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Stability of Boolean networks with state-dependent random impulses*

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Abstract: We investigate the stability of Boolean networks (BNs) with impulses triggered by both states and random factors. A hybrid index model is used to describe impulsive BNs. First, several necessary and sufficient conditions for forward completeness are obtained. Second, based on the stability criterion of probabilistic BNs and the forward completeness criterion, the necessary and sufficient conditions for the finite-time stability with probability one and the asymptotical stability in distribution are presented. The relationship between these two kinds of stability is discussed. Last, examples and time-domain simulations are provided to illustrate the obtained results.

Key words: Boolean network with impulses; Forward completeness; Finite-time stability with probability one; Asymptotical stability in distribution

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1 Introduction

Boolean networks (BNs) were first proposed by Kauffman (1969) to describe genetic regulatory networks, and have attracted a lot of attention since then. However, due to the lack of powerful mathematical tools for logical systems, studies of BNs were limited to a certain extent until Cheng and Qi (2010) proposed the semi-tensor product (STP) of matrices. Under the framework of STP, many challenging problems in BNs have been studied (Cheng et al., 2009, 2011b, 2018; Cheng, 2011; Li F and Sun, 2011a; Zhao et al., 2011; Li R and Chu, 2012; Laschov et al., 2013; Li HT and Wang, 2013; Xu XR and Hong, 2013; Fornasini and Valcher, 2014; Liu Y et al., 2014, 2015, 2016, 2017; Guo et al., 2015, 2017; Lu et al., 2015; Li HT et al., 2016; Meng et al., 2018; Zhu QX et al., 2018; Li YY et al., 2019; Wu et al., 2019).

In nature, social life, and production, many dynamic systems may experience abrupt changes of state (Shah et al., 2018). For example, injection of a drug causes abrupt changes in the number of bacteria in an organism. When we describe a system in a relatively large timescale, these abrupt changes can be regarded as ideal impulsive disturbances. BNs with impulsive effects have been studied over the last decade (Li F and Sun, 2011b; Zhong et al., 2014; Chen H et al., 2015; Chen HW et al., 2016; Xu XJ et al., 2018). In particular, Guo et al. (2019a) studied the finite-time stability of BNs with state-triggered impulses. Note that most of the research is concerned with time-triggered impulsive systems. However, in fact, impulses may be triggered by not only time, but also the state and environmental factors (Chellaboina and Haddad, 2002; Ambrosino et al., 2008; Jiao and Zheng, 2016; Li LL et al., 2019). In addition, environmental factors introduce randomness into the state-triggered impulses. For instance, in a state-triggered impulsive system, the state space

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is usually partitioned into two disjoint subsets, called the flow and jump sets. An impulse is triggered whenever the state falls in the jump set. However, near the boundary of the jump set, due to the disturbance and noise, both the flow and jump behaviors are possible when one detects that the state is in the jump or flow set. To the best of our knowledge, BNs with impulses triggered by both state and random factors have not been investigated. Besides, it is well known that stability is a fundamental requirement of any real system. Motivated by these factors, we investigate the stability of state-triggered random impulsive BNs (SRIBNs).

In this study, a hybrid index model is used to characterize SRIBNs, based on which the finite-time stability with probability one and the asymptotical stability in distribution are investigated. Compared with conventional models for impulsive BNs, the hybrid index model can better characterize the instantaneousness of an impulse, which is the essential characteristic of impulsive systems. In addition to the time index, the impulse index is introduced into the hybrid index model to indicate the number of impulses. Assume that the whole state space consists of jump and step subsets. The system might be disturbed by impulses only when the state falls into the jump subset. In addition, to characterize the randomness of the state-triggered impulses, we allow that there exists an overlap between these two subsets. When the state falls into the overlap, the system randomly triggers the impulses subject to a state-dependent probability distribution.

A state-triggered impulsive system may exist with the Zeno phenomenon; that is, there are infinitely many successive impulses occurring at a single time instant. However, a practical system should be forward complete in the sense that there is no Zeno phenomenon. One natural problem is under the condition that an impulsive BN described by the hybrid index model is forward complete. We prove that a BN is forward complete if and only if it can step to the next moment with probability one. We decompose the transition probability matrix (TPM) into the stepping and jumping parts, called the stepping probability matrix (SPM) and jumping probability matrix (JPM), respectively, and prove that an SRIBN is forward complete if and only if the JPM is a nilpotent matrix.

The finite-time stability with probability one and the asymptotical stability in distribution for SRIBNs are investigated herein. Note that the stability of ordinary probabilistic BNs (PBNs) has been studied in recent years (Chen H and Sun, 2014; Li R et al., 2014; Li Z et al., 2014; Zhao and Cheng, 2014; Guo et al., 2019b; Liu JY et al., 2019; Zhu SY et al., 2019; Huang et al., 2020). The main difference between the stabilities of a PBN and an SRIBN is that the stability of an SRIBN requires the system be forward complete. Based on the forward completeness obtained in this study and the existing stability analysis methods for PBNs in the literature, several necessary and sufficient conditions are obtained for the finite-time stability with probability one and the asymptotical stability in distribution for SRIBNs. Notations used in this paper are listed in Table 1.

Table 1 Notations used in this paper

Notation	Definition
$\mathcal{D}$	Logic domain $\{0,1\}$
$\ltimes$	STP of matrices
$\mathcal{L}_{m\times n}$	Set of all $m\times n$ logical matrices
$\boldsymbol{\delta}_n^i$	The i^{th} column of identity matrix $\boldsymbol{I}_n$
$\varDelta_n$	Set of $\{\boldsymbol{\delta}_n^i \mid i=1,2,\cdots,n\}$
$\mathcal{M}\setminus\mathcal{N}$	Set of $\{x\notin\mathcal{N} \mid x\in\mathcal{M}\}$
$\mathbb{Z}^+$	Nonnegative integers
$[M:N]$	Set of integers n satisfying $M\le n\le N$
$\mathrm{Col}_i(\boldsymbol{M})$	The i^{th} column of matrix $\boldsymbol{M}$
$\mathrm{Row}_j(\boldsymbol{M})$	The j^{th} row of matrix $\boldsymbol{M}$
$[M]_{i,j}$	The (i,j)-element of matrix $\boldsymbol{M}$

2 Preliminaries and problem setting

2.1 A hybrid index model for random impulsive BNs

An n-node BN can be described as

$$\begin{cases} X_1(t+1)=f_1[X_1(t),X_2(t),\ldots,X_n(t)],\\ X_2(t+1)=f_2[X_1(t),X_2(t),\ldots,X_n(t)],\\ \qquad\vdots\\ X_n(t+1)=f_n[X_1(t),X_2(t),\ldots,X_n(t)], \end{cases} \tag{1}$$

where $X_i\in\mathcal{D}$ ($\mathcal{D}=\{0,1\}, i=1,2,\ldots,n$) denotes the state of the i^{th} node and $f_i:\mathcal{D}^n\rightarrow\mathcal{D}$ is a logical function. The vector of the logic variable X_i is defined as $\boldsymbol{x}_i:=\boldsymbol{\delta}_2^{2-X_i}\in\varDelta_2$. Using the theory of STP of matrices (Cheng et al., 2011a), Eq. (1) can be expressed in the equivalent algebraic form as

$$\boldsymbol{x}(t+1)=\boldsymbol{L}\boldsymbol{x}(t), \tag{2}$$

where $\boldsymbol{x} := \boldsymbol{x}_1 \ltimes \boldsymbol{x}_2 \ltimes \ldots \ltimes \boldsymbol{x}_n \in \Delta_{2^n}$ and $\boldsymbol{L} \in \mathcal{L}_{2^n\times 2^n}$.

In this study, we consider BNs with random impulsive disturbances subject to the state-dependent probability distribution. When the system is not disturbed by impulses, the state updates according to Eq. (2), which is called the "step process." When impulses occur, the state updating time is much less than the unit time in the step process. Thus, we call this instantaneous change of the state the "jump process" and use $\boldsymbol{x}^+(t)$ to denote the new state after jumping. The jump process can be described as

$$\boldsymbol{x}^+(t)=\boldsymbol{G}\boldsymbol{x}(t), \tag{3}$$

where $\boldsymbol{G} \in \mathcal{L}_{2^n\times 2^n}$ is a logical matrix. Assume that the system can enter the jump process when the state belongs to the jump subset $\mathcal{J} \subset \Delta_{2^n}$. When the state belongs to the step subset $\mathcal{S} \subset \Delta_{2^n}$, the system can enter the step process. The whole state space is covered by the union of the jump and step subsets, i.e., $\mathcal{J}\cup\mathcal{S}=\Delta_{2^n}$. Aside from that, we consider that there exists an overlap between the jump and step subsets, i.e., $\mathcal{J}\cap\mathcal{S}\neq\varnothing$. When the state falls into the overlapping part of the jump and step subsets, the system randomly enters the jump or step process. In general, assume that the state of the network is updated by the following laws:

1. When $\boldsymbol{x}(t)=\boldsymbol{\delta}_{2^n}^i \in \mathcal{J}\cap\mathcal{S}$, the system enters the jump process with probability p_i $(0<p_i<1)$, and enters the step process with probability $q_i := 1-p_i$.

2. When $\boldsymbol{x}(t)\in\mathcal{J}\setminus\mathcal{S}$, the system enters the jump process with probability one.

3. When $\boldsymbol{x}(t)\in\mathcal{S}\setminus\mathcal{J}$, the system enters the step process with probability one.

Although Eq. (3) describes the instantaneousness of the jump process well, it is inconvenient for analysis. Thus, we introduce an impulse index j which represents the number of impulses, and redenote the state $\boldsymbol{x}(t)$ as $\boldsymbol{x}(t,j)$, which is defined on the hybrid index space. Then, the SRIBN considered in this study can be described as the following hybrid index model:

$$\begin{cases}\boldsymbol{x}(t+1,j)=\boldsymbol{L}\boldsymbol{x}(t,j),\ \gamma_{t,j}=0,\\ \boldsymbol{x}(t,j+1)=\boldsymbol{G}\boldsymbol{x}(t,j),\ \gamma_{t,j}=1,\end{cases} \tag{4}$$

where $\gamma_{t,j}$ is a series of double-indexed Boolean random variables that obey the following state-dependent probability distribution:

$$p_i=\Pr\{\gamma_{t,j}=1|\boldsymbol{x}(t,j)=\boldsymbol{\delta}_{2^n}^i\},\ \forall i\in[1:2^n]. \tag{5}$$

Note that $p_i=1$ for any $\boldsymbol{\delta}_{2^n}^i\in\mathcal{J}\setminus\mathcal{S}$ and $p_i=0$ for any $\boldsymbol{\delta}_{2^n}^i\in\mathcal{S}\setminus\mathcal{J}$. Obviously, the value of $\gamma_{t,j}$ determines whether the state triggers the impulses. When $\gamma_{t,j}=0$, the system is not affected by impulses and enters the step process where the state updates according to the first equation of Eq. (4). Besides, the number of impulses remains unchanged and the time index t is added by 1 during each step process. When $\gamma_{t,j}=1$, a state-triggered impulse occurs and the system enters the jump process, where the state updates according to the second equation of Eq. (4). Due to the instantaneousness of the jump process, assume that the time index t remains unchanged and that the impulse index j is added by 1 during each jump process.

2.2 Definitions and problem setting

Definition 1 A stochastic sequence $\boldsymbol{x}(t_k,j_k)$, $k\in\mathbb{Z}^+$, is called a solution to SRIBN given by Eq. (4), if the following two conditions are satisfied:

1. If $\gamma_{t,j}=0$, then $t_{k+1}=t_k+1$, $j_{k+1}=j_k$, and $\boldsymbol{x}(t_{k+1},j_{k+1})=\boldsymbol{L}\boldsymbol{x}(t_k,j_k)$.

2. If $\gamma_{t,j}=1$, then $t_{k+1}=t_k$, $j_{k+1}=j_k+1$, and $\boldsymbol{x}(t_{k+1},j_{k+1})=\boldsymbol{G}\boldsymbol{x}(t_k,j_k)$.

Denote the solution with the initial state $\boldsymbol{x}_0$ by $\boldsymbol{x}(t_k,j_k;\boldsymbol{x}_0)$ and the stochastic sequence of time index t_k for the solution with the initial state $\boldsymbol{x}_0$ by $t_k(\boldsymbol{x}_0)$.

Definition 2 SRIBN is said to be forward complete if any $\boldsymbol{x}_0\in\Delta_{2^n}$ and any $T\in\mathbb{Z}^+$, and there exists a nonnegative integer K such that

$$\Pr\{t_k(\boldsymbol{x}_0)\geq T\}=1,\ \forall k\geq K.$$

Remark 1 For a system with state-dependent impulses, the Zeno phenomenon means that infinitely many jumps may happen at a single time instant. However, for a practical system, the process of a sudden change in the state is not ideally instantaneous. Instead, each jump process takes some time. Thus, the Zeno phenomenon should not occur in a practical system. Before stability analysis, we should investigate under which conditions an SRIBN is forward complete (that is, no Zeno phenomenon occurs).

Definition 3 SRIBN is said to be finite-time $\boldsymbol{x}_d$-stable with probability one, if it is forward complete and if, for any $\boldsymbol{x}_0 \in \Delta_{2^n}$, there exists a nonnegative integer K such that

$$\Pr\{\boldsymbol{x}(t_k, j_k; \boldsymbol{x}_0) = \boldsymbol{x}_d\} = 1, \ \forall k \geq K.$$

Definition 4 SRIBN is said to be finite-time $\mathcal{M}$-stable with probability one, if it is forward complete and if, for any $\boldsymbol{x}_0 \in \Delta_{2^n}$, there exists a nonnegative integer K such that

$$\Pr\{\boldsymbol{x}(t_k, j_k; \boldsymbol{x}_0) \in \mathcal{M}\} = 1, \ \forall k \geq K.$$

Definition 5 SRIBN is said to be asymptotically $\boldsymbol{x}_d$-stable in distribution, if it is forward complete and if

$$\lim_{k\to\infty} \Pr\{\boldsymbol{x}(t_k, j_k; \boldsymbol{x}_0) = \boldsymbol{x}_d\} = 1, \ \forall \boldsymbol{x}_0 \in \Delta_{2^n}.$$

Definition 6 SRIBN is said to be asymptotically $\mathcal{M}$-stable in distribution, if it is forward complete and if

$$\lim_{k\to\infty} \Pr\{\boldsymbol{x}(t_k, j_k; \boldsymbol{x}_0) \in \mathcal{M}\} = 1, \ \forall \boldsymbol{x}_0 \in \Delta_{2^n}.$$

Definition 7 Suppose that $\boldsymbol{x}_k$ is a homogeneous Markov chain, which is characterized by the 1-step TPM $\boldsymbol{P}$, expressed as

$$[P]_{i,j} = \Pr\{\boldsymbol{x}_{k+1} = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}_k = \boldsymbol{\delta}_{2^n}^j\}.$$

A subset $\mathcal{C} \subseteq \Delta_{2^n}$ is called a $\boldsymbol{P}$-invariant subset of $\boldsymbol{x}_k$, if

$$\Pr\{\boldsymbol{x}_k \in \mathcal{C} | \boldsymbol{x}_0 \in \mathcal{C}\} = 1, \ \forall k \in \mathbb{Z}^+.$$

The union of all $\boldsymbol{P}$-invariant subsets contained in $\mathcal{M} \subseteq \Delta_{2^n}$ is called the largest $\boldsymbol{P}$-invariant subset, denoted by $I_{\boldsymbol{P}}(\mathcal{M})$.

2.3 Transition probability matrices

The jump probability of the state $\boldsymbol{\delta}_{2^n}^i$ can be expressed as

$$\begin{aligned} p_i =& \Pr\{t_{k+1} = t_k | \boldsymbol{x}(t_k, j_k) = \boldsymbol{\delta}_{2^n}^i\} \\ =& \Pr\{\gamma_{t_k, j_k} = 1 | \boldsymbol{x}(t_k, j_k) = \boldsymbol{\delta}_{2^n}^i\}. \end{aligned} \tag{6}$$

For simplification, denote

$$\boldsymbol{p}_J = [p_1, p_2, \ldots, p_{2^n}]$$

and

$$\boldsymbol{p}_S = [1 - p_1, 1 - p_2, \ldots, 1 - p_{2^n}].$$

Define the 1-step JPM $\boldsymbol{P}_J$ as

$$\begin{aligned} [P_J]_{i,j} =& \Pr\big\{\gamma_{t_k, j_k} = 1, \boldsymbol{x}(t_{k+1}, j_{k+1}) = \boldsymbol{\delta}_{2^n}^i \ \big| \ \boldsymbol{x}(t_k, j_k) \\ &= \boldsymbol{\delta}_{2^n}^j\big\}, \ \forall k \in \mathbb{Z}^+ \text{ and } \forall i, j \in [1 : 2^n]. \end{aligned}$$

It is easy to see that

$$\boldsymbol{P}_J = \boldsymbol{G} \cdot \mathrm{diag}(\boldsymbol{p}_J).$$

In the same way, define the 1-step SPM $\boldsymbol{P}_S$ as

$$\begin{aligned} [P_S]_{i,j} =& \Pr\big\{\gamma_{t_k, j_k} = 0, \boldsymbol{x}(t_{k+1}, j_{k+1}) = \boldsymbol{\delta}_{2^n}^i \ \big| \ \boldsymbol{x}(t_k, j_k) \\ &= \boldsymbol{\delta}_{2^n}^j\big\}, \ \forall k \in \mathbb{Z}^+ \text{ and } \forall i, j \in [1 : 2^n]. \end{aligned}$$

Then,

$$\boldsymbol{P}_S = \boldsymbol{L} \cdot \mathrm{diag}(\boldsymbol{p}_S).$$

Since at each step, the state is updated through either a jump process or a step process, the 1-step TPM $\boldsymbol{P}$ can be expressed as

$$\boldsymbol{P} = \boldsymbol{P}_J + \boldsymbol{P}_S. \tag{7}$$

3 Forward completeness

In this section, we investigate the forward completeness of the SRIBN.

Lemma 1 For any $k \in \mathbb{N}^+$ and any $i, j \in [1 : 2^n]$, it holds that

$$[P_J^k]_{i,j} = \Pr\{t_k = t_0, \boldsymbol{x}(t_k, j_k) = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}(t_0, j_0) = \boldsymbol{\delta}_{2^n}^j\}. \tag{8}$$

Proof Obviously, Eq. (8) holds for $k = 1$. Assume that Eq. (8) holds for $k = \xi > 1$, that is,

$$[P_J^\xi]_{i,j} = \Pr\{t_\xi = t_0, \boldsymbol{x}(t_\xi, j_\xi) = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}(t_0, j_0) = \boldsymbol{\delta}_{2^n}^j\},$$

for any $i, j \in [1 : 2^n]$. Then for any $i, j \in [1 : 2^n]$,

$$\begin{aligned} & \Pr\{t_{\xi+1} = t_0, \boldsymbol{x}(t_{\xi+1}, j_{\xi+1}) = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}(t_0, j_0) = \boldsymbol{\delta}_{2^n}^j\} \\ =& \sum_{\alpha=1}^{2^n} \Pr\{t_\xi = t_0, \boldsymbol{x}(t_\xi, j_\xi) = \boldsymbol{\delta}_{2^n}^\alpha | \boldsymbol{x}(t_0, j_0) = \boldsymbol{\delta}_{2^n}^j\} \\ & \cdot \Pr\{t_{\xi+1} = t_\xi, \boldsymbol{x}(t_{\xi+1}, j_{\xi+1}) = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}(t_\xi, j_\xi) = \boldsymbol{\delta}_{2^n}^\alpha\} \\ =& \sum_{\alpha=1}^{2^n} [P_J^\xi]_{\alpha,j} [P_J]_{i,\alpha} \\ =& \sum_{\alpha=1}^{2^n} [P_J]_{i,\alpha} [P_J^\xi]_{\alpha,j} \\ =& [P_J^{\xi+1}]_{i,j}, \end{aligned}$$

implying that Eq. (8) holds for $k = \xi + 1$. Therefore, Eq. (8) holds for any $k \in \mathbb{N}^+$ and any $i, j \in [1 : 2^n]$.

Theorem 1 SRIBN is forward complete if and only if there exists a nonnegative integer K such that

$$\Pr\{t_k(\boldsymbol{x}_0) \geq 1\} = 1,\ \forall k \geq K, \forall \boldsymbol{x}_0 \in \Delta_{2^n}. \tag{9}$$

Proof The necessity is obviously true. We need to prove only the sufficiency. We suppose that Eq. (9) holds and prove that for any $T \in \mathbb{Z}^+$,

$$\Pr\{t_{Tk}(\boldsymbol{x}_0) \geq T\} = 1,\ \forall k \geq K,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n}. \tag{10}$$

Obviously, Eq. (10) holds for $T = 1$. Provided that Eq. (10) holds for $T = s > 1$, that is,

$$\Pr\{t_{sk}(\boldsymbol{x}_0) \geq s\} = 1,$$

then

$$\begin{aligned}
&\Pr\{t_{(s+1)k}(\boldsymbol{x}_0) \geq s+1\} \\
&\geq \sum_{i=1}^{2^n} \Pr\{t_{sk} \geq s, \boldsymbol{x}(t_{sk}, j_{sk}) = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}(t_0, j_0) = \boldsymbol{x}_0\} \\
&\quad \cdot \Pr\{t_{(s+1)k} \geq t_{sk} + 1 | \boldsymbol{x}(t_{sk}, j_{sk}) = \boldsymbol{\delta}_{2^n}^i\} \\
&= \sum_{i=1}^{2^n} \Pr\{t_{sk} \geq s, \boldsymbol{x}(t_{sk}, j_{sk}) = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}(t_0, j_0) = \boldsymbol{x}_0\} \\
&= \Pr\{t_{sk} \geq s | \boldsymbol{x}(t_0, j_0) = \boldsymbol{x}_0\} \\
&= 1,
\end{aligned}$$

implying that Eq. (10) holds for $T = s+1$. Therefore, Eq. (10) holds for any $T \in \mathbb{Z}^+$. Then we have

$$\Pr\{t_K(\boldsymbol{x}_0) \geq T\} = 1,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n}, \forall K \geq Tk,$$

implying that SRIBN is forward complete.

Theorem 2 SRIBN is forward complete if and only if $\boldsymbol{P}_J^{2^n} = \mathbf{0}$.

Proof We need to prove only that $\boldsymbol{P}_J^{2^n} = \mathbf{0}$ is equivalent to the condition in Theorem 1. According to the condition in Theorem 1, SRIBN is forward complete if and only if

$$\Pr\{t_k(\boldsymbol{\delta}_{2^n}^j) = 0\} = 1 - \Pr\{t_k(\boldsymbol{\delta}_{2^n}^j) \geq 1\} = 0$$

holds for any $k \geq K$ and any $j \in [1 : 2^n]$. Besides,

$$\begin{aligned}
&\Pr\{t_k(\boldsymbol{\delta}_{2^n}^j) = 0\} \\
&= \sum_{i=1}^{2^n} \Pr\{t_k = 0, \boldsymbol{x}(t_k, j_k) = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}(t_0, j_0) = \boldsymbol{\delta}_{2^n}^j\} \\
&= \sum_{i=1}^{2^n} [P_J^k]_{i,j}.
\end{aligned}$$

Then, the forward completeness is equivalent to $[P_J^k]_{i,j} = 0$ for any $k \geq K$ and any $i, j \in [1 : 2^n]$, and thus is equivalent to $\boldsymbol{P}_J^k = \mathbf{0}\ \ \forall k > K$. Thus, we need to prove only that there exists a nonnegative integer K such that $\boldsymbol{P}_J^k = \mathbf{0}\ \forall k \geq K$ if and only if $\boldsymbol{P}_J^{2^n} = \mathbf{0}$.

Sufficiency. Suppose that $\boldsymbol{P}_J^{2^n} = \mathbf{0}$. Then

$$\boldsymbol{P}_J^k = \boldsymbol{P}_J^{2^n} \cdot \boldsymbol{P}_J^{k-2^n} = \mathbf{0} \cdot \boldsymbol{P}_J^{k-2^n} = \mathbf{0}$$

holds for any $k \geq 2^n$, implying that there exists a nonnegative integer K such that $\boldsymbol{P}_J^k = \mathbf{0}\ \forall k \geq K$.

Necessity. Assume that there exists a nonnegative integer K such that $\boldsymbol{P}_J^k = \mathbf{0}\ \forall k \geq K$. It is obvious that $\boldsymbol{P}_J$ is a nilpotent matrix. As we know, the nilpotent exponent of a nilpotent matrix is less than or equal to its order. Note that $\boldsymbol{P}_J$ is a 2^n-order matrix, and we observe that there exists a nonnegative integer $s \leq 2^n$ such that $\boldsymbol{P}_J^s = \mathbf{0}$. Then,

$$\boldsymbol{P}_J^{2^n} = \boldsymbol{P}_J^s \cdot \boldsymbol{P}_J^{2^n-s} = \mathbf{0} \cdot \boldsymbol{P}_J^{2^n-s} = \mathbf{0}.$$

Example 1 Consider the SRIBN with $\boldsymbol{L} = \boldsymbol{\delta}_8[5, 6, 3, 5, 1, 4, 1, 7]$, $\boldsymbol{G} = \boldsymbol{\delta}_8[8, 7, 7, 4, 1, 7, 3, 2]$, $\mathcal{S} = \{\boldsymbol{\delta}_8^1, \boldsymbol{\delta}_8^2, \boldsymbol{\delta}_8^3, \boldsymbol{\delta}_8^4, \boldsymbol{\delta}_8^7\}$, $\mathcal{J} = \{\boldsymbol{\delta}_8^2, \boldsymbol{\delta}_8^3, \boldsymbol{\delta}_8^5, \boldsymbol{\delta}_8^6, \boldsymbol{\delta}_8^8\}$, $p_2 = 0.4$, and $p_3 = 0.5$. The state transfer of the SRIBN is shown in Fig. 1. Then, $\mathcal{J} \cap \mathcal{S} = \{\boldsymbol{\delta}_8^2, \boldsymbol{\delta}_8^3\}$, $\mathcal{J} \setminus \mathcal{S} = \{\boldsymbol{\delta}_8^5, \boldsymbol{\delta}_8^6, \boldsymbol{\delta}_8^8\}$, and $\mathcal{S} \setminus \mathcal{J} = \{\boldsymbol{\delta}_8^1, \boldsymbol{\delta}_8^4, \boldsymbol{\delta}_8^7\}$. By the definition of p_i, we have $p_1 = p_4 = p_7 = 0$, $p_5 = p_6 = p_8 = 1$, and $\boldsymbol{p}_J = [p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8] = [0, 0.4, 0.5, 0, 1, 1, 0, 1]$.

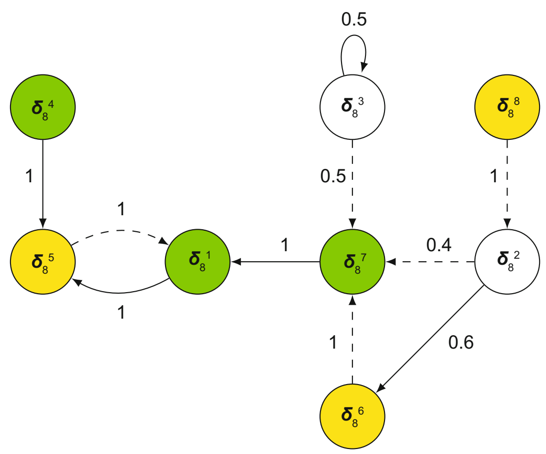

Fig. 1 State transfer graph of the SRIBN in Example 1
Solid and dashed arrows represent the step and jump processes, respectively

It is easy to check that the JPM of this SRIBN is

$$\begin{aligned}\boldsymbol{P}_J &= \boldsymbol{G}\cdot\mathrm{diag}(\boldsymbol{p}_J)\\ &= \boldsymbol{\delta}_8[8,7,7,4,1,7,3,2]\\ &\quad\cdot\mathrm{diag}(0,0.4,0.5,0,1,1,0,1)\\ &= \begin{bmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0.4&0.5&0&0&1&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.\end{aligned}$$

A simple calculation shows that $\boldsymbol{P}_J^8 = \mathbf{0}$. By Theorem 2, this SRIBN is forward complete.

4 Stability analysis

In this section, we consider two kinds of stability problems: finite-time stability with probability one and asymptotical stability in distribution. According to the definition of the TPM,

$$[P]_{i,j} = \Pr\{\boldsymbol{x}(t_{s+1}, j_{s+1}) = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}(t_s, j_s) = \boldsymbol{\delta}_{2^n}^j\}$$

holds for any $s \in \mathbb{Z}^+$ and any $i, j \in [1:2^n]$. Then, it is obvious that

$$[P^k]_{i,j} = \Pr\{\boldsymbol{x}(t_k, j_k) = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}(t_0, j_0) = \boldsymbol{\delta}_{2^n}^j\} \quad (11)$$

holds for any $k \in \mathbb{Z}^+$ and any $i, j \in [1:2^n]$.

4.1 Finite-time stability with probability one

Theorem 3 Given $\boldsymbol{x}_d \in \Delta_{2^n}$, SRIBN is finite-time $\boldsymbol{x}_d$-stable with probability one, if and only if $\boldsymbol{x}_d \in \mathcal{S} \setminus \mathcal{J}$ and $\mathrm{Col}_j[\boldsymbol{P}^{2^n}] = \boldsymbol{x}_d\ \forall j \in [1:2^n]$.

Proof Sufficiency. Assume that $\mathrm{Col}_j[\boldsymbol{P}^{2^n}] = \boldsymbol{x}_d$ holds for any $j \in [1:2^n]$, which implies that $\mathrm{Row}_i[\boldsymbol{P}^{2^n}] = \mathbf{1}_{2^n}^{\mathrm{T}}$ and $\boldsymbol{x}_d = \boldsymbol{\delta}_{2^n}^i$. Then,

$$\begin{aligned}\mathrm{Row}_i[\boldsymbol{P}^k] &= \mathrm{Row}_i[\boldsymbol{P}^{2^n}\cdot\boldsymbol{P}^{k-2^n}]\\ &= \mathrm{Row}_i[\boldsymbol{P}^{2^n}]\cdot\boldsymbol{P}^{k-2^n}\\ &= \mathbf{1}_{2^n}^{\mathrm{T}}\cdot\boldsymbol{P}^{k-2^n}\\ &= \mathbf{1}_{2^n}^{\mathrm{T}}\end{aligned}$$

holds for any $k \geq 2^n$, and is equivalent to

$$[P_J^k]_{i,j} = 1,\ \forall j \in [1:2^n],\ \forall k \geq 2^n.$$

According to Eq. (11), we find that

$$\Pr\{\boldsymbol{x}(t_k, j_k; \boldsymbol{x}_0) = \boldsymbol{\delta}_{2^n}^i\} = 1,\ \forall k \geq 2^n, \forall \boldsymbol{x}_0 \in \Delta_{2^n}.$$

Due to $\boldsymbol{x}_d \in \mathcal{S} \setminus \mathcal{J}$, it is clear that $t_1(\boldsymbol{x}_d) = 1$. Note that

$$\boldsymbol{x}(t_{2^n}, j_{2^n}; \boldsymbol{x}_0) = \boldsymbol{x}_d,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n},$$

and we can easily have

$$\Pr\{t_{2^n+1}(\boldsymbol{x}_0) \geq 1\} = 1,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n},$$

meaning that SRIBN is forward complete. Thus, SRIBN is finite-time $\boldsymbol{x}_d$-stable with probability one.

Necessity. Suppose that SRIBN is finite-time $\boldsymbol{x}_d$-stable with probability one and $\boldsymbol{x}_d = \boldsymbol{\delta}_{2^n}^i$. By Definition 3 and Eq. (11), we find that there exists a nonnegative integer K such that

$$[P_J^k]_{i,j} = \Pr\{\boldsymbol{x}(t_k, j_k) = \boldsymbol{\delta}_{2^n}^i | \boldsymbol{x}(t_0, j_0) = \boldsymbol{\delta}_{2^n}^j\} = 1$$

holds for any $j \in [1:2^n]$ and any $k \geq K$. Assume that K is the smallest nonnegative integer k satisfying

$$\mathrm{Col}_j[\boldsymbol{P}^k] = \boldsymbol{x}_d,\ \forall j \in [1:2^n] \text{ and } K \geq 2^n.$$

Then, there exists a K-length path from $\boldsymbol{x}_0$ to $\boldsymbol{x}_d$ and this path reaches $\boldsymbol{x}_d$ only once. Because there are only 2^n elements in the state space, then there exist $k_1, k_2 \in [0, K)$ ($k_1 < k_2$) and α, i with $\alpha \neq i$, such that

$$\boldsymbol{x}(t_{k_1}, j_{k_1}; \boldsymbol{x}_0) = \boldsymbol{x}(t_{k_2}, j_{k_2}; \boldsymbol{x}_0) = \boldsymbol{\delta}_{2^n}^\alpha$$

and

$$\Pr\{\boldsymbol{x}(t_{k_2-k_1}, j_{k_2-k_1}) = \boldsymbol{\delta}_{2^n}^\alpha | \boldsymbol{x}(t_0, j_0) = \boldsymbol{\delta}_{2^n}^\alpha\} > 0.$$

Then,

$$\begin{aligned}&\Pr\{\boldsymbol{x}(t_{k_1+(k_2-k_1)s}, j_{k_1+(k_2-k_1)s}; \boldsymbol{x}_0) = \boldsymbol{\delta}_{2^n}^\alpha\}\\ &\geq \Pr\{\boldsymbol{x}(t_{k_1}, j_{k_1}; \boldsymbol{x}_0) = \boldsymbol{\delta}_{2^n}^\alpha\}\\ &\quad\cdot\left[\Pr\{\boldsymbol{x}(t_{k_2-k_1}, j_{k_2-k_1}; \boldsymbol{\delta}_{2^n}^\alpha) = \boldsymbol{\delta}_{2^n}^\alpha\}\right]^s\\ &> 0\end{aligned}$$

holds for any $s \in \mathbb{Z}^+$. Thus, there exists an s such that $k_1 + (k_2 - k_1)s > K$ and

$$\Pr\{\boldsymbol{x}(t_{k_1+(k_2-k_1)s}, j_{k_1+(k_2-k_1)s}; \boldsymbol{x}_0) = \boldsymbol{\delta}_{2^n}^\alpha\} > 0,$$

which contradicts

$$\Pr\{\boldsymbol{x}(t_k, j_k; x_0) = \boldsymbol{\delta}_{2^n}^i\} = 1,\ \forall k \geq K.$$

Thus, K must be less than or equal to 2^n. According to

$$[P^k]_{i,j} = 1,\ \forall j \in [1:2^n],\ \forall k \geq K,$$

it is obvious that

$$[P^{2^n}]_{i,j} = 1,\ \forall j \in [1:2^n],$$

which is equivalent to condition 2.

If SRIBN is finite-time $\boldsymbol{x}_d$-stable with probability one, then it is forward complete and $\boldsymbol{x}_d$ is a fixed point. It is clear that $\Pr\{t_1(\boldsymbol{x}_d) = 1\} = 1$, which implies that condition 1 holds.

Next, we consider finite-time set stability with probability one. First, we give an algorithm to calculate $I_{\boldsymbol{P}}(\mathcal{M})$.

Lemma 2 (Guo et al., 2019b) Given $\mathcal{M} = \{\boldsymbol{\delta}_{2^n}^j | j \in \Lambda_0\}$, we suppose that $\boldsymbol{P}$ is 1-step TPM. Define a sequence of subsets as

$$\Lambda_s = \left\{ j \in \Lambda_{s-1} | \sum_{i \in \Lambda_{s-1}} [P]_{i,j} = 1 \right\},\ s = 1, 2, \ldots.$$

There must exist a k satisfying $k \leq |\mathcal{M}|$ such that $\Lambda_k = \Lambda_{k-1}$. In addition, $I_{\boldsymbol{P}}(\mathcal{M}) = \{\boldsymbol{\delta}_{2^n}^j | j \in \Lambda_k\}$.

Theorem 4 Given $\mathcal{M} \in \Delta_{2^n}$ with $I_{\boldsymbol{P}}(\mathcal{M}) = \{\boldsymbol{\delta}_{2^n}^j | j \in \Lambda\}$, then SRIBN is finite-time $\mathcal{M}$-stable with probability one, if and only if $\boldsymbol{P}_J^{2^n} = \boldsymbol{0}$ and $\sum_{i \in \Lambda} \mathrm{Row}_i[\boldsymbol{P}^{2^n}] = \boldsymbol{1}_{2^n}^{\mathrm{T}}$.

Proof Sufficiency. Suppose that conditions 1 and 2 hold. By Theorem 2 and condition 1, it is easy to see that SRIBN is forward complete. In addition, condition 2 implies

$$\Pr\{\boldsymbol{x}(t_{2^n}, j_{2^n}; \boldsymbol{x}_0) \in I_{\boldsymbol{P}}(\mathcal{M})\} = 1,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n}.$$

According to the property of $I_{\boldsymbol{P}}(\mathcal{M})$, it holds that

$$\Pr\{\boldsymbol{x}(t_k, j_k; \boldsymbol{x}_0) \in \mathcal{M}\} = 1,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n},\ \forall k \geq 2^n.$$

As a result, the SRIBN is finite-time $\mathcal{M}$-stable with probability one.

Necessity. Suppose that the SRIBN is finite-time $\mathcal{M}$-stable with probability one. Then, there must exist a nonnegative integer K such that

$$\Pr\{\boldsymbol{x}(t_k, j_k; \boldsymbol{x}_0) \in I_{\boldsymbol{P}}(\mathcal{M})\} = 1,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n},\ \forall k \geq K.$$

Otherwise, there exists $\boldsymbol{x}(t_\alpha, j_\alpha; \boldsymbol{x}_0) \in \mathcal{M} \backslash I_{\boldsymbol{P}}(\mathcal{M})$ for $\alpha \geq K$. Then, there must exist an integer $\beta > \alpha$ such that

$$\Pr\{\boldsymbol{x}(t_\beta, j_\beta; \boldsymbol{x}_0) \in \mathcal{M}\} < 1,$$

which contradicts the finite-time $\mathcal{M}$-stability with probability one. Thus, it holds that

$$\Pr\{\boldsymbol{x}(t_k, j_k; \boldsymbol{x}_0) \in I_{\boldsymbol{P}}(\mathcal{M})\} = 1,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n},\ \forall k \geq K,$$

implying that each initial state $\boldsymbol{x}_0$ can reach $I_{\boldsymbol{P}}(\mathcal{M})$ with probability one. Note that there are only 2^n states in the whole state space. Then the length of the shortest path from $\boldsymbol{x}_0$ to $I_{\boldsymbol{P}}(\mathcal{M})$ is less than 2^n. That is, there exists an integer $k < 2^n$ satisfying

$$\Pr\{\boldsymbol{x}(t_k, j_k; \boldsymbol{x}_0) \in I_{\boldsymbol{P}}(\mathcal{M})\} = 1,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n}.$$

Obviously, it holds that

$$\Pr\{\boldsymbol{x}(t_{2^n}, j_{2^n}; \boldsymbol{x}_0) \in I_{\boldsymbol{P}}(\mathcal{M})\} = 1,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n},$$

which is equivalent to condition 2. In addition, condition 1 directly follows the fact that SRIBN is forward complete.

4.2 Asymptotical stability in distribution

Note that SRIBN can be regarded as a special finite Markov chain evolving in the k-domain; that is, the step number k instead of the time t is viewed as the time parameter. Thus, the asymptotical stability analysis is closely related to that of ordinary finite Markov chains in Guo et al. (2019b).

Lemma 3 (Guo et al., 2019b) A finite Markov chain $\boldsymbol{x}(t) \in \Delta_{2^n}$ is asymptotically $\boldsymbol{x}_d$-stable in distribution; that is,

$$\lim_{t \to \infty} \Pr\{\boldsymbol{x}(t; \boldsymbol{x}_0) = \boldsymbol{x}_d\} = 1,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n},$$

if and only if $\boldsymbol{x}_d$ is a fixed point and for any $\boldsymbol{x}_0 \in \Delta_{2^n}$, there is an admissible path from $\boldsymbol{x}_0$ to $\boldsymbol{x}_d$.

Lemma 4 (Guo et al., 2019b) A finite Markov chain $\boldsymbol{x}(t) \in \Delta_{2^n}$ with TPM $\boldsymbol{P}$ is asymptotically $\mathcal{M}$-stable in distribution; that is,

$$\lim_{t \to \infty} \Pr\{\boldsymbol{x}(t; \boldsymbol{x}_0) \in \mathcal{M}\} = 1,\ \forall \boldsymbol{x}_0 \in \Delta_{2^n},$$

if and only if $I_{\boldsymbol{P}}(\mathcal{M})$ is nonempty and for any $\boldsymbol{x}_0 \in \Delta_{2^n}$, there is an admissible path from $\boldsymbol{x}_0$ to $I_{\boldsymbol{P}}(\mathcal{M})$.

Based on Lemmas 3 and 4 and the forward completeness criterion, the following necessary and sufficient conditions for asymptotical stability in distribution are obviously true.

Theorem 5 SRIBN is asymptotically $\boldsymbol{\delta}_{2^n}^i$-stable in distribution, if and only if $\boldsymbol{P}_J^{2^n} = \boldsymbol{0}$, $[P]_{i,i} = 1$, and $\mathrm{Row}_i[\boldsymbol{P}^{2^n}] > \boldsymbol{0}$.

Theorem 6 Given $\mathcal{M} \in \Delta_{2^n}$ with $I_{\boldsymbol{P}}(\mathcal{M}) = \{\boldsymbol{\delta}_{2^n}^j | j \in \Lambda\}$, SRIBN is asymptotically $\mathcal{M}$-stable in distribution if and only if $\boldsymbol{P}_J^{2^n} = \mathbf{0}$ and $\sum_{i\in\Lambda} \mathrm{Row}_i[\boldsymbol{P}^{2^n}] > \mathbf{0}$.

Example 2 Revisit the SRIBN in Example 1 with the target subset $\mathcal{M} = \{\boldsymbol{\delta}_8^1, \boldsymbol{\delta}_8^2, \boldsymbol{\delta}_8^5\}$. Check the $\mathcal{M}$-stability of the SRIBN. TPM of this SRIBN is calculated as

$$\begin{aligned}
\boldsymbol{P} &= \boldsymbol{P}_J + \boldsymbol{P}_S \\
&= \boldsymbol{G} \cdot \mathrm{diag}(\boldsymbol{p}_J) + \boldsymbol{L} \cdot \mathrm{diag}(\boldsymbol{p}_S) \\
&= \boldsymbol{\delta}_8[8,7,7,4,1,7,3,2] \\
&\quad \cdot\mathrm{diag}(0, 0.4, 0.5, 0, 1, 1, 0, 1) \\
&\quad +\boldsymbol{\delta}_8[5,6,3,5,1,4,1,7] \\
&\quad \cdot\mathrm{diag}(1, 0.6, 0.5, 1, 0, 0, 1, 0) \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Then,

$$\boldsymbol{P}^8 = \begin{bmatrix} 1 & 0.4 & 0.664 & 1 & 0 & 1 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.004 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0.328 & 0 & 1 & 0 & 1 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.004 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Lemma 2, we have $I_{\boldsymbol{P}}(\mathcal{M}) = \{\boldsymbol{\delta}_8^1, \boldsymbol{\delta}_8^5\}$ and $\Lambda = \{1, 5\}$. It can be easily checked that

$$\sum_{i\in\Lambda} \mathrm{Row}_i[\boldsymbol{P}^{2^n}] = [1, 1, 0.992, 1, 1, 1, 1, 1].$$

Besides, it holds that SRIBN is forward complete. Thus, by Theorems 4 and 6, SRIBN is not $\mathcal{M}$-stable with probability one in finite time, but it is asymptotically $\mathcal{M}$-stable in distribution. The time-domain simulation has been done, and three sample trajectories of the solution with initial state $\boldsymbol{\delta}_8^3$ are shown in Fig. 2, which verifies the convergence of the solution to the target subset.

Remark 2 Note that $\sum_{i\in\Lambda} \mathrm{Row}_i[\boldsymbol{P}^{2^n}] = \mathbf{1}_{2^n}$ implies $\sum_{i\in\Lambda} \mathrm{Row}_i[\boldsymbol{P}^{2^n}] > \mathbf{0}$. Thus, finite-time set stability with probability one implies asymptotical set stability in distribution. However, the inverse claim does not hold in general, as shown by Example 2.

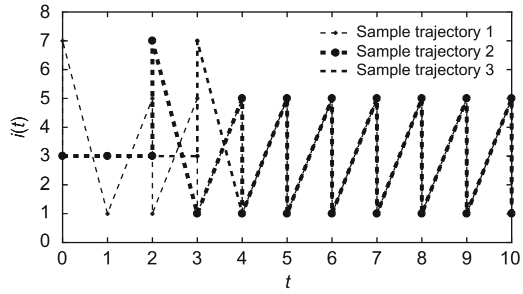

Fig. 2 Three sample trajectories of solution $\boldsymbol{x}(t; \boldsymbol{\delta}_8^3) = \boldsymbol{\delta}_8^{i(t)}$ over $0 \le t \le 10$ for Example 2

5 Summary

In this study, we studied the forward completeness and stability of the state-triggered random impulsive Boolean networks (SRIBNs). First, a hybrid index model was introduced to describe SRIBNs. Then, based on this model, several necessary and sufficient conditions were presented for the forward completeness of SRIBNs. Furthermore, several necessary and sufficient conditions were provided for finite-time stability with probability one and asymptotical stability in distribution. Finally, the relationships between these two kinds of stability were discussed.

Contributors

Yu-qian GUO proposed the main idea. Ya-wen SHEN derived the main results and drafted the manuscript. Wei-hua GUI guided the research. Yu-qian GUO revised and finalized the paper.

Compliance with ethics guidelines

Ya-wen SHEN, Yu-qian GUO, and Wei-hua GUI declare that they have no conflict of interest.

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