the n-step honeycomb network in Fig. 1 is reduced to an n-step rectangular resistor parallel network connected by two rectangles (Fig. 6). According to Eq. (4), we obtain

$$\begin{cases} b = 1 + \dfrac{2r}{r_1} + \dfrac{r}{r_3}, \\ d = 1 + \dfrac{2(r_0 + r)}{r_1} + \dfrac{r_1^2 + 2r_0(r_3 + r_1)}{r_1^2 r_3} r. \end{cases} \tag{28}$$

The equivalent resistance formula in this case is still expressed by Eqs. (1) and (5), and the parameters are degenerated into Eq. (28).

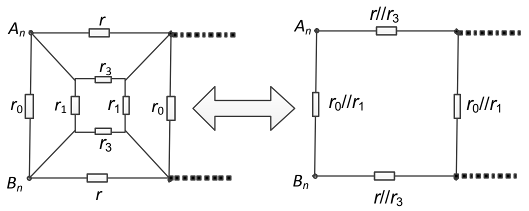

Fig. 6 The double-rectangular resistor network and equivalent single rectangular resistor network

Case 7: r_3=r_1=∞ or r_2=∞. In this case, the n-step honeycomb network in Fig. 1 is reduced to a simple n-step rectangular resistor network (Fig. 7).

When the limit r_2→∞ in Eq. (4) is taken, we have

$$\begin{cases} b = 1, \\ d = 1, \end{cases} \tag{29}$$

$$\lambda = b/d = 1, \tag{30}$$

$$\begin{cases} \alpha = r_0 + r + \sqrt{r(2r_0 + r)}, \\ \beta = r_0 + r - \sqrt{r(2r_0 + r)}. \end{cases} \tag{31}$$

Putting them into Eq. (1), we obtain

$$R_n = r_0 - r_0^2 \frac{F_n + (R_0 - r_0)F_{n-1}}{F_{n+1} + (R_0 - r_0)F_n}. \tag{32}$$

The network model in Fig. 7 has been studied before (Tan, 2011; 2016). Comparing Eq. (32) with the results in the literature, we find that the two conclusions are completely the same, which also indirectly verifies our conclusion.

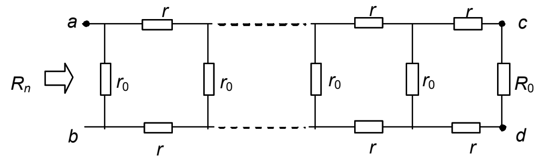

Fig. 7 A simple n-step rectangular resistor network model

Case 8: n→∞. From Eq. (3), we have 0<β/α<1; therefore,

$$\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n = \lim_{n\to\infty}\left(\frac{r + br_0 - \sqrt{r^2 + 4adr_0}}{r + br_0 + \sqrt{r^2 + 4adr_0}}\right)^n = 0. \tag{33}$$

According to Eq. (1), we have

$$R_{A_nB_n}(\infty) = \lambda r_0 - \beta. \tag{34}$$

Substituting Eq. (3) into Eq. (34), we obtain

$$R_{A_nB_n}(\infty) = \frac{1}{d}\left(\sqrt{r^2 + 2dr_0 r} - r\right). \tag{35}$$

The above result shows that the equivalent resistance in Eq. (1) is a finite constant when n→∞.

Case 9: R_0=λr_0−β. From Eq. (17), we have

$$R_{A_nB_n}(n) = \lambda r_0 - \beta. \tag{36}$$

We find that Eq. (36) is the same as Eq. (34). However, they are different in origin. Eq. (36) gives the characteristic resistance of the resistor network. Such characteristic resistance is the same as the limit resistance. Obviously, when the load resistance R_0=λr_0−β is the characteristic resistance, the equivalent resistance is a finite constant, equal to the resistance in the limit case. This is an interesting finding.

6 Characteristics of an LC network

The method and conclusion in this paper are also suitable for a complex impedance network. As long as we use the variable substitution method, we can obtain the equivalent complex impedance formula derived from the equivalent resistance formula. Of course, the characteristics of complex impedance are completely different from those of the resistance. Since complex impedance is related to the frequency of the input current, the expression for complex impedance is more complex. In particular, we should discuss the case where the root of the characteristic equation is a complex number. Here we study a type of LC complex impedance network whose sub-network model is shown in Fig. 8. The oscillation characteristic and resonance property are discussed.

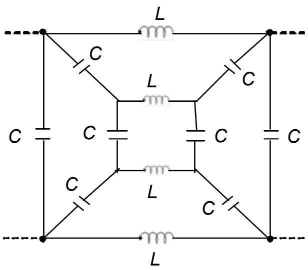

Fig. 8 Sub-network model of a type of LC complex impedance network

Consider a multifunctional n-step LC network whose complex impedance parameters are shown in Fig. 8. From Eq. (1), we can obtain the equivalent complex impedance formula. According to the theory of complex impedance, the relationships between the resistors in Fig. 1 and the complex impedance elements in Fig. 8 are $r_3=r=\mathrm{i}\omega L$, $r_2=r_1=r_0=1/(\mathrm{i}\omega C)$. Assuming $R_0=qr_0=q/(\mathrm{i}\omega C)$, $\omega^2LC=x$, the corresponding parameters from Eqs. (2) and (4) are given by

$$b=-2\cdot\frac{3x^2-7x+3}{3(3x-2)}, \tag{37}$$

$$d=-2\cdot\frac{4x^2-11x+5}{3(3x-2)}, \tag{38}$$

$$\lambda=\frac{b}{d}=\frac{3x^2-7x+3}{4x^2-11x+5}. \tag{39}$$

Substituting Eqs. (37), (38), and (39) into Eq. (1) gives the equivalent complex impedance formula:

$$\begin{aligned}\frac{Z_{A_nB_n}}{r_0}=&\lambda-\left(\frac{5x-3}{4x^2-11x+5}\right)^2\\&\cdot\frac{\delta^n-\rho^n+(q-\lambda)(\delta^{n-1}-\rho^{n-1})}{\delta^{n+1}-\rho^{n+1}+(q-\lambda)(\delta^n-\rho^n)},\end{aligned} \tag{40}$$

where $\delta=\alpha/r_0$, $\rho=\beta/r_0$. Substituting Eqs. (37), (38), and (39) into the characteristic roots (3), we obtain

$$\begin{cases}\delta=\dfrac{15x^2-20x+6-\sqrt{15x(3x-2)(5x^2-10x+4)}}{2(4x^2-11x+5)},\\ \rho=\dfrac{15x^2-20x+6+\sqrt{15x(3x-2)(5x^2-10x+4)}}{2(4x^2-11x+5)}.\end{cases} \tag{41}$$

Obviously, there are three parameters x, n, and q in the complex impedance. From Eq. (41), we should discuss the cases where $(5x^2-10x+4)(3x-2)=(>,<)0$.

Solving $(3x-2)(5x^2-10x+4)=0$, we obtain

$$x_1=\frac{2}{3}\approx 0.667,\ \ x_2=\frac{5-\sqrt{5}}{5}\approx 0.553,$$
$$x_3=\frac{5+\sqrt{5}}{5}\approx 1.447.$$

Thus, when $(3x-2)(5x^2-10x+4)\geq 0$, we obtain

$$\frac{5-\sqrt{5}}{5}\leq x\leq\frac{2}{3}\ \text{ or }\ x\geq\frac{5+\sqrt{5}}{5}. \tag{42}$$

When $(3x-2)(5x^2-10x+4)<0$, we have

$$0<x<\frac{5-\sqrt{5}}{5}\ \text{ or }\ \frac{2}{3}<x<\frac{5+\sqrt{5}}{5}. \tag{43}$$

Therefore, to study the characteristics of complex impedance, we should consider the four intervals above.

In particular, when $q=\lambda$, from Eq. (40) we have

$$\frac{Z_{A_nB_n}}{r_0}=\frac{3x^2-7x+3}{4x^2-11x+5}-\left(\frac{5x-3}{4x^2-11x+5}\right)^2\frac{\delta^n-\rho^n}{\delta^{n+1}-\rho^{n+1}}, \tag{44}$$

where $x=\omega^2LC$. In the following, we plot 3D graphs to further clarify the change characteristics of complex impedance at different intervals.

6.1 Case for $(3x-2)(5x^2-10x+4)\geq 0$

When $(3x-2)(5x^2-10x+4)\geq 0$, we have δ, $\rho\in$ |. The complex impedance is expressed as Eq. (40). From Eq. (40), we can draw the 3D graphs of $Z_{A_nB_n}\sim(x,n), Z_{A_nB_n}\sim(x,q)$, where $x=\omega^2LC$. The characteristics of complex impedance are shown in Figs. 9–12.

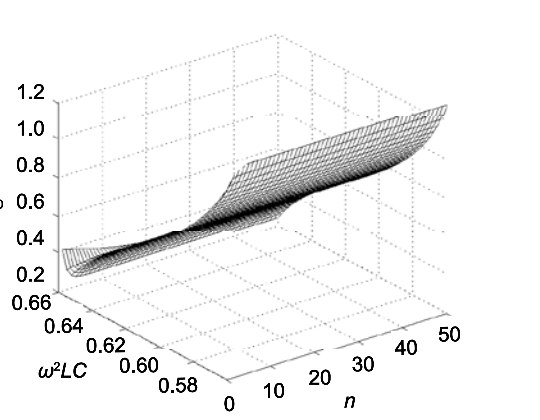

Fig. 9 A 3D graph showing the impedance changes with ω and n in the case of $(5-\sqrt{5})/5 \le x \le 2/3$

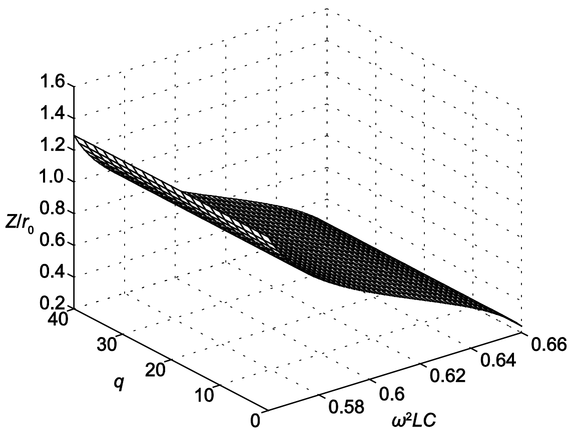

Fig. 10 A 3D graph showing the impedance changes with ω and q in the case of $(5-\sqrt{5})/5 \le x \le 2/3$ when n=10

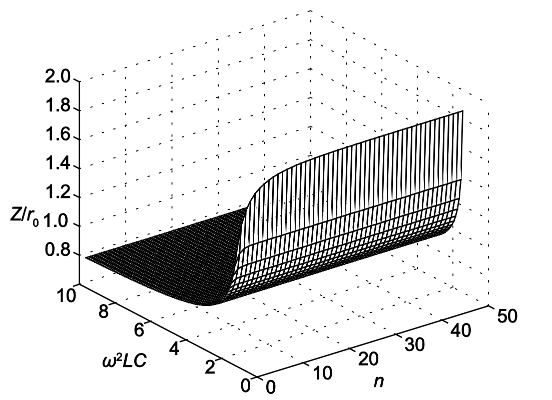

Fig. 11 A 3D graph showing the impedance changes with x and n in the case of $x \ge (5+\sqrt{5})/5$

Fig. 9 shows that the complex impedance in Eq. (40) changes gradually with increasing x and n when $(5-\sqrt{5})/5 \le x \le 2/3$. Fig. 10 shows that the complex impedance in Eq. (40) decreases gradually with increasing x and q when $(5-\sqrt{5})/5 \le x \le 2/3$.

Fig. 11 shows that the complex impedance of Eq. (40) gradually decreases with increasing x and n when $x \ge (5+\sqrt{5})/5$. Fig. 12 shows that the complex impedance of Eq. (40) is gradually decreasing with increasing x and q when $x \ge (5+\sqrt{5})/5$.

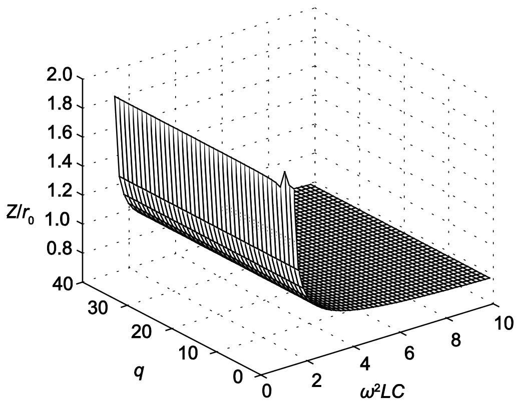

Fig. 12 A 3D graph showing the impedance changes with x and q in the case of $x \ge (5+\sqrt{5})/5$ when n=10

6.2 Case for $(5x^2-10x+4)(3x-2)<0$

When $(5x^2-10x+4)(3x-2)<0$, we have $\delta, \rho \in \mathbb{Z}$. Eq. (40) can be further simplified to obtain a practical result. From Eq. (41) we have

$$\begin{aligned} &15x^2-20x+6-\sqrt{15x(3x-2)(5x^2-10x+4)} \\ &\quad =2(5x-3)(\cos\theta-\mathrm{i}\sin\theta), \end{aligned} \tag{45}$$

where $\theta=\arccos[(15x^2-20x+6)/(5x-3)]$. Substituting Eq. (45) into Eq. (41), we obtain

$$\begin{cases} \delta=\dfrac{5x-3}{4x^2-11x+5}(\cos\theta-\mathrm{i}\sin\theta), \\ \rho=\dfrac{5x-3}{4x^2-11x+5}(\cos\theta+\mathrm{i}\sin\theta). \end{cases} \tag{46}$$

Substituting Eq. (46) into Eq. (40), we obtain

$$\frac{Z_{A_nB_n}}{r_0}=\lambda-\mu\cdot\frac{\mu\sin(n\theta)+(q-\lambda)\sin((n-1)\theta)}{\mu\sin((n+1)\theta)+(q-\lambda)\sin(n\theta)}, \tag{47}$$

where $\theta=\arccos[(15x^2-20x+6)/(5x-3)]$ and

$$\begin{cases} \lambda = \dfrac{3x^2 - 7x + 3}{4x^2 - 11x + 5}, \\ \mu = \dfrac{5x - 3}{4x^2 - 11x + 5}. \end{cases} \tag{48}$$

From Eq. (47), we can draw the 3D graphs of $Z_{A_nB_n} \sim (x,n), Z_{A_nB_n} \sim (x,q)$, where $x=\omega^2LC$. The characteristics of the complex impedance are shown in Figs. 13 and 14.

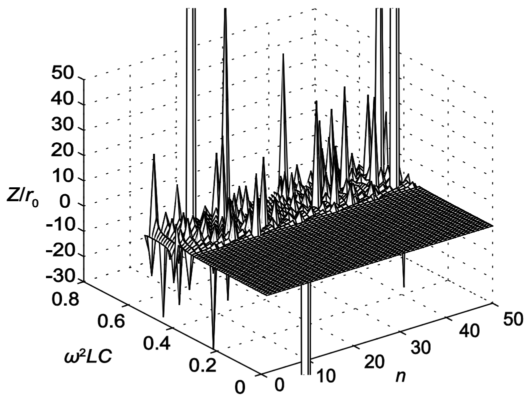

Fig. 13 A 3D graph showing the impedance changes with ω and n in the case of 0<x<(5−√5)/5

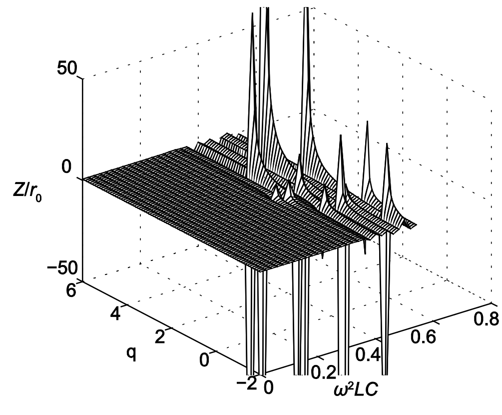

Fig. 14 A 3D graph showing the impedance changes with x and q in the case of 0<x<(5−√5)/5 when n=10

Fig. 13 shows that the complex impedance in Eq. (47) changes irregularly with increasing x and n when 0<x<(5−√5)/5, where the complex impedance changes gradually when 0<x<1/3, and it has an irregular oscillation characteristic and resonance property when 1/3<x<(5−√5)/5.

Fig. 14 shows that the complex impedance in Eq. (47) changes irregularly with increasing x and q when 0<x<(5−√5)/5 and n=10, where the complex impedance changes gradually when 0<x<1/3, and it has an irregular oscillation characteristic and resonance property when 1/3<x<(5−√5)/5.

Fig. 15 shows that the complex impedance in Eq. (47) changes irregularly with increasing x and n when 2/3<x<(5+√5)/5, where the complex impedance has an irregular oscillation characteristic and resonance property when 2/3<x<1, and it changes gradually when 1<x<(5+√5)/5.

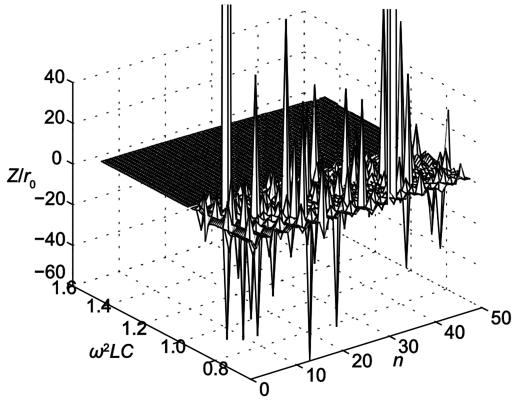

Fig. 15 A 3D graph showing the impedance changes with x and q in the case of 2/3<x<(5+√5)/5

Fig. 16 shows that the complex impedance in Eq. (47) changes irregularly with increasing x and q when 2/3<x<(5+√5)/5, where the complex impedance has an irregular oscillation characteristic and resonance property when 2/3<x<1, and it changes gradually when 1<x<(5+√5)/5.

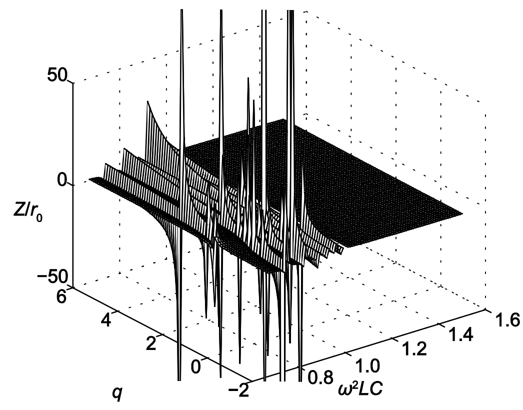

Fig. 16 A 3D graph showing the impedance changes with x and q in the case of 2/3<x<(5+√5)/5 when n=10

From Figs. 13–16, we find that the characteristics of the complex impedance are different from those of the resistance. The complex impedance often oscillates and resonates with the AC frequency ω. Note that the seven graphs above (Figs. 9–16) have been given for the first time.

7 Summary and discussion

It is difficult to calculate the resistance of complex resistor networks with multiple parameters without an innovative approach. The transform method established in our study obtains the resistance and impedance of a type of n-step honeycomb circuit network, which has never been solved before. The method includes three steps primarily. First, an equivalent network model is established. Then using Kirchhoff's laws, the recursive relation of the equivalent resistance is found. Finally, the general solution for the differential equation is given by the transform method. In this study, we obtain the fractional-order concise solution in the form of formula (1) though the network contains five independent parameters. In practical applications, a number of interesting results are obtained from the general formula (1) by considering some parameters as special values. In particular, the research method and conclusion of this study are also suitable for a complex impedance network. We obtain the equivalent impedance formula of a complex impedance network by using the variable substitution method. As an example, we consider an LC impedance network. The resonance and oscillatory characteristics are found. Considering that the complex impedance is related to the frequency of the input current, the positive and negative of the factor $(3x-2)(5x^2-10x+4)$ in the characteristic roots (41) are changed with x ($x=\omega^2LC$). Therefore, for cases where $(3x-2)(5x^2-10x+4)\geq 0$ and $(3x-2)(5x^2-10x+4)<0$ are considered, the seven 3D graphs are obtained (Figs. 9–16). Obviously, the complex impedance network is different from the resistor network. These interesting results show that it is possible to use our formulae to discuss these resonant circuits with practical applications.

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