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A new auxiliary information based cumulative sum median control chart for location monitoring*#

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Abstract: Control charts are commonly used tools in statistical process control for the detection of shifts in process parameters. Shewhart-type charts are efficient for large shift values, whereas cumulative sum (CUSUM) charts are effective in detecting medium and small shifts. Control chart use commonly assumes that data are free of outliers and parameters are known or correctly estimated based on an in-control process. In practice, these assumptions are not often true because some processes occasionally have outliers. Monitoring the location parameter is usually based on mean charts, which are seriously affected by violations of these assumptions. In this paper we propose several CUSUM median control charts based on auxiliary variables, and offer comparisons with their corresponding mean control charts. To monitor the location parameter, we examined the performance of mean and median control charts in the presence and absence of outliers. Both symmetric and non-symmetric processes were studied to examine the properties of the proposed control charts to monitor the location parameter using CUSUM control charts. We used different run length measures to study in-control and out-of-control performances of CUSUM charts. Results revealed that our proposed control charts perform much better than the traditional charts in the presence of outliers. A real application of our study was provided using data on concrete compressive strength as it relates to the quality of cement manufacturing.

Key words: Average run length; Auxiliary information; CUSUM control charts; Location parameter; Median control charts
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1 Introduction

Statistical process control (SPC) is commonly used to monitor manufacturing and non-manufacturing process parameters. Variations in the manufacturing production process always exist. Common and/or special causes are the main reason for the variation in processes (Oakland, 2007). Control charts are well-known and commonly used tools for identifying unusual variations in such parameters (mainly location and dispersion). In the response of any process that deals with quality characteristics, control charts are the most important tools that help differentiate between special (also known as assignable or non-random) cause and common (also known as random) cause variations. Control charts can be categorized as memory and memoryless control charts. The Shewhart control chart is a well-known and commonly used control chart proposed by Shewhart (1924) and is categorized as a memoryless control chart. Memoryless control charts are famous for efficient detection of large shift values, but these charts

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are less effective in identifying moderate and small shift values of the shift, so memory control charts are the best answer to address this issue. Roberts (1959) and Page (1954) suggested exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) charts, respectively, which are considered more capable than traditional Shewhart charts in detecting moderate and small shifts.

The CUSUM scheme is frequently used in industrial production as the best alternative to the Shewhart scheme. The CUSUM chart has been helpful in SPC for process monitoring (Qiu and Hawkins, 2001; Rao, 2013; Abujiya et al., 2015a; Shafae et al., 2015; Huang et al., 2016). The design strategies and properties of CUSUM control charts can be meticulously explored in the literature (Qiu and Hawkins, 2001; Nazir et al., 2013, 2015; Hawkins and Wu, 2014; Rakitzis et al., 2018). Many researchers proposed other methods to monitor the location parameter (Haridy and Elshabrawy, 1996; Umble, 2001; Ryu et al., 2010; Riaz et al., 2011; Mukherjee et al., 2013; Abujiya et al., 2015a), but less consideration has been given to monitoring by median control charts.

Additional information associated with the variable of interest can be used to improve the execution of a control chart. For example, in the process monitoring of missile testing, some extra information like momentum or the weight of the carrier can enhance the efficiency of the projectile. In general, precision can be increased using auxiliary characteristics. To enhance the monitoring of quality characteristics, auxiliary information is used in various charting procedures as in regression adjustments and cause-selecting charts suggested by Hawkins (1993) and Shu et al. (2005). Some control charts studies are based on auxiliary information used for estimation of parameters (Riaz, 2008a, 2008b; Riaz and Does, 2009; Abbasi and Riaz, 2013; Abbas et al., 2014). Some studies are based on auxiliary information for ranking purposes (Riaz et al., 2013; Mehmood et al., 2013; Abujiya et al., 2015a, 2015b). The information about auxiliary variables may be extended to more variables. For example, if we consider a bearing manufacturing process and our study variable is the inner diameter, then the weight of the bearing and its outer diameter may be considered auxiliary characteristics. A variety of literature can be seen in this direction (Ahmad et al., 2013, 2014a, 2014b; Riaz et al., 2013; Riaz, 2015). Recently, Sanusi et al. (2017, 2018) proposed CUSUM charts using auxiliary information to monitor the location parameter.

The literature addresses monitoring of the location parameter with mean charts and their modifications, but rarely addresses the use of median control charts for location monitoring. There are many techniques for estimating population parameters in survey sampling literature. Detailed literature is available in which authors developed different estimators (or class of estimators) based on auxiliary information to estimate the population mean (Kadilar and Cingi, 2003, 2005a, 2005b; Singh HP et al., 2004; 2008; Singh R et al., 2007, 2009; Tailor and Sharma, 2009; Singh R and Kumar, 2011; Singh HP and Solanki, 2012; Solanki et al., 2012; Tailor et al., 2012). Adebola et al. (2015) suggested a class of regression-type estimators based on auxiliary information to estimate the population mean. However, in the real world, some of the variables show highly skewed distribution, where the median is the most suitable measure for estimating the location parameter as compared to the mean. In the quality control literature, location parameters are usually monitored with mean charts (e.g., the $\overline{X}$ chart). Although the sample median is used in statistical process control applications, less attention has been paid to using this statistic as a tool for monitoring process parameters in quality control. In process monitoring, some researchers have suggested median-based EWMA control charts, for example, median Rankit control charts (Kanji and Arif, 2000, 2001; Chen and Chiou, 2008) and CUSUM median charts (Yang et al., 2010). Sheu and Yang (2006a, 2006b) and Sheu et al. (2009) proposed generally weighted moving average (GWMA) median control charts; Castagliola (2001) suggested mixed EWMA median and range control charts, and Castagliola et al. (2015) proposed EWMA median control charts based on estimated parameters. Yang et al. (2010) compared the usual median chart with Shewhart, EWMA, and CUSUM structures. However, to the best of our knowledge, the quality control literature has not addressed CUSUM techniques with the design of auxiliary information based median control charts.

Khoo (2005) presented a sample median Shewhart control structure based on a normally distributed process. Ahmad et al. (2014a) added some more median estimators under the normal distribution, which are based on two auxiliary variables, but focused only on processes that follow a normal distribution without any comparison with corresponding mean control charts. Because the CUSUM control charting structure is more efficient than the Shewhart control chart, we took inspiration from this work and proposed the study on CUSUM control charts for these estimators for symmetric and non-symmetric processes and compared the performance with that of mean control charts. For these charting structures, we consider the properties of understudy estimators using simple random sampling. We have also studied the run length properties of understudy CUSUM median and mean charts to compare their performances and carried out a contamination study for comparison.

2 Median estimators and CUSUM control charting structure

Let Y denote the quality characteristic of interest (for example, inner diameter, the amount of total power generated, and tensile strength) and auxiliary characteristics be denoted by X and Z (for example, pressure air temperature, the amount of fuel, and outer diameter). The population quantities are defined as follows: μ_y, μ_x, and μ_z are the means of Y, X, and Z, respectively; σ_y^2, σ_x^2, and σ_z^2 are the variances of Y, X, and Z, respectively; M_y, M_x, and M_z are the medians of Y, X, and Z, respectively; σ_{xy}, σ_{yz}, and σ_{zx} are the co-variances between X and Y, Y and Z, and Z and X, respectively; ρ_{xy}, ρ_{yz}, and ρ_{zx} are the correlations between X and Y, Y and Z, and Z and X, respectively; C_y, C_x, and C_z are the coefficient of variations of Y, X, and Z, respectively; $\beta_2(y)$, $\beta_2(x)$, and $\beta_2(z)$ are the coefficient of kurtosis of Y, X, and Z, respectively. Similarly, let (y_i, x_i, z_i) $(i=1, 2, ..., n)$ be a trivariate sample of size n. The sample quantities are defined as follows: $\bar{y}$, $\bar{x}$, and $\bar{z}$ are the means; s_y^2, s_x^2, and s_z^2 are the variances; $\hat{M}_y$, $\hat{M}_x$, and $\hat{M}_z$ are the medians of y, x, and z, respectively. Considering these preliminaries, we used a set of mean estimators (Gupta and Shabbir, 2007) to estimate μ_y, and median estimators (Ahmad et al., 2014a) to estimate M_y based on auxiliary characteristics (Table 1).

In Table 1, $f_y(M_y)$, $f_x(M_x)$, and $f_z(M_z)$ are the ordinates of the distributions of Y, X, and Z at M_y, M_x, and M_z, respectively. Also, $\phi_{yx}=\phi(\hat{M}_y, \hat{M}_x)=4P_{11}(y, x)-1$, where $P_{11}(y, x)$ is the proportion of units in the population with $Y\le M_y$ and $X\le M_x$. Similarly, $\phi_{yz}=\phi(\hat{M}_y, \hat{M}_z)=4P_{11}(y, z)-1$ and $\phi_{zx}=\phi(\hat{M}_z, \hat{M}_x)=4P_{11}(z, x)-1$ where $P_{11}(y, z)$ and $P_{11}(z, x)$ are the proportions of units in the population with $(Y\le M_y, Z\le M_z)$ and $(Z\le M_z, X\le M_x)$, respectively. Also, we have

$$\varDelta_{1x}=\frac{M_x}{M_x+C_x},\ \varDelta_{2x}=\frac{M_x}{M_x+\beta_2(x)},$$
$$\varDelta_{3x}=\frac{M_x\beta_2(x)}{M_x\beta_2(x)+C_x},\ \varDelta_{4x}=\frac{M_xC_x}{M_xC_x+\beta_2(x)},$$
$$\varDelta_{1z}=\frac{M_z}{M_z+C_z},\ \varDelta_{2z}=\frac{M_z}{M_z+\beta_2(z)},$$
$$\varDelta_{3z}=\frac{M_z\beta_2(z)}{M_z\beta_2(z)+C_z},\ \varDelta_{4z}=\frac{M_zC_z}{M_zC_z+\beta_2(z)}.$$

Similarly, we have

$$\xi_{1x}=\frac{\mu_x}{\mu_x+C_x},\ \xi_{2x}=\frac{\mu_x}{\mu_x+\beta_2(x)},$$
$$\xi_{3x}=\frac{\mu_x\beta_2(x)}{\mu_x\beta_2(x)+C_x},\ \xi_{4x}=\frac{\mu_xC_x}{\mu_xC_x+\beta_2(x)},$$
$$\xi_{1z}=\frac{\mu_z}{\mu_z+C_z},\ \xi_{2z}=\frac{\mu_z}{\mu_z+\beta_2(z)},$$
$$\xi_{3z}=\frac{\mu_z\beta_2(z)}{\mu_z\beta_2(z)+C_z},\ \xi_{4z}=\frac{\mu_zC_z}{\mu_zC_z+\beta_2(z)}.$$

The means and standard deviations (up to first-order approximation) of median estimators are given in Table S1, which is available in the supplementary materials.

Page (1954) was the first to introduce the concept of the CUSUM control chart to address small parameter shifts. The CUSUM procedure works by accumulating the up and down deviations from the target value, denoted by upper CUSUM (S_i^+) and lower CUSUM (S_i^-), respectively. For monitoring of the location parameter, a traditional two-sided CUSUM charting structure can be described as follows: suppose that for the normally distributed process Y_i denotes the i^{th} observation of the study variable, with known parameters μ_0 and σ_0. The two-sided

Table 1 Median and mean estimators used in this study

Usual estimators	
$M_1=\hat{M}_y$	$T_1=\bar{y}$
Estimators based on one auxiliary variable	
$M_2=\hat{M}_y(M_x/\hat{M}_x)$	$T_2=\bar{y}(\mu_x/\bar{x})$
$M_3=\hat{M}_y\{(M_x+C_x)/(\hat{M}_x+C_x)\}$	$T_3=\bar{y}_y\{(\mu_x+C_x)/(\bar{x}+C_x)\}$
$M_4=\hat{M}_y\left\{\left(M_x+\beta_2(x)\right)/\left(\hat{M}_x+\beta_2(x)\right)\right\}$	$T_4=\bar{y}\left\{\left(\mu_x+\beta_2(x)\right)/\left(\bar{x}+\beta_2(x)\right)\right\}$
$M_5=\hat{M}_y\left\{\left(M_x\beta_2(x)+C_x\right)/\left(\hat{M}_x\beta_2(x)+C_x\right)\right\}$	$T_5=\bar{y}\left\{\left(\mu_x\beta_2(x)+C_x\right)/\left(\bar{x}\beta_2(x)+C_x\right)\right\}$
$M_6=\hat{M}_y\left\{\left(M_xC_x+\beta_2(x)\right)/\left(\hat{M}_xC_x+\beta_2(x)\right)\right\}$	$T_6=\bar{y}\left\{\left(\mu_xC_x+\beta_2(x)\right)/\left(\bar{x}C_x+\beta_2(x)\right)\right\}$
Estimators based on two auxiliary variables	
$M_7=\hat{M}_y\left(\frac{M_x}{\hat{M}_x}\right)^{\theta_1}\left(\frac{M_z}{\hat{M}_z}\right)^{\theta_2}$ where $\theta_1=\frac{(\phi_{xy}-\phi_{yz}\phi_{xz})M_xf_x(M_x)}{(1-\phi_{xz}^2)M_yf_y(M_y)}$ and $\theta_2=\frac{(\phi_{yz}-\phi_{xy}\phi_{xz})M_zf_z(M_z)}{(1-\phi_{xz}^2)M_yf_y(M_y)}$	$T_7=\bar{y}\left(\frac{\mu_x}{\bar{x}}\right)^{R_1}\left(\frac{\mu_z}{\bar{z}}\right)^{R_2}$ where $R_1=\frac{C_y(\rho_{xy}-\rho_{yz}\rho_{xz})}{C_x(1-\rho_{xz}^2)}$ and $R_2=\frac{C_y(\rho_{xy}-\rho_{yz}\rho_{xz})}{C_z(1-\rho_{xz}^2)}$
$M_8=\hat{M}_y\left(\frac{M_x+C_x}{\hat{M}_x+C_x}\right)^{\gamma_1}\left(\frac{M_z+C_z}{\hat{M}_z+C_z}\right)^{\gamma_2}$ where $\gamma_1=\frac{(\phi_{xy}-\phi_{yz}\phi_{xz})M_xf_x(M_x)}{\varDelta_{1x}(1-\phi_{xz}^2)M_yf_y(M_y)}$ and $\gamma_2=\frac{(\phi_{yz}-\phi_{xy}\phi_{xz})M_zf_z(M_z)}{\varDelta_{1z}(1-\phi_{xz}^2)M_yf_y(M_y)}$	$T_8=\bar{y}_{m(SD)}=\bar{y}\left(\frac{\mu_x+C_x}{\bar{x}+C_x}\right)^{J_1}\left(\frac{\mu_z+C_z}{\bar{z}+C_z}\right)^{J_2}$ where $J_1=\frac{C_y(\rho_{xy}-\rho_{yz}\rho_{xz})}{\xi_{1x}C_x(1-\rho_{xz}^2)}$ and $J_2=\frac{C_y(\rho_{xy}-\rho_{yz}\rho_{xz})}{\xi_{1z}C_z(1-\rho_{xz}^2)}$
$M_9=\hat{M}_y\left(\frac{M_x+\beta_2(x)}{\hat{M}_x+\beta_2(x)}\right)^{\gamma_1}\left(\frac{M_z+\beta_2(z)}{\hat{M}_z+\beta_2(z)}\right)^{\gamma_2}$ where $\gamma_1=\frac{(\phi_{xy}-\phi_{yz}\phi_{xz})M_xf_x(M_x)}{\varDelta_{2x}(1-\phi_{xz}^2)M_yf_y(M_y)}$ and $\gamma_2=\frac{(\phi_{yz}-\phi_{xy}\phi_{xz})M_zf_z(M_z)}{\varDelta_{3z}(1-\phi_{xz}^2)M_yf_y(M_y)}$	$T_9=\bar{y}_{m(SK)}=\bar{y}\left(\frac{\mu_x+\beta_2(x)}{\bar{x}+\beta_2(x)}\right)^{J_1}\left(\frac{\mu_z+\beta_2(z)}{\bar{z}+\beta_2(z)}\right)^{J_2}$ where $J_1=\frac{C_y(\rho_{xy}-\rho_{yz}\rho_{xz})}{\xi_{2x}C_x(1-\rho_{xz}^2)}$ and $J_2=\frac{C_y(\rho_{xy}-\rho_{yz}\rho_{xz})}{\xi_{2z}C_z(1-\rho_{xz}^2)}$
$M_{10}=\hat{M}_y\left(\frac{M_x\beta_2(x)+C_x}{\hat{M}_x\beta_2(x)+C_x}\right)^{\gamma_1}\left(\frac{M_z\beta_2(z)+C_z}{\hat{M}_z\beta_2(z)+C_z}\right)^{\gamma_2}$ where $\gamma_1=\frac{(\phi_{xy}-\phi_{yz}\phi_{xz})M_xf_x(M_x)}{\varDelta_{3x}(1-\phi_{xz}^2)M_yf_y(M_y)}$ and $\gamma_2=\frac{(\phi_{yz}-\phi_{xy}\phi_{xz})M_zf_z(M_z)}{\varDelta_{2z}(1-\phi_{xz}^2)M_yf_y(M_y)}$	$T_{10}=\bar{y}_{m(US1)}=\bar{y}\left(\frac{\mu_x\beta_2(x)+C_x}{\bar{x}\beta_2(x)+C_x}\right)^{J_1}\left(\frac{\mu_z\beta_2(z)+C_x}{\bar{z}\beta_2(z)+C_x}\right)^{J_2}$ where $J_1=\frac{C_y(\rho_{xy}-\rho_{yz}\rho_{xz})}{\xi_{3x}C_x(1-\rho_{xz}^2)}$ and $J_2=\frac{C_y(\rho_{xy}-\rho_{yz}\rho_{xz})}{\xi_{3z}C_z(1-\rho_{xz}^2)}$
$M_{11}=\hat{M}_y\left(\frac{M_xC_x+\beta_2(x)}{\hat{M}_xC_x+\beta_2(x)}\right)^{\gamma_1}\left(\frac{M_zC_z+\beta_2(z)}{\hat{M}_zC_z+\beta_2(z)}\right)^{\gamma_2}$ where $\gamma_1=\frac{(\phi_{xy}-\phi_{yz}\phi_{xz})M_xf_x(M_x)}{\varDelta_{4x}(1-\phi_{xz}^2)M_yf_y(M_y)}$ and $\gamma_2=\frac{(\phi_{yz}-\phi_{xy}\phi_{xz})M_zf_z(M_z)}{\varDelta_{4z}(1-\phi_{xz}^2)M_yf_y(M_y)}$	$T_{10}=\bar{y}_{m(US2)}=\bar{y}\left(\frac{\mu_xC_x+\beta_2(x)}{\bar{x}C_x+\beta_2(x)}\right)^{J_1}\left(\frac{\mu_zC_z+\beta_2(z)}{\bar{z}C_z+\beta_2(z)}\right)^{J_2}$ where $J_1=\frac{C_y(\rho_{xy}-\rho_{yz}\rho_{xz})}{\xi_{4x}C_x(1-\rho_{xz}^2)}$ and $J_2=\frac{C_y(\rho_{xy}-\rho_{yz}\rho_{xz})}{\xi_{4z}C_z(1-\rho_{xz}^2)}$

CUSUM statistics for a particular location estimator (M) from Table 1 are defined as follows:

$$\begin{cases} S_i^+=\max\left(0, M_i-(\mu_0+K)+S_{i-1}^+\right), \\ S_i^-=\max\left(0, (\mu_0-K)-M_i+S_{i-1}^+\right), \end{cases} \tag{1}$$

where the initial values of S_i^+ and S_i^- are equal to zero in a standard CUSUM chart. The statistics S_i^+ and S_i^- are plotted against the control limit H_M. A process is considered out of control if either S_i^+ or S_i^- exceeds the control limit. The parameters of the

CUSUM chart (H_M and K_M) are defined as

$$K_M = k \times \sigma_M, H_M = h \times \sigma_M, \tag{2}$$

where K_M is the slack or reference value of the CUSUM structure, usually chosen as half of the magnitude of shift (τ) at the mean level of the quality characteristic, i.e., $K_M=\tau/2=|\mu_1-\mu_0|/2$. Here, μ_1 is the mean value of the out-of-control process. The values S_i^+ and S_i^- are used to detect positive and negative shifts, respectively, in the process mean. The term σ_M is the standard deviation of a particular estimator M, whereas k and h are constants, and their values are chosen with the end goal that the in-control ARL of the CUSUM structure comes to the pre-specified ARL_0. Modifications in the CUSUM structure have been suggested by several researchers to enhance its performance (Montgomery, 2007). In the CUSUM structure, the decision is based on the cumulative sum of a number of observations by considering the process history in an implicit way. The ARL values for the CUSUM structures are calculated at different shift values, using k=0.5, h, and correlation combinations.

3 Performance measures and simulation procedure

To evaluate the performance of CUSUM mean and median charts, we used the average run length (ARL) at each single shift point, the relative average run length (RARL), extra quadratic loss (EQL), and performance comparison index (PCI) for overall performance over the whole range of shifts. For performance evaluation, there are two ARL values: (1) ARL_0 is defined as the average number of subgroups to obtain an out-of-control signal from the in-control process; (2) ARL_1 is defined as the average number of subgroups to obtain an out-of-control signal from the shifted state of the process. In the control chart setting, the large values of ARL_0 and small values of ARL_1 are preferred. A chart is said to be more efficient than others if it has a smaller ARL_1 at fixed choices of ARL_0. The decision intervals h are chosen to fix the in-control ARL (ARL_0) at the desired level for all charts (in our study we fix ARL_0 at 200).

Similarly, measures EQL, RARL, and PCI are considered to examine the detection ability of a control chart over a range of τ values, and any chart is considered a better choice if it has smaller values of these measures. Symbolically, these measures can be defined as

$$\text{EQL} = \frac{1}{\tau_{\max} - \tau_{\min}} \int_{\tau_{\min}}^{\tau_{\max}} \tau^2 \text{ARL}(\tau) \mathrm{d}\tau, \tag{3}$$

$$\text{RARL} = \frac{1}{\tau_{\max} - \tau_{\min}} \int_{\tau_{\min}}^{\tau_{\max}} \frac{\text{ARL}(\tau)}{\text{ARL}_{\text{basechart}}(\tau)} \mathrm{d}\tau, \tag{4}$$

$$\text{PCI} = \frac{\text{EQL}}{\text{EQL}_{\text{basechart}}}, \tag{5}$$

where $\tau_{\max}$ and $\tau_{\min}$ are the maximum and minimum values of shift (τ), respectively. EQL measures the overall performance of a control chart over the whole range of shifts (Wu et al., 2009) and explains the overall detection ability of a control structure. Some reforms of EQL may be seen in Ou et al. (2012b), Ryu et al. (2010), Sanusi et al. (2018), Wu et al. (2009), and Zhang and Wu (2006). Similarly, RARL describes how close the efficiency of a given chart is to that of the base chart (Wu et al., 2009; Ryu et al., 2010; Sanusi et al., 2018). A chart having a minimum EQL value is used as a base chart. PCI is a ratio between the EQL of a particular control chart and the EQL of the base control chart. The PCI value will be equal to 1 for the control chart having minimum EQL results and for other control charts, PCI>1.

In our study, the median and mean charts in a CUSUM setup are compared under normal and non-normal distributed processes in terms of the above performance measures. We consider a trivariate setup that can be described as follows: (Y, X, Z) follows a trivariate distribution with $\boldsymbol{\mu}=(\mu_y\ \mu_x\ \mu_z)$ and $\boldsymbol{\Sigma}=\begin{pmatrix} \sigma_y^2 & \sigma_{yx} & \sigma_{yz} \\ \sigma_{xy} & \sigma_x^2 & \sigma_{xz} \\ \sigma_{zy} & \sigma_{zx} & \sigma_z^2 \end{pmatrix}$, which can be replaced by a correlation matrix in case of standardized variables $\boldsymbol{\rho}=\begin{pmatrix} 1 & \rho_{yx} & \rho_{yz} \\ \rho_{xy} & 1 & \rho_{xz} \\ \rho_{zy} & \rho_{zx} & 1 \end{pmatrix}$. We have considered both normal and non-normal distributions, namely, normal, Student's t, and log-normal distributions. Generally, the p-dimensional multivariate PDFs are given below:

Multivariate normal distribution:

$$f(r_1,r_2,\ldots,r_p)=\frac{\exp\left[-\frac{1}{2}(\boldsymbol{r}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{r}-\boldsymbol{\mu})\right]}{\sqrt{(2\pi)^p\left\|\boldsymbol{\Sigma}\right\|}},\quad(6)$$

where $\boldsymbol{r}=(r_1, r_2, ..., r_p)^{\mathrm{T}}$, $-\infty<r_1, r_2, ..., r_p<\infty$, $\boldsymbol{\mu}=[E[r]=E[r_1], E[r_2], ..., E[r_p]]^{\mathrm{T}}\in\mathbb{R}^p$, $\boldsymbol{\Sigma}=E[(\boldsymbol{r}-\boldsymbol{\mu})(\boldsymbol{r}-\boldsymbol{\mu})^p]\in\mathbb{R}^{p\times p}$ (non-negative and definite matrix), $-1\le\rho_{xi}, \rho_{xj}\le1$.

Symbolically we may write this as $\boldsymbol{r}\sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Multivariate t distribution:

$$f(r)=c\left[1+\frac{1}{\upsilon}(\boldsymbol{r}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{r}-\boldsymbol{\mu})\right]^{-\frac{\upsilon+p}{2}},\quad(7)$$

where $c=(\upsilon\pi)^{-p/2}\frac{\Gamma[(\upsilon+p)/2]}{\Gamma(\upsilon/2)}\left\|\boldsymbol{\Sigma}\right\|^{-1/2}$, $\boldsymbol{r}=(r_1, r_2, \ldots, r_p)^{\mathrm{T}}$, $\boldsymbol{\mu}\in\mathbb{R}^p$, $\boldsymbol{\Sigma}\in\mathbb{R}^{p\times p}$ (non-negative and definite matrix), $-\infty<r_1, r_2, ..., r_p<\infty$, $-1\le\rho_{xi}, \rho_{xj}\le1$. Symbolically we may write this as $\boldsymbol{r}\sim t_v(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Multivariate log-normal distribution:

$$f(\boldsymbol{s})=\frac{\exp\left[-\frac{1}{2}(\ln\boldsymbol{s}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\ln\boldsymbol{s}-\boldsymbol{\mu})\right]}{\left[(2\pi)^p\left\|\boldsymbol{\Sigma}\right\|\right]^{1/2}(s_1,s_2,\ldots,s_p)},\quad(8)$$

where $\boldsymbol{s}=\exp(\boldsymbol{r})$, and $\boldsymbol{r}\sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. $\boldsymbol{\mu}\in\mathbb{R}^p$, $\boldsymbol{\Sigma}\in\mathbb{R}^{p\times p}$ (non-negative and definite matrix), $0<s_1, s_2, ..., s_p<\infty$, $-1\le\rho_{xi}, \rho_{xj}\le1$.

Different approaches can be used to calculate the run length (RL) values, for example, the integral equations approach (Page, 1954), Markov chain approximations (Brook and Evans, 1972), and Monte Carlo simulations (Hawkins, 1981; Mehmood et al., 2013; Abujiya et al., 2015b; Sanusi et al., 2017). Advancements in computer technology allow us to use simulation techniques to estimate run length properties (Woodall and Montgomery, 1999).

In our study, the values of all estimators are obtained by applying a simple random sampling technique using the Monte Carlo simulation approach (Sepulveda and Nachlas, 1997; Mundform et al., 2011) using R software (Team, 2015). We obtain an RL value by simulating 100 000 values of a particular estimator through Monte Carlo simulation. We obtain 5000 RL values repeatedly to obtain an ARL for a particular estimator. For the appropriate number of Monte Carlo simulations in quality control, the reader is referred to Schaffer and Kim (2007) and Hawkins and Olwell (2012). We consider three probability distributions for evaluation and comparison of our proposed median control charts with their corresponding mean control charts.

4 Comparative analysis

To analyze the run length characteristics of our charts, simple random samples were generated from a trivariate normal distribution of size n using different high, moderate, and weak correlation combinations ρ_{xy}, ρ_{yz}, and ρ_{zx}. We determined the CUSUM control structures defined in Eq. (1) for both median (M_1–M_{11}) and mean (T_1–T_{11}) control charts using ARL_0=200, n=5, 10, 15, and k=0.5. We calculated the value of h for each chart that fixes the control limits (2) to obtain in-control ARL (ARL_0=200) and out-of-control ARL (ARL_1) values at discrete points of shift as τ=0.05, 0.10, 0.15, …, 3.0.

4.1 Comparison of control charts in an uncontaminated scenario

The uncontaminated scenario is the basic assumption for the comparison of control charts. Here, we considered a perfectly normal scenario with μ_0 and σ_0 to evaluate the performance of control charts. We used μ_0=5 and σ_0=1 for the normal distribution, whereas both symmetric and non-symmetric processes were considered for comparison. The results of our study in the uncontaminated scenario are presented in Tables 2–4 and Figs. 1 and 2. For some other combinations, see Tables S2–S5. The ARL results for a randomly selected correlation combination of both median and mean control charts for the process following a normal distribution are presented in Table 2. Similarly, the ARL results for the process following Student's t and log-normal distributions are presented in Tables 3 and 4, respectively.

The performance measures EQL, RARL, and PCI of median charts for normal, t, and log-normal distributions are presented in Tables S2, S3, and S4, respectively. Table S5 represents the analysis median and mean charts for all symmetric and non-symmetric processes for some selective correlation combinations. For some selective charts, the ARL curves for

Table 2 Average run length values of CUSUM median and mean charts for the process following the normal distribution for ρ_{xy}=0.30, ρ_{yz}=0.30, ρ_{zx}=0.30, and n=5

	Average run length														
	τ=0	0.05	0.10	0.15	0.20	0.25	0.40	0.50	0.60	0.75	1.00	1.25	1.50	2.00	3.00
M_1	200.19	120.49	74.68	49.24	33.87	24.90	11.59	8.10	6.41	4.68	3.24	2.55	2.12	1.63	1.06
T_1	200.17	109.29	62.32	38.91	26.08	18.55	8.86	6.33	4.94	3.71	2.66	2.12	1.80	1.33	1.00
M_2	200.19	136.62	97.68	68.55	50.29	37.37	18.48	13.20	10.08	7.34	5.02	3.90	3.18	2.36	1.67
T_2	200.05	116.37	75.43	51.31	35.18	25.52	12.31	8.88	6.91	5.14	3.64	2.85	2.38	1.84	1.23
M_3	200.05	135.11	95.32	67.00	48.94	36.09	18.24	12.92	9.78	7.05	4.89	3.74	3.07	2.33	1.62
T_3	200.05	115.60	73.71	49.97	34.32	24.51	11.99	8.53	6.58	4.98	3.53	2.76	2.33	1.78	1.19
M_4	200.28	124.30	81.39	55.01	38.16	28.02	13.14	9.45	7.28	5.24	3.74	2.90	2.39	1.84	1.21
T_4	200.03	106.01	62.49	39.62	26.81	18.84	9.13	6.63	5.09	3.88	2.81	2.23	1.89	1.41	1.01
M_5	200.05	135.99	96.85	67.75	49.89	36.96	18.29	13.07	10.07	7.24	4.96	3.85	3.14	2.34	1.65
T_5	200.15	116.11	74.56	50.79	34.92	25.20	12.15	8.81	6.75	5.10	3.60	2.84	2.35	1.83	1.22
M_6	200.10	114.89	72.48	48.02	32.69	23.31	11.22	7.82	6.11	4.49	3.17	2.52	2.10	1.60	1.05
T_6	200.07	103.90	59.00	35.41	24.26	17.22	8.13	5.94	4.60	3.51	2.54	2.04	1.73	1.24	1.00
M_7	200.01	123.21	75.79	51.27	35.22	25.49	12.05	8.65	6.46	4.82	3.34	2.61	2.18	1.68	1.07
T_7	199.89	109.72	65.32	41.34	28.01	19.77	9.37	6.74	5.12	3.91	2.80	2.23	1.88	1.40	1.01
M_8	200.14	116.77	70.53	47.96	32.12	22.79	10.95	7.78	6.03	4.48	3.18	2.48	2.08	1.60	1.04
T_8	199.93	103.62	58.26	35.20	22.98	16.41	7.81	5.75	4.54	3.48	2.52	2.02	1.71	1.24	1.00
M_9	199.93	116.95	70.80	48.03	32.19	22.77	10.99	7.72	5.98	4.46	3.16	2.47	2.07	1.59	1.04
T_9	200.11	104.42	58.66	35.46	23.12	16.42	7.85	5.73	4.52	3.45	2.49	2.01	1.71	1.23	1.00
M_{10}	200.07	116.81	70.48	47.93	32.12	22.76	10.94	7.81	6.02	4.48	3.17	2.48	2.08	1.60	1.04
T_{10}	200.07	103.62	58.18	35.19	22.98	16.41	7.81	5.74	4.54	3.49	2.52	2.03	1.71	1.24	1.00
M_{11}	199.86	117.13	71.11	48.16	32.44	22.85	11.00	7.68	5.99	4.44	3.16	2.45	2.07	1.57	1.03
T_{11}	199.93	104.16	58.92	35.62	23.22	16.44	7.83	5.71	4.51	3.43	2.47	1.99	1.69	1.21	1.00

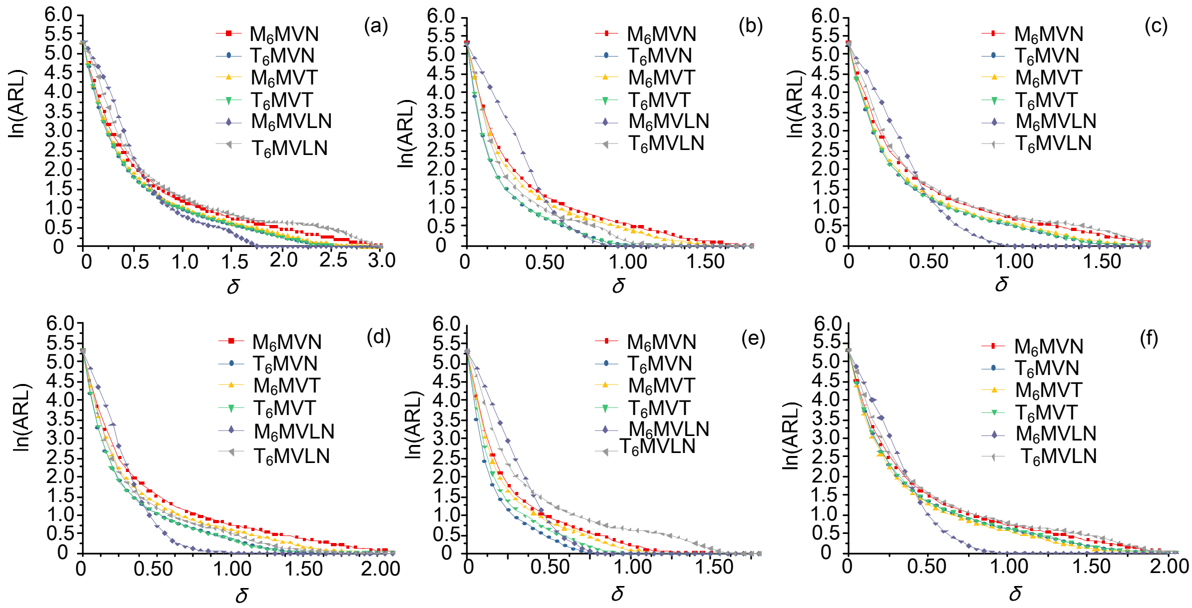

Fig. 1 The average run length (ARL) curves for CUSUM median and mean control charts at ARL$_0$=200: (a) n=5, ρ_{xy}=0.20, ρ_{yz}=0.10, ρ_{zx}=0.70; (b) n=5, ρ_{xy}=0.90, ρ_{yz}=0.90, ρ_{zx}=0.90; (c) n=10, ρ_{xy}=0.50, ρ_{yz}=0.40, ρ_{zx}=0.70; (d) n=10, ρ_{xy}=0.75, ρ_{yz}=0.10, ρ_{zx}=0.10; (e) n=15, ρ_{xy}=0.90, ρ_{yz}=0.75, ρ_{zx}=0.50; (f) n=15, ρ_{xy}=0.50, ρ_{yz}=0.50, ρ_{zx}=0.10

Table 3 Average run length values of CUSUM median and mean charts for the process following Student's *t* distribution for ρ_{xy}=0.20, ρ_{yz}=0.70, ρ_{zx}=0.10, and n=10

	Average run length														
	τ=0	0.05	0.10	0.15	0.20	0.25	0.40	0.50	0.60	0.75	1.00	1.25	1.50	2.00	3.00
M_1	199.92	87.51	41.57	23.32	14.88	10.50	5.18	3.90	3.17	2.49	1.89	1.53	1.21	1.01	1.00
T_1	200.32	91.26	43.77	25.02	15.79	10.80	5.33	4.05	3.25	2.53	1.95	1.56	1.24	1.01	1.00
M_2	199.92	105.68	60.31	36.94	24.07	16.98	8.28	6.00	4.76	3.67	2.68	2.17	1.84	1.39	1.01
T_2	199.95	104.31	58.79	37.11	23.80	16.60	8.07	5.93	4.66	3.59	2.64	2.12	1.81	1.34	1.01
M_3	200.26	103.62	58.89	35.62	23.38	16.10	8.04	5.91	4.65	3.58	2.61	2.12	1.81	1.34	1.01
T_3	199.59	102.99	57.52	35.76	23.32	15.89	7.82	5.81	4.49	3.47	2.55	2.07	1.77	1.31	1.01
M_4	200.29	89.33	43.50	25.33	15.85	11.37	5.62	4.16	3.38	2.63	2.00	1.63	1.30	1.02	1.00
T_4	200.53	87.26	43.99	24.98	15.67	11.25	5.58	4.19	3.32	2.59	1.97	1.60	1.29	1.02	1.00
M_5	200.93	105.43	60.03	36.76	23.94	16.92	8.25	5.97	4.76	3.65	2.67	2.16	1.84	1.38	1.01
T_5	199.38	103.88	58.56	36.76	23.87	16.44	7.96	5.93	4.66	3.56	2.64	2.12	1.82	1.34	1.01
M_6	199.74	84.82	40.48	23.41	14.50	10.10	5.14	3.89	3.11	2.43	1.86	1.50	1.19	1.01	1.00
T_6	199.93	85.94	41.75	23.84	14.55	10.25	5.22	3.91	3.15	2.47	1.88	1.51	1.20	1.01	1.00
M_7	200.44	107.45	59.64	35.85	23.53	16.58	7.96	5.83	4.57	3.47	2.54	2.04	1.73	1.26	1.00
T_7	200.44	123.21	74.22	49.49	32.88	22.89	10.81	7.63	5.82	4.34	3.09	2.44	2.06	1.58	1.03
M_8	199.89	74.72	32.54	17.25	11.09	7.83	4.14	3.17	2.59	2.11	1.62	1.22	1.04	1.00	1.00
T_8	200.51	64.69	26.02	13.92	8.79	6.46	3.53	2.72	2.26	1.87	1.37	1.07	1.01	1.00	1.00
M_9	200.21	74.92	32.96	17.34	11.09	7.85	4.13	3.16	2.59	2.09	1.61	1.21	1.04	1.00	1.00
T_9	200.35	65.23	26.33	13.91	8.76	6.43	3.48	2.69	2.24	1.84	1.34	1.06	1.01	1.00	1.00
M_{10}	200.17	74.73	32.53	17.29	11.09	7.82	4.14	3.17	2.59	2.11	1.63	1.23	1.04	1.00	1.00
T_{10}	200.35	64.59	25.99	13.90	8.79	6.45	3.52	2.73	2.27	1.87	1.38	1.08	1.01	1.00	1.00
M_{11}	200.17	75.38	33.17	17.45	11.13	7.84	4.15	3.14	2.56	2.08	1.59	1.21	1.03	1.00	1.00
T_{11}	200.21	65.43	26.62	14.00	8.82	6.39	3.44	2.68	2.22	1.82	1.31	1.05	1.01	1.00	1.00

distributional comparison are presented in Fig. 1. The bar graphs of EQL measures for sample size comparison using median control charts are presented in Fig. 2. From these tables and figures, we provide some findings:

1. We evaluated the performance of median and mean charts under normal and non-normal processes, and as expected, when there is no contamination, the performance of mean charts is better in symmetric processes (followed by normal and *t* distributions). For example, in the case of a normal distribution, at n=5 and τ=0.05, the ARL_1 values for three randomly selected median control charts (M_1, M_5, and M_7) are 120.49, 135.99, and 123.21, respectively, whereas the ARL_1 values for corresponding mean control charts (T_1, T_5, and T_7) are 109.29, 116.11, and 109.72, respectively. Similar findings can be observed in Table 2 for other control charts and Table 3 for the *t* distributed process.

2. For the case of log-normal distribution, median control charts have smaller ARL_1 values than mean control charts at τ>0.50. For example, the ARL_1 values for M_1, M_4, and M_8 are 3.60, 3.53, and 2.51, respectively, whereas the ARL_1 values for T_1, T_4, and T_8 are 5.12, 4.91, and 3.18, respectively. One may observe similar ARL_1 behavior for all median and mean control charts (Table 4 and Fig. 1), which indicates that median control charts perform more efficiently at large process shifts, following the log-normal distribution.

3. Within the comparison of median control charts, it can also be seen that the overall detection ability of a process following a log-normal distribution model is superior to those of other symmetric models (Tables S2–S4). For example, the EQL values for three randomly selected median control charts (M_1, M_2, and M_{10}) are (4.894, 7.854, 4.871), (4.268, 6.737, 4.246), and (3.913, 4.161, 3.901) for normal, Student's *t*, and log-normal distributions, respectively. One may observe similar EQL behavior for all median and mean control charts (Tables S2–S4).

4. Another observation is that the detection ability of charts based on two auxiliary variables is better than that of charts based on a single auxiliary variable or usual median or mean estimators. If we used the average EQL values for estimators 2–6 and 7–11,

Table 4 Average run length values of CUSUM median and mean charts for the process following the log-normal distribution for ρ_{xy}=0.75, ρ_{yz}=0.10, ρ_{zx}=0.10, and n=5

	Average run length														
	τ=0	0.05	0.10	0.15	0.20	0.25	0.40	0.50	0.60	0.75	1.00	1.25	1.50	2.00	3.00
M_1	200.06	162.24	131.80	105.15	77.64	58.58	19.94	9.78	5.86	3.60	2.27	1.77	1.43	1.00	1.00
T_1	200.04	154.76	114.45	80.87	55.13	36.42	14.39	9.56	7.16	5.12	3.58	2.80	2.34	1.89	1.02
M_2	199.97	161.51	124.14	95.19	70.88	50.87	14.34	6.55	3.88	2.40	1.46	1.15	1.05	1.01	1.00
T_2	199.90	121.54	65.47	35.42	21.12	14.14	6.56	4.92	3.92	3.04	2.27	1.96	1.61	1.12	1.01
M_3	200.06	162.54	125.08	96.59	71.89	51.85	14.68	6.70	3.85	2.43	1.48	1.14	1.05	1.01	1.00
T_3	199.72	122.81	67.28	36.08	21.82	14.15	6.63	4.89	3.90	3.06	2.30	1.94	1.62	1.13	1.01
M_4	200.01	168.21	132.60	106.05	79.96	60.11	19.97	9.62	5.75	3.53	2.22	1.73	1.37	1.00	1.00
T_4	199.86	153.42	108.88	78.78	50.95	33.94	13.10	9.02	6.77	4.91	3.42	2.71	2.23	1.87	1.00
M_5	200.23	161.71	124.18	95.42	70.95	51.10	14.23	6.53	3.90	2.39	1.48	1.15	1.06	1.01	1.00
T_5	199.90	121.68	65.48	35.38	21.16	14.12	6.57	4.92	3.92	3.04	2.27	1.95	1.61	1.12	1.01
M_6	199.90	168.13	132.21	106.37	80.00	60.57	20.20	9.81	5.93	3.59	2.25	1.76	1.44	1.00	1.00
T_6	199.94	154.75	110.06	80.74	52.37	35.78	13.74	9.43	6.99	5.08	3.54	2.77	2.31	1.88	1.01
M_7	199.90	155.49	120.48	96.31	70.46	50.78	15.38	7.06	4.09	2.55	1.70	1.20	1.00	1.00	1.00
T_7	200.51	131.66	77.06	43.63	25.64	16.30	7.21	5.27	4.15	3.20	2.36	1.96	1.79	1.09	1.00
M_8	200.18	155.54	119.59	95.73	70.12	50.70	15.39	7.09	4.08	2.51	1.68	1.17	1.00	1.00	1.00
T_8	200.23	130.78	77.16	43.23	25.28	16.25	7.24	5.24	4.13	3.18	2.36	1.96	1.78	1.08	1.00
M_9	200.18	154.64	119.79	95.77	69.87	50.59	15.66	7.08	4.07	2.49	1.65	1.14	1.00	1.00	1.00
T_9	200.33	134.65	81.29	46.00	27.42	16.80	7.28	5.24	4.10	3.15	2.29	1.93	1.73	1.07	1.00
M_{10}	199.90	155.35	119.75	95.65	70.08	50.74	15.45	7.05	4.10	2.52	1.67	1.17	1.00	1.00	1.00
T_{10}	200.17	130.74	76.95	43.13	25.19	16.21	7.25	5.25	4.14	3.19	2.36	1.96	1.79	1.08	1.00
M_{11}	200.32	154.48	119.72	95.53	69.90	50.63	15.61	7.10	4.07	2.50	1.65	1.14	1.00	1.00	1.00
T_{11}	200.38	135.07	81.53	45.93	27.54	16.84	7.33	5.24	4.12	3.14	2.29	1.93	1.74	1.06	1.00

the averages for Table S2 are 6.82 and 4.67, respectively, for the normal process with n=5. Similarly, for Student's t and log-normal distributions, these averages are 5.77 and 4.24, respectively. The same averages for n=10 and n=15 are (4.53, 3.62), (3.87, 3.37), (3.27, 3.24) and (3.98, 3.39), (3.53, 3.22), (3.15, 3.14), respectively, for all three distributions. These findings can also be observed in Tables S2-S4.

We have observed that the performance in all charts increases with increments in sample size (Tables S2–S4 and Fig. 2).

4.2 Comparison of control charts in a contaminated scenario

In the current study, we also compared the performance of median charts and mean charts in contaminated scenarios for normal and non-normal distributions. We used four different scenarios in this comparison, which are defined as follows:

1. Uncontaminated scenario: In this scenario, we present the performance of the median and mean charts under a perfectly normal scenario with 100(1−α)% data from μ_y=μ_x=μ_z=5 and σ_y^2=σ_x^2=σ_z^2=1.

2. Localized contaminated scenario: In this scenario, we introduce a process disturbance in the mean of the process. 100(1−α)% observations are from μ_y=μ_x=μ_z=5 and σ_y^2=σ_x^2=σ_z^2=1 and 100α% observations are from μ_y=6, μ_x=μ_z=5 and σ_y^2=σ_x^2=σ_z^2=1.

3. Variance contaminated scenario: In this case, the disturbance is introduced in the variance of the process. That is, 100(1−α)% observations in a sample are from μ_y=μ_x=μ_z=5 and σ_y^2=σ_x^2=σ_z^2=1 and 100α% observations in the sample are from μ_y=μ_x=μ_z=5 and σ_y^2=2, σ_x^2=σ_z^2=1.

4. Simultaneous contamination scenario: In this scenario, the mean and variance of the process are disturbed with some outliers. That is, 100(1–α)% observations in the sample are from μ_y=μ_x=μ_z=5 and

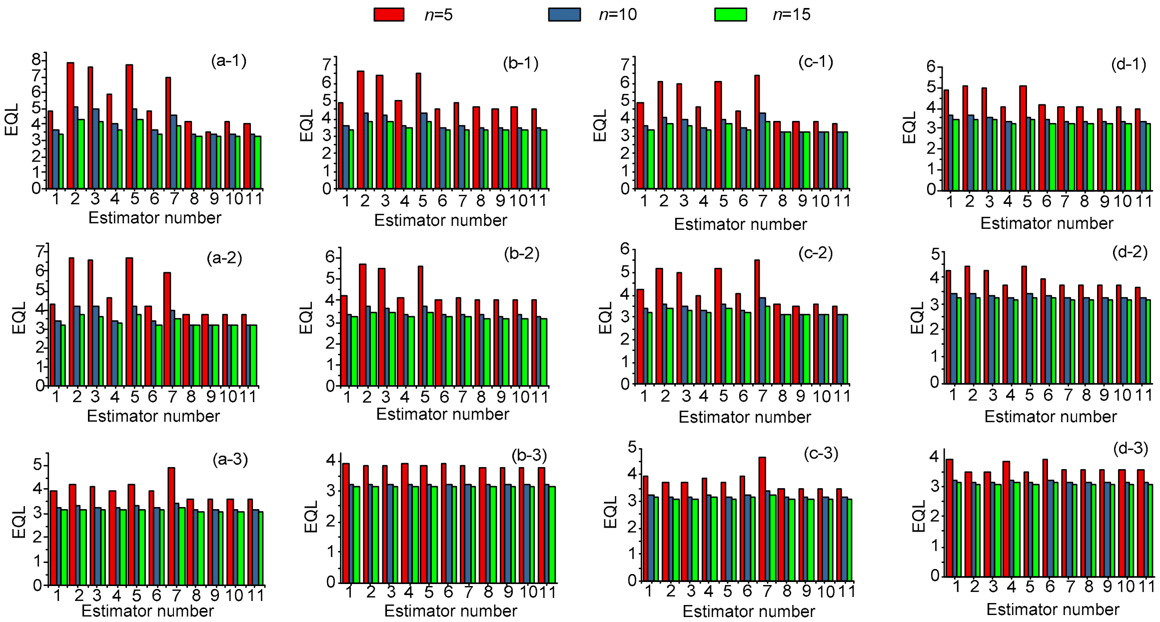

Fig. 2 Bar charts EQL values for CUSUM median control charts for the normal distribution (-1), Student's *t* distribution (-2), and log-normal distribution (-3) at ARL_0=200: (a) ρ_{xy}=0.20, ρ_{yz}=0.70, ρ_{zx}=0.10; (b) ρ_{xy}=0.50, ρ_{yz}=0.40, ρ_{zx}=0.70; (c) ρ_{xy}=0.60, ρ_{yz}=0.75, ρ_{zx}=0.30; (d) ρ_{xy}=0.75, ρ_{yz}=0.10, ρ_{zx}=0.10

$\sigma_y^2=\sigma_x^2=\sigma_z^2=1$ and 100α% observations in the sample are from μ_y=6, $\mu_x=\mu_z$=5 and $\sigma_y^2=2$, $\sigma_x^2=\sigma_z^2=1$.

Note that these scenarios are commonly used in studies to check the robustness of control charts (Ou et al., 2012a; Nazir et al., 2013, 2015). We used α=0.05 to examine the contamination effects on the ARL. For this, we generated data from processes followed by different distributions under all four scenarios for n=5, 10, 15 at selected choices of ρ_{xy}, ρ_{yz}, ρ_{zx}, which are presented in Tables 5–7 and also in Tables S6 and S7.

The following points cover the findings from Tables 10–14:

The ARL values of the contaminated process following a normal distribution for both median and mean control charts are presented in Table 5 for some selected correlation combinations. It can be observed from Table 5 that the deviation of all median charts is less than that of the mean charts from the uncontaminated scenario. To show this finding numerically, we calculate the average of deviations of all median and mean charts from the uncontaminated scenario. For all three scenarios at ρ=0.20, ρ_{yz}=0.10, ρ_{zx}=0.70 and n=5, the averages of all deviations of the median and mean control charts at shift values 0.0, 0.10, 0.15, 0.20, and 0.25 are:

Scenario II Median: 40.1%, 31.0%, 27.4%, 22.1%; 19.1%; mean: 48.7%, 36.3%, 31.8%, 26.5%, 20.5%.

Scenario III Median: 8.6%, 1.3%, 1.5%, 2.1%, 2.9%; mean: 11.3%, 2.7%, 1.6%, 4.5%, 5.3%.

Scenario IV Median: 38.1%, 28.8%, 25.0%, 20.2%, 16.9%; mean: 51.2%, 39.3%, 33.8%, 27.8%, 23.2%.

From these percentages, we can conclude that median control charts are more robust as compared to mean control charts.

The CUSUM ARL contaminated values for the processes following a *t* distribution for some selected correlation combinations are presented in Table 6. The same effect on the performance of median and mean charts can be observed from Table 6; that is, the median chart structures are more robust against mean charts in the presence of outliers.

Similarly, the contaminated CUSUM ARL values for the processes following a skewed distribution (log-normal) are presented in Table 7. Although the ARL values are not greatly affected by outliers as compared to symmetric distributions, here again, median charts are more robust compared to mean charts.

The comparison of the median control charts for all symmetric and skewed distributions using the same selected correlation combinations is presented

Table 5 Contaminated average run length values of CUSUM median and mean charts for the process following the normal distribution for ρ_{xy}=0.20, ρ_{yz}=0.10, ρ_{zx}=0.70, and n=5

	Average run length														
	Scenario II					Scenario III					Scenario IV				
	τ=0	0.10	0.15	0.20	0.25	0	0.10	0.15	0.20	0.25	0	0.10	0.15	0.20	0.25
M_1	110.7	49.9	34.6	25.5	19.4	189.4	75.6	49.8	34.0	25.4	113.3	50.7	35.3	26.1	19.6
T_1	96.5	38.6	26.8	19.0	14.5	184.6	63.7	40.0	27.5	19.4	90.9	37.3	26.3	18.7	14.3
M_2	135.8	70.6	54.7	40.8	33.5	190.2	97.4	72.5	53.8	40.6	142.1	73.5	55.8	42.3	34.1
T_2	116.9	54.2	39.3	29.7	23.1	183.5	82.1	55.5	40.3	29.8	109.7	52.4	37.0	29.0	22.5
M_3	133.3	69.3	52.8	39.9	31.9	190.4	96.1	71.1	52.4	39.3	140.0	71.5	54.6	40.6	33.0
T_3	116.1	53.0	38.1	28.5	22.2	183.2	80.5	54.0	39.4	28.6	108.1	51.0	36.0	28.1	21.7
M_4	120.6	56.3	40.8	31.0	23.9	181.3	80.6	56.8	41.2	30.6	125.2	58.2	42.7	32.0	24.3
T_4	103.3	42.4	29.0	21.4	16.7	173.2	66.1	43.7	30.4	22.4	91.7	40.1	28.9	20.6	16.1
M_5	135.0	70.2	54.1	40.7	32.9	190.3	96.9	72.0	53.4	40.1	141.2	72.7	55.8	41.5	33.8
T_5	116.8	53.9	38.8	29.4	22.8	183.1	81.9	55.0	39.8	29.4	109.0	51.8	36.8	28.6	22.2
M_6	110.6	48.4	33.9	25.3	19.4	174.3	70.6	48.5	34.9	24.9	114.2	50.0	35.1	26.3	19.9
T_6	96.0	36.9	25.8	18.0	14.0	167.7	59.5	38.3	25.7	18.9	82.7	35.3	24.5	18.1	13.6
M_7	110.2	49.2	34.7	26.0	18.6	178.8	72.8	48.5	33.7	24.5	113.4	51.0	36.0	26.4	19.4
T_7	98.3	38.6	26.0	18.8	14.5	175.9	61.2	38.5	27.1	19.4	85.2	36.7	25.3	18.4	14.0
M_8	110.2	49.2	34.7	26.0	18.6	178.7	72.9	48.5	33.7	24.5	113.4	51.0	36.0	26.4	19.4
T_8	95.8	38.3	25.4	18.3	14.1	174.6	59.9	37.6	26.5	18.6	82.7	36.1	25.1	18.0	13.4
M_9	44.8	24.7	19.3	15.4	12.1	62.2	33.7	24.7	19.3	14.6	45.9	25.3	19.9	15.7	12.3
T_9	96.1	38.3	25.5	18.3	14.1	174.7	60.4	37.7	26.5	18.7	82.8	36.2	25.1	18.1	13.4
M_{10}	110.1	49.2	34.7	25.9	18.7	178.7	72.8	48.6	33.7	24.6	113.4	51.0	35.9	26.4	19.4
T_{10}	95.9	38.3	25.5	18.3	14.1	174.6	60.0	37.6	26.5	18.6	82.7	36.1	25.1	18.0	13.4
M_{11}	110.2	49.4	34.8	26.0	18.6	178.0	73.0	48.8	33.7	24.7	113.5	51.2	36.0	26.4	19.5
T_{11}	96.1	38.5	25.6	18.3	14.1	175.5	60.4	37.9	26.5	18.9	82.9	36.4	25.2	18.0	13.5

in Tables S6 and S7. From these tables, one can conclude that median control charts are more robust for processes following a skewed distribution against the outliers as compared with processes following symmetric distributions.

5 Case study

To evaluate the significance of our proposed auxiliary information based CUSUM median control charts (M) in a real scenario, in this section we provide a real-life example comparing median control charts based on auxiliary variables. We used real data from the cement manufacturing industry in which the concrete compressive strength is important for quality cement manufacturing. This process can be described as follows: Portland cement is prepared by two heating processes, calcining and burning, after the execution of operations such as crushing, grinding, and blending of ingredients into a fine powder. After being heated at a high temperature during calcining, the fine powder is partially fused to form clinkers, which are pulverized in the ball mill. The manufactured cement is then mixed with aggregates such as blast furnace slag, fly ash, sand, gravel, admixtures, fibers, and water. Finally, the cement is shifted to the workstation. Segregation is avoided to achieve full compaction with the goal of eliminating bubbles from the product. The prime attribute—the strength of the concrete—is controlled by drawing quality control charts at the workstation. We used CUSUM median control charts to control the concrete compressive strength. For our example, we selected one median chart based on two auxiliary variables, M_{11}, and one median chart based on a single auxiliary variable M_6, and compared the usual median control chart M_1. For this purpose, we considered the dataset based on the process of concrete compressive strength, which is important in civil engineering processes. This dataset was originally proposed by Yeh (1998) and then used in many research articles (Yeh, 2003, 2006).

Table 6 Contaminated average run length values of CUSUM median and mean charts for the process following Student's *t* distribution for ρ_{xy}=0.20, ρ_{yz}=0.70, ρ_{zx}=0.10, and *n*=10

	Contaminated average run length														
	Scenario II					Scenario III					Scenario IV				
	τ=0	0.10	0.15	0.20	0.25	0	0.10	0.15	0.20	0.25	0	0.10	0.15	0.20	0.25
M_1	34.2	16.4	12.3	9.5	7.7	53.5	23.2	16.6	12.0	9.5	35.2	16.8	12.7	9.8	7.8
T_1	34.8	16.8	12.5	9.7	7.9	33.6	16.5	12.3	9.6	7.9	33.6	16.5	12.3	9.6	7.9
M_2	38.0	21.8	17.3	13.3	11.4	52.3	28.2	21.1	17.0	13.3	40.0	22.5	17.8	13.8	11.6
T_2	37.5	21.3	16.8	12.8	11.0	36.4	21.0	16.6	12.7	10.8	36.4	21.0	16.6	12.7	10.8
M_3	37.8	21.4	16.8	13.0	11.2	52.2	28.1	20.9	16.4	13.1	39.9	22.3	17.5	13.5	11.2
T_3	37.2	20.8	16.5	12.6	10.7	36.0	20.8	16.3	12.4	10.5	36.0	20.8	16.3	12.4	10.5
M_4	34.5	17.3	12.9	9.9	8.0	51.8	22.4	16.3	12.6	10.4	36.5	18.0	13.3	10.1	8.3
T_4	34.4	17.8	12.8	9.7	7.8	32.5	18.2	12.3	9.7	7.8	32.5	18.2	12.3	9.7	7.8
M_5	38.0	21.6	17.2	13.3	11.3	52.3	28.4	21.2	16.7	13.3	40.0	22.3	17.7	13.8	11.5
T_5	37.4	21.5	16.8	12.8	10.9	36.3	21.3	16.5	12.7	10.8	36.3	21.3	16.5	12.7	10.8
M_6	33.6	16.3	12.0	9.4	7.5	51.2	22.4	15.7	12.0	9.2	35.5	17.2	12.6	9.7	7.8
T_6	33.4	16.6	12.2	9.4	7.6	31.2	16.3	12.0	9.3	7.5	31.2	16.3	12.0	9.3	7.5
M_7	38.9	21.2	16.7	13.4	10.9	52.6	27.6	%1.	16.2	13.1	40.6	22.0	17.2	14.0	11.1
T_7	41.3	24.5	20.0	16.2	13.4	39.3	23.9	19.6	16.1	13.2	39.3	23.9	19.6	16.1	13.2
M_8	31.5	13.7	10.0	7.6	6.3	52.7	19.5	13.6	9.9	7.6	34.3	14.7	10.5	8.1	6.5
T_8	28.0	11.6	8.4	6.4	5.2	26.5	11.5	8.3	6.4	5.2	26.5	11.5	8.3	6.4	5.2
M_9	31.8	13.9	10.1	7.7	6.3	53.5	19.8	13.9	10.1	7.6	34.6	14.6	10.6	8.0	6.5
T_9	28.6	11.8	8.5	6.4	5.2	26.9	11.7	8.3	6.3	5.1	26.9	11.7	8.3	6.3	5.1
M_{10}	31.5	13.8	9.9	7.6	6.3	52.8	19.4	13.6	9.9	7.6	34.2	14.7	10.5	8.0	6.5
T_{10}	28.0	11.6	8.4	6.4	5.2	26.5	11.4	8.3	6.4	5.2	26.5	11.4	8.3	6.4	5.2
M_{11}	32.2	13.8	10.1	7.7	6.3	54.0	19.8	13.9	10.0	7.6	34.8	14.8	10.6	8.0	6.5
T_{11}	29.1	12.0	8.5	6.5	5.2	27.3	11.7	8.4	6.4	5.2	27.3	11.7	8.4	6.4	5.2

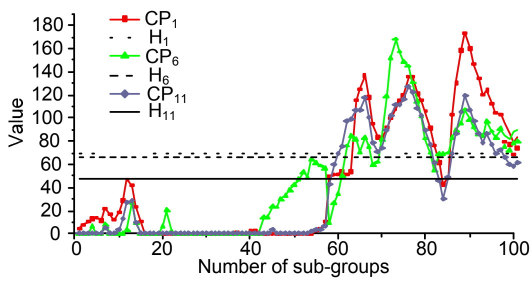

Fig. 3 Control charting structure of charts of the concrete compressive strength process

The dataset includes 1030 samples and the concrete compressive strength (process variable) is measured by eight different factors.

This study emphasizes the application of CUSUM median control charts to concrete compressive strength in efficiently monitoring any potential change. In this example, *Y* is the concrete compressive strength (measured in MPa), *X* is the cement mixture (measured in kg/m^3), and *Z* is the fine aggregate mixture (measured in kg/m^3). The density plot of the study variable is shown in Fig. S1 (available in supplementary materials). In addition, some descriptive measures of the dataset are presented in Table 8. The dataset originally contained 1030 (for the exact number of samples/subgroups, we used 1020). The initial 510 (the first half of the data) came from the in-control state with shift zero, whereas a shift (τ=0.15) was introduced in μ_y for the second half of the dataset to monitor the process location parameter. We considered the trivariate dataset in the form of 102 subgroups, each of size *n*=10. The control limits for all charts were constructed for ARL$_0$=200 based on in-control data. The values of three plotting statistics (M_1, M_6, and M_{11}) were computed and the results are displayed in the form of a control chart, Fig. 3, by plotting charting statistic values on the vertical axis and sample numbers (subgroups) on the horizontal axis. After the 51st subgroup, we can observe that the M_{11}, M_6, and M_1 control charts detect 42, 38, and 36 out-of-control points, respectively. Also, the M_{11}, M_6, and M_1 control charts detect the first out-of-control signal at the 59th, 62nd, and 64th samples, respectively.

Table 7 Contaminated average run length values of CUSUM median and mean charts for the process following the log-normal distribution for ρ_{xy}=0.75, ρ_{yz}=0.10, ρ_{zx}=0.10, and n=5

	Contaminated average run length														
	Scenario II					Scenario III					Scenario IV				
	τ=0	0.10	0.15	0.20	0.25	0	0.10	0.15	0.20	0.25	0	0.10	0.15	0.20	0.25
M_1	152.9	76.4	52.9	35.2	24.9	191.8	109.7	79.3	53.7	37.6	145.3	75.6	52.1	35.6	25.2
T_1	127.5	77.7	59.1	44.3	33.5	202.1	137.6	107.2	81.2	62.6	134.7	80.1	60.3	45.2	33.6
M_2	99.6	32.7	20.0	13.9	10.2	182.7	65.2	35.6	21.1	13.9	97.6	33.0	20.0	13.9	10.2
T_2	81.0	49.3	37.1	29.3	21.6	213.1	133.3	99.3	75.3	53.8	76.8	47.4	36.0	28.4	21.9
M_3	101.8	33.3	20.3	14.1	10.3	182.2	66.8	36.7	21.5	14.1	99.5	33.6	20.5	14.1	10.4
T_3	82.8	50.4	37.7	29.7	21.8	212.9	133.9	100.7	76.2	54.4	79.0	48.1	36.7	29.1	21.7
M_4	142.5	71.2	48.7	33.2	23.0	184.8	107.0	75.3	50.0	34.8	136.6	70.4	48.4	33.3	23.3
T_4	128.1	77.4	58.7	45.3	32.6	211.5	138.1	108.3	82.5	63.8	136.2	79.9	60.6	45.6	32.3
M_5	99.7	32.7	20.0	13.9	10.2	182.7	65.2	35.6	21.2	13.9	97.7	33.0	20.1	13.9	10.2
T_5	81.2	49.5	37.1	29.4	21.6	213.8	133.6	99.3	75.7	53.8	77.1	47.5	36.0	28.6	21.8
M_6	144.4	73.5	50.2	35.0	24.4	185.1	108.1	78.0	51.9	36.2	138.2	72.5	50.2	35.0	24.4
T_6	129.7	78.3	59.4	46.1	33.0	210.8	137.6	108.5	82.6	64.0	137.0	80.7	61.5	46.0	32.6
M_7	116.2	39.9	24.1	15.9	11.4	191.3	76.1	44.2	25.3	16.3	111.8	39.6	24.3	16.0	11.4
T_7	93.9	54.9	39.6	30.4	22.7	193.4	121.3	92.6	71.3	50.9	101.7	56.0	39.8	29.8	22.1
M_8	115.8	39.3	23.9	15.8	11.4	190.5	75.9	44.0	25.1	16.4	110.8	39.4	24.2	15.9	11.3
T_8	92.9	54.4	39.3	30.3	22.6	192.0	120.1	91.9	71.0	51.0	100.6	55.5	39.5	29.3	22.1
M_9	118.8	41.9	25.4	16.4	11.8	191.5	79.5	46.8	26.9	17.0	113.8	41.9	25.3	16.5	11.8
T_9	92.2	53.9	39.2	30.1	22.4	192.9	120.4	91.5	71.3	51.2	99.6	54.7	39.2	28.9	22.2
M_{10}	115.7	39.1	23.9	15.7	11.4	190.6	75.6	43.8	25.0	16.3	110.6	39.3	24.1	15.8	11.3
T_{10}	92.9	54.4	39.4	30.4	22.6	192.3	120.6	91.8	71.2	50.9	100.9	55.5	39.7	29.3	22.1
M_{11}	119.0	41.9	25.5	16.4	11.8	191.5	79.7	46.9	27.0	17.1	113.9	42.0	25.4	16.6	11.8
T_{11}	92.1	53.7	39.1	30.0	22.4	192.6	120.0	91.5	71.2	51.1	99.4	54.3	39.0	29.0	22.0

Table 8 Descriptive measures for concrete compressive strength

Statistic	Value	Percentile	Value
Mean	35.817961	100.0% (maximum)	82.600 00
Std dev	16.705 742	99.5%	79.384 50
Std err mean	0.520 531 7	97.5%	73.059 75
Upper 95% mean	36.839 386	90.0%	58.980 00
Lower 95% mean	34.796 536	75.0% (Q_3)	46.207 50
N	1030	50.0% (median)	34.445 00
Variance	279.081 81	25.0% (Q_1)	23.697 50
Skewness	0.416 977 3	10.0%	14.200 00
Kurtosis	−0.313 725	2.5%	8.528 75
Mode	33.4	0.5%	4.840 85
Range	80.27	0% (minimum)	2.330 00

These results show that while monitoring the performance of concrete compressive strength using CUSUM median control charts, if we consider the cement mixture and fine aggregate mixture as auxiliary variables, the ability of the process to detect unnatural variations is better than that of the process of monitoring concrete compressive strength using a CUSUM median control chart that considers only the cement mixture as an auxiliary variable. From this case study, we can conclude that if we use more auxiliary variables in the monitoring of concrete compressive strength using CUSUM median control charts, the process will detect unnatural variations as early as possible and we can improve the compressive strength of concrete by identifying these unnatural variations in the process, which resembles the results in Section 4. The concrete compressive strength depends on two factors, composition of the concrete and its mixture lifetime.

If in composition we assume that all raw materials are similar and fine, then unnatural variations can be classifiers; for example, the manufactured cement is mixed with aggregates afterward, such as blast furnace slag, fly ash, sand, gravel, admixtures, fibers, and water. Variations in the heating temperature of the blast furnace, the quality of fly ash or gravel, the quantity of items used in the mixture, etc., can have unnatural variations, and the compressive strength of the concrete can be improved by detecting these unnatural variations as early as possible. Therefore, the different variables of interest in manufacturing processes can be monitored efficiently using the proposed charting techniques in this study.

6 Conclusions and recommendations

Variation is an essential part of any ongoing process and it needs our attention to improve the quality of the process. Control charts help classify these variations as natural and unnatural. Mean charts are commonly used for location monitoring, but the presence of outliers harms their performance. The median serves as a better alternative when processes face sudden outliers. Moreover, information about auxiliary variables helps enhance the precision of the estimators and hence the charting structure. In this paper, we proposed median-based CUSUM control charts to monitor the location parameter. With the help of one and two auxiliary variables, we designed mean- and median-based CUSUM control charts for the processes, following symmetric and non-symmetric distributions, and compared their performances in contaminated and uncontaminated process environments. We performed detailed run length analysis using different performance measures: ARL, EQL, RARL, and PCI. Our evaluation shows that, in the uncontaminated environment, mean control charts are superior to median control charts for symmetric distributions. For skewed distributions, the detection ability of the proposed median charts is more promising. In contaminated environments, the proposed median charts have outstanding performance compared to the classical mean control charts for all normal and non-normal processes. The resistance ability of the proposed median charts is higher than that of the traditional mean control charts in the presence of outliers. Within median charts, we established the overall dominance of control charting structures based on two auxiliary variables. The resistance of median control charts for the non-normal processes is higher than that for the processes following symmetric distributions.

A case study from the cement manufacturing industry is also provided in support of our study proposals and indicates that the proposed CUSUM control charts using auxiliary information based median estimators are very efficient in detecting shifts in process parameters. The scope of this study can be further widened to other types of control charts, such as multivariate EWMA and CUSUM control charts, mixed EWMA-CUSUM control charts, and runs rules-based median auxiliary charts to increase the performance of these charting structures.

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