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Controllability of fractional-order damped systems with time-varying delays in control*

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Abstract: In this study, we focus on the controllability of fractional-order damped systems in linear and nonlinear cases with multiple time-varying delays in control. For the linear system based on the Mittag-Leffler matrix function, we define a controllability Gramian matrix, which is useful in judging whether the system is controllable or not. Furthermore, in two special cases, we present serval equivalent controllable conditions which are easy to verify. For the nonlinear system, under the controllability of its corresponding linear system, we obtain a sufficient condition on the nonlinear term to ensure that the system is controllable. Finally, two examples are given to illustrate the theory.

Key words: Controllability; Fractional-order damped systems; Time-varying delays; Gramian matrix
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1 Introduction

As a generalization of classical integer-order calculus, fractional-order calculus, or more appropriately, arbitrary real-order calculus was born at almost the same time as classical calculus. Recently, fractional-order differential equations have been widely used in assorted fields, including control theory, viscoelastic materials, and blood flow phenomena. Their hereditary property makes them precise in modeling the dynamic process (Miller and Ross, 1993; Podlubny, 1998; Hilfer, 2000; Kilbas et al., 2006; Monje et al., 2010).

Fractional control theory is an important and basic topic of application-oriented fractional-order calculus for information science. In recent years, the controllability of fractional-order systems has been widely researched (Balachandran and Park, 2009; Balachandran et al., 2015; Ge et al., 2016). It is worth remarking that time delays often occur in practical systems. There are many achievements concerning systems with delays (Gu et al., 2003; Zhou, 2014; Sikora and Klamka, 2017; Liu et al., 2019).

The controllability of fractional-order systems with time-varying delays in control was analyzed by Balachandran et al. (2012a, 2012b). Balachandran et al. (2012a) derived the necessary and sufficient conditions for the following system with $0 < p < 1$:

$$ {}_0^C D_t^p \boldsymbol{y}(t) = \boldsymbol{A}\boldsymbol{y}(t) + \sum_{i=0}^{M} \boldsymbol{B}_i \boldsymbol{u}[h_i(t)],\ t \in [0, T], $$

where $\boldsymbol{u}$ represents the control input and $h_i(t)$ ($i = 0, 1, \ldots, M$) are continuous functions denoting the time-varying delays. The control in Balachandran et al. (2012a) was explicitly constructed. In our

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previous work (He et al., 2016a), we investigated the following system:

$$ {}_0^C D_t^p \boldsymbol{y}(t) - \boldsymbol{A}{}_0^C D_t^q \boldsymbol{y}(t) = \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{C}\boldsymbol{u}(t-\tau),\ t \geq 0, $$

where $\boldsymbol{A}{}_0^C D_t^q \boldsymbol{y}(t)$ is the fractional-order damped term, $\boldsymbol{u}$ the control input, and τ the constant delay. A necessary and sufficient condition for controllability has been explored.

It is well-known that for a moving spring oscillator in oil or in a thicker liquid, the resistance is proportional to its speed. By Newton's Second Law, the location y of the oscillator can be expressed as $\ddot{y}(t) + a\dot{y}(t) + by(t) = 0$, where a is a constant related to the liquid and b a constant related to the spring. This system is called a damped system. In fractional-order calculus, the system ${}_0^C D_t^p \boldsymbol{y}(t) - a{}_0^C D_t^q \boldsymbol{y}(t) + b\boldsymbol{y}(t) = \boldsymbol{0}$ is often called a fractional damped system because the fractional derivative replaces the integer-order derivative. From a theoretical perspective, this extension from the integer order to the fractional order is meaningful.

Motivated mainly by Balachandran et al. (2012a) and He et al. (2016a), we concern the controllability of the following fractional-order damped systems with multiple time-varying delays in control:

$$ \begin{cases} {}_0^C D_t^p \boldsymbol{y}(t) - \boldsymbol{A}{}_0^C D_t^q \boldsymbol{y}(t) = \sum\limits_{i=0}^{M} \boldsymbol{B}_i \boldsymbol{u}[t-\tau_i(t)] \\ \qquad\qquad + f(t,\boldsymbol{y},\boldsymbol{u}),\ t \geq 0, \\ \boldsymbol{y}(0) = \boldsymbol{y}_0,\ \boldsymbol{y}'(0) = \boldsymbol{y}'_0, \\ \boldsymbol{u}(t) = \boldsymbol{\psi}(t),\ -\tau_M(0) \leq t \leq 0, \end{cases} \tag{1} $$

where $0 < q \leq 1 < p \leq 2$, $\boldsymbol{y} \in \mathbb{R}^n$ is a state vector, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is a control input vector, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B}_i \in \mathbb{R}^{n\times m}$ $(i = 0, 1, ..., M)$ are any matrices, $\tau_i(t) \geq 0$ $(i = 0, 1, \ldots, M)$ are the time-varying delays satisfying $0 = \tau_0(t) < \tau_1(t) < \ldots < \tau_M(t)$ and $t - \tau_i(t) \geq 0$ for $t \geq 0$, $\boldsymbol{\psi}(t)$ is the initial control function, and ${}_0^C D_t^p \boldsymbol{y}(t)$ and ${}_0^C D_t^q \boldsymbol{y}(t)$ denote the Caputo fractional derivatives of orders p and q for function $\boldsymbol{y}(t)$, respectively.

In particular, if we choose $\tau_i(t)$ as τ_i, $M = 1$, and $f(t, \boldsymbol{y}, \boldsymbol{u}) = \boldsymbol{0}$, then system (1) becomes

$$ {}_0^C D_t^p \boldsymbol{y}(t) - \boldsymbol{A}{}_0^C D_t^q \boldsymbol{y}(t) = \boldsymbol{B}_0\boldsymbol{u}(t) + \boldsymbol{B}_1\boldsymbol{u}(t-\tau_1),\ t \geq 0, $$

which is just the system in He et al. (2016a). Thus, system (1) is more general and the results can be regarded as a generalization of the theorem in He et al. (2016a).

2 Preliminaries

In this section, serval basic concepts and lemmas are presented, which are used throughout this study.

Definition 1 The fractional derivative of order p $(0 \leq n < p \leq n+1)$ in the Caputo sense for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$ {}_0^C D_t^p f(t) = \frac{1}{\Gamma(n-p+1)} \int_0^t \frac{f^{(n+1)}(\theta)}{(t-\theta)^{p-n}} \mathrm{d}\theta. $$

For fractional-order systems, a kind of function called the Mittag-Leffler function is important for obtaining the solutions.

Definition 2 The one-parameter Mittag-Leffler function $E_p(z)$ with $p > 0$ and the two-parameter Mittag-Leffler function $E_{p,q}(z)$ with $p, q > 0$ are defined as

$$ E_p(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(pj+1)},\ p > 0,\ z \in \mathbb{C}, $$

$$ E_{p,q}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(pj+q)},\ p > 0,\ q > 0,\ z \in \mathbb{C}. $$

Definition 3 We call system (1) controllable on $J = [0, T]$ if, for vectors $\boldsymbol{y}_0, \boldsymbol{y}'_0, \boldsymbol{y}_1 \in \mathbb{R}^n$, we can find a control input function $\boldsymbol{u} \in C(J)$ such that the corresponding solution to system (1) with $\boldsymbol{y}(0) = \boldsymbol{y}_0$ and $\boldsymbol{y}'(0) = \boldsymbol{y}'_0$ satisfies $\boldsymbol{y}(T) = \boldsymbol{y}_1$.

For further discussion, we need to present two assumptions about the delays.

Assumption 1 The delay functions $\tau_i : [0, T] \to \mathbb{R}$ $(i = 0, 1, \ldots, M)$ are continuously differentiable. Moreover,

$$ \tau_i(t) \geq 0,\ t \in [0, T],\ i = 0, 1, \ldots, M. $$

Assumption 2 There exist functions $r_i : [-\tau_i(0), T - \tau_i(T)] \to [0, T]$ $(i = 0, 1, \ldots, M)$ such that $r_i[t - \tau_i(t)] = t$ for $t \in [0, T]$.

Lemma 1 (Balachandran et al., 2015) For a continuous function $f : [0, T] \to \mathbb{R}^n$, the general solution to system

$$ \begin{cases} {}_0^C D_t^p \boldsymbol{y}(t) - \boldsymbol{A}{}_0^C D_t^q \boldsymbol{y}(t) = f(t),\ t \geq 0, \\ \boldsymbol{y}(0) = \boldsymbol{y}_0,\ \boldsymbol{y}'(0) = \boldsymbol{y}'_0 \end{cases} $$

with $0 < q \leq 1 < p \leq 2$ can be written as

$$ \begin{aligned} \boldsymbol{y}(t) = & \boldsymbol{E}_{p-q}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 - \boldsymbol{A}t^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 \\ & + t\boldsymbol{E}_{p-q,2}(\boldsymbol{A}t^{p-q})\boldsymbol{y}'_0 \\ & + \int_0^t (t-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(t-s)^{p-q}]f(s)\mathrm{d}s. \end{aligned} $$

Lemma 2 (He et al., 2016b) If $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, then matrix $\boldsymbol{E}_\gamma(\boldsymbol{A}t^\gamma) - \boldsymbol{A}t^\gamma \boldsymbol{E}_{\gamma,\gamma+1}(\boldsymbol{A}t^\gamma)$ is invertible for $\gamma > 0$.

Lemma 3 (He et al., 2016a) Let $\mu > -1$. Assume that $\varphi(\tilde{T}, s)$ is a nonnegative continuous function on $[\bar{T}, \tilde{T}]$ with s and $\varphi(\tilde{T}, s)$ meeting

$$\int_{\bar{T}}^{\tilde{T}} (\tilde{T} - s)^\mu \varphi(\tilde{T}, s)\mathrm{d}s = 0.$$

Then, $\varphi(\tilde{T}, s) \equiv 0$ for $s \in [\bar{T}, \tilde{T}]$.

3 Controllability of linear systems

Consider the following linear damped system with multiple time-varying delays in control:

$$\begin{cases} {}_0^C D_t^p \boldsymbol{y}(t) - \boldsymbol{A}\,{}_0^C D_t^q \boldsymbol{y}(t) = \displaystyle\sum_{i=0}^{M} \boldsymbol{B}_i \boldsymbol{u}[t-\tau_i(t)],\ t\geq 0, \\ \boldsymbol{y}(0) = \boldsymbol{y}_0,\ \boldsymbol{y}'(0) = \boldsymbol{y}_0', \\ \boldsymbol{u}(t) = \boldsymbol{\psi}(t),\ -\tau_M(0) \leq t \leq 0. \end{cases} \tag{2}$$

From Lemma 1, Lemma 4 is immediately derived.

Lemma 4 The general solution to system (2) can be written as

$$\begin{aligned} \boldsymbol{y}(t) =& \boldsymbol{E}_{p-q}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 - \boldsymbol{A}t^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 \\ &+ t\boldsymbol{E}_{p-q,2}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0' \\ &+ \int_0^t (t-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(t-s)^{p-q}] \\ &\cdot \sum_{i=0}^{M} \boldsymbol{B}_i \boldsymbol{u}[s-\tau_i(s)]\mathrm{d}s. \end{aligned} \tag{3}$$

Using the properties of functions $r_i(t)$, the solution to system (3) becomes

$$\begin{aligned} \boldsymbol{y}(t) =& \boldsymbol{E}_{p-q}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 - \boldsymbol{A}t^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 \\ &+ t\boldsymbol{E}_{p-q,2}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0' \\ &+ \sum_{i=0}^{M} \int_{-\tau_i(0)}^{t-\tau_i(t)} [t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i \dot{r}_i(s)\boldsymbol{u}(s)\mathrm{d}s. \end{aligned}$$

Note that $\tau_i(t)$ is increasing with i. Then $t-\tau_i(t)$ is decreasing with i. Now using the properties of $\tau_i(t)$, state $\boldsymbol{y}(t)$ can be expressed as

$$\begin{aligned} \boldsymbol{y}(t) =& \boldsymbol{E}_{p-q}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 - \boldsymbol{A}t^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 \\ &+ t\boldsymbol{E}_{p-q,2}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0' \\ &+ \sum_{i=0}^{M} \int_{-\tau_i(0)}^{0} [t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i \dot{r}_i(s)\boldsymbol{\psi}(s)\mathrm{d}s \\ &+ \sum_{i=0}^{M} \int_{0}^{t-\tau_i(t)} [t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i \dot{r}_i(s)\boldsymbol{u}(s)\mathrm{d}s \\ =& \boldsymbol{E}_{p-q}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 - \boldsymbol{A}t^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 \\ &+ t\boldsymbol{E}_{p-q,2}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0' \\ &+ \sum_{i=0}^{M} \int_{-\tau_i(0)}^{0} [t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i \dot{r}_i(s)\boldsymbol{\psi}(s)\mathrm{d}s \\ &+ \int_0^{t-\tau_M(t)} \sum_{i=0}^{M} [t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i \dot{r}_i(s)\boldsymbol{u}(s)\mathrm{d}s \\ &+ \int_{t-\tau_M(t)}^{t-\tau_{M-1}(t)} \sum_{i=0}^{M-1} [t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i \dot{r}_i(s)\boldsymbol{u}(s)\mathrm{d}s + \ldots \\ &+ \int_{t-\tau_1(t)}^{t} [t-r_0(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_0(s)]^{p-q}\}\boldsymbol{B}_0 \dot{r}_0(s)\boldsymbol{u}(s)\mathrm{d}s. \end{aligned} \tag{4}$$

For simplification, we introduce the notations as

$$\begin{cases} \boldsymbol{G}_j(t,s) = \displaystyle\sum_{i=0}^{j} [t-r_i(s)]^{p-1}\boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\} \\ \qquad\qquad \cdot \boldsymbol{B}_i \dot{r}_i(s),\ j = 0,1,\ldots,M, \\ \boldsymbol{H}(t) = \boldsymbol{E}_{p-q}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 - \boldsymbol{A}t^{p-q}\boldsymbol{E}_{p-q,p-q+1} \\ \qquad\qquad \cdot (\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 + t\boldsymbol{E}_{p-q,2}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0', \\ \boldsymbol{I}(t) = \displaystyle\sum_{i=0}^{M} \int_{-\tau_i(0)}^{0} [t-r_i(s)]^{p-1} \\ \qquad\qquad \cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i \dot{r}_i(s)\boldsymbol{\psi}(s)\mathrm{d}s, \end{cases}$$

and the controllability Gramian matrix as

$$\begin{aligned} \boldsymbol{W} =& \sum_{j=0}^{M-1} \int_{T-\tau_{j+1}(T)}^{T-\tau_j(T)} \boldsymbol{G}_j(T,s)\boldsymbol{G}_j^{\mathrm{T}}(T,s)\mathrm{d}s \\ &+ \int_0^{T-\tau_M(T)} \boldsymbol{G}_M(T,s)\boldsymbol{G}_M^{\mathrm{T}}(T,s)\mathrm{d}s. \end{aligned} \tag{5}$$

Therefore, the solution $\boldsymbol{y}(t)$ can be rewritten as

$$\begin{aligned}\boldsymbol{y}(t) =&\boldsymbol{H}(t)+\boldsymbol{I}(t)+\int_0^{t-\tau_M(t)}\boldsymbol{G}_M(t,s)\boldsymbol{u}(s)\mathrm{d}s\\&+\int_{t-\tau_M(t)}^{t-\tau_{M-1}(t)}\boldsymbol{G}_{M-1}(t,s)\boldsymbol{u}(s)\mathrm{d}s+\ldots\\&+\int_{t-\tau_1(t)}^{t}\boldsymbol{G}_0(t,s)\boldsymbol{u}(s)\mathrm{d}s.\end{aligned}$$

Theorem 1 The linear fractional damped system (2) is controllable on $[0,T]$, if and only if the controllability Gramian matrix $\boldsymbol{W}$ is positive definite.

Proof First, if $\boldsymbol{W}$ is positive definite, then $\boldsymbol{W}$ is invertible. For any given vectors $\boldsymbol{y}_0,\boldsymbol{y}_0',\boldsymbol{y}_1\in\mathbb{R}^n$, we can construct the control input $\boldsymbol{u}(t)$ as

$$\boldsymbol{u}(t)=\begin{cases}\boldsymbol{G}_M^{\mathrm{T}}(T,t)\boldsymbol{W}^{-1}\big[\boldsymbol{y}_1-\boldsymbol{H}(t)-\boldsymbol{I}(t)\big],\\\qquad t\in(0,T-\tau_M(T)],\\\boldsymbol{G}_{M-1}^{\mathrm{T}}(T,t)\boldsymbol{W}^{-1}\big[\boldsymbol{y}_1-\boldsymbol{H}(t)-\boldsymbol{I}(t)\big],\\\qquad t\in(T-\tau_M(T),T-\tau_{M-1}(T)],\\\qquad\vdots\\\boldsymbol{G}_0^{\mathrm{T}}(T,t)\boldsymbol{W}^{-1}\big[\boldsymbol{y}_1-\boldsymbol{H}(t)-\boldsymbol{I}(t)\big],\\\qquad t\in(T-\tau_1(T),T].\end{cases}\tag{6}$$

Substituting Eq. (6) into Eq. (4) and using Eq. (5), we can obtain

$$\begin{aligned}\boldsymbol{y}(T) =&\boldsymbol{H}(T)+\boldsymbol{I}(T)\\&+\int_0^{T-\tau_M(T)}\boldsymbol{G}_M(T,s)\boldsymbol{G}_M^{\mathrm{T}}(T,s)\\&\cdot\boldsymbol{W}^{-1}\big[\boldsymbol{y}_1-\boldsymbol{H}(T)-\boldsymbol{I}(T)\big]\mathrm{d}s\\&+\int_{T-\tau_M(T)}^{T-\tau_{M-1}(T)}\boldsymbol{G}_{M-1}(T,s)\boldsymbol{G}_{M-1}^{\mathrm{T}}(T,s)\\&\cdot\boldsymbol{W}^{-1}\big[\boldsymbol{y}_1-\boldsymbol{H}(T)-\boldsymbol{I}(T)\big]\mathrm{d}s+\ldots\\&+\int_{T-\tau_1(T)}^{T}\boldsymbol{G}_0(T,s)\boldsymbol{G}_0^{\mathrm{T}}(T,s)\\&\cdot\boldsymbol{W}^{-1}\big[\boldsymbol{y}_1-\boldsymbol{H}(T)-\boldsymbol{I}(T)\big]\mathrm{d}s\\=&\boldsymbol{y}_1.\end{aligned}$$

This means that system (2) is controllable.

Now we show the necessity by reductio ad absurdum. If $\boldsymbol{W}$ is not positive definite, then there exists $\boldsymbol{x}\neq\boldsymbol{0}$ such that

$$\begin{aligned}\boldsymbol{0} =&\boldsymbol{x}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{x}\\=&\sum_{j=0}^{M-1}\boldsymbol{x}^{\mathrm{T}}\int_{T-\tau_{j+1}(T)}^{T-\tau_j(T)}\boldsymbol{G}_j(T,s)\boldsymbol{G}_j^{\mathrm{T}}(T,s)\boldsymbol{x}\mathrm{d}s\\&+\boldsymbol{x}^{\mathrm{T}}\int_0^{T-\tau_M(T)}\boldsymbol{G}_M(T,s)\boldsymbol{G}_M^{\mathrm{T}}(T,s)\boldsymbol{x}\mathrm{d}s.\end{aligned}$$

Hence, for $s\in[0,T]$ and $j=0,1,\ldots,M$, we have

$$\begin{aligned}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{G}_j(T,s) =&\boldsymbol{x}^{\mathrm{T}}\sum_{i=0}^{j}\boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[T-r_i(s)]^{p-q}\}\\&\cdot[T-r_i(s)]^{p-1}\boldsymbol{B}_i\dot{r}_i(s)=\boldsymbol{0}.\end{aligned}\tag{7}$$

By Lemma 2, $[\boldsymbol{E}_{p-q}(\boldsymbol{A}T^{p-q})-\boldsymbol{A}T^{p-q}\cdot\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}T^{p-q})]^{-1}$ exists. Now we choose $\boldsymbol{y}_0$ and $\boldsymbol{y}_0'$ such that

$$\begin{aligned}\boldsymbol{y}_0 =&[\boldsymbol{E}_{p-q}(\boldsymbol{A}T^{p-q})-\boldsymbol{A}T^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}T^{p-q})]^{-1}\\&\cdot[\boldsymbol{x}-T\boldsymbol{E}_{p-q,2}(\boldsymbol{A}T^{p-q})\boldsymbol{y}_0'],\end{aligned}$$

and choose $\boldsymbol{\psi}(s)=\boldsymbol{0}$. By hypothesis, system (2) is controllable. That is, there exists a control input $\boldsymbol{u}\in C[0,T]$ such that the state can be steered to the origin from the initial state in the interval of $[0,T]$. It follows that

$$\begin{aligned}\boldsymbol{y}(T) =&\boldsymbol{E}_{p-q}(\boldsymbol{A}T^{p-q})\boldsymbol{y}_0\\&-\boldsymbol{A}T^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}T^{p-q})\boldsymbol{y}_0\\&+T\boldsymbol{E}_{p-q,2}(\boldsymbol{A}T^{p-q})\boldsymbol{y}_0'\\&+\int_0^{T-\tau_M(T)}\boldsymbol{G}_M(T,s)\boldsymbol{u}(s)\mathrm{d}s\\&+\int_{T-\tau_M(T)}^{T-\tau_{M-1}(T)}\boldsymbol{G}_{M-1}(T,s)\boldsymbol{u}(s)\mathrm{d}s+\ldots\\&+\int_{T-\tau_1(T)}^{T}\boldsymbol{G}_0(T,s)\boldsymbol{u}(s)\mathrm{d}s\\=&\boldsymbol{x}+\int_0^{T-\tau_M(T)}\boldsymbol{G}_M(T,s)\boldsymbol{u}(s)\mathrm{d}s\\&+\int_{T-\tau_M(T)}^{T-\tau_{M-1}(T)}\boldsymbol{G}_{M-1}(T,s)\boldsymbol{u}(s)\mathrm{d}s+\ldots\\&+\int_{T-\tau_1(T)}^{T}\boldsymbol{G}_0(T,s)\boldsymbol{u}(s)\mathrm{d}s\\=&\boldsymbol{0}.\end{aligned}$$

Therefore, we can obtain

$$\begin{aligned}\mathbf{0}=&\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}+\int_0^{T-\tau_M(T)}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{G}_M(T,s)\boldsymbol{u}(s)\mathrm{d}s\\&+\int_{T-\tau_M(T)}^{T-\tau_{M-1}(T)}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{G}_{M-1}(T,s)\boldsymbol{u}(s)\mathrm{d}s+\ldots\\&+\int_{T-\tau_1(T)}^{T}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{G}_0(T,s)\boldsymbol{u}(s)\mathrm{d}s.\end{aligned}\tag{8}$$

It follows from Eqs. (7) and (8) that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}=\mathbf{0}$. This is a contradiction to the fact that $\boldsymbol{x}$ is nonzero. Thus, $\boldsymbol{W}$ is invertible.

4 Two special cases of the linear system

Suppose that the delays in system (2) are independent of t. Then system (2) becomes

$$\begin{cases}{}_0^CD_t^p\boldsymbol{y}(t)-\boldsymbol{A}_0^CD_t^q\boldsymbol{y}(t)=\sum\limits_{i=0}^{M}\boldsymbol{B}_i\boldsymbol{u}(t-\tau_i),\ t\geq0,\\ \boldsymbol{y}(0)=\boldsymbol{y}_0,\ \boldsymbol{y}'(0)=\boldsymbol{y}'_0,\\ \boldsymbol{u}(t)=\boldsymbol{\psi}(t),\ -\tau_M(0)\leq t\leq0,\end{cases}\tag{9}$$

where $0=\tau_0<\tau_1<\ldots<\tau_M<T$.

Then for $j=0,1,\ldots,M$, functions $\boldsymbol{G}_j(t,s)$ can be expressed as

$$\begin{aligned}\boldsymbol{G}_j(t,s)=&\sum_{i=0}^{j}(t-s-\tau_i)^{p-1}\\&\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(t-s-\tau_i)^{p-q}]\boldsymbol{B}_i,\end{aligned}$$

and we can obtain the controllability Gramian matrix as

$$\begin{aligned}\boldsymbol{W}=&\sum_{j=0}^{M-1}\int_{T-\tau_{j+1}}^{T-\tau_j}\Big\{\sum_{i=0}^{j}(T-s-\tau_i)^{p-1}\\&\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_i)^{p-q}]\boldsymbol{B}_i\Big\}\\&\cdot\Big\{\sum_{i=0}^{j}(T-s-\tau_i)^{p-1}\\&\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_i)^{p-q}]\boldsymbol{B}_i\Big\}^{\mathrm{T}}\mathrm{d}s\\&+\int_0^{T-\tau_M}\Big\{\sum_{i=0}^{M}(T-s-\tau_i)^{p-1}\\&\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_i)^{p-q}]\boldsymbol{B}_i\Big\}\\&\cdot\Big\{\sum_{i=0}^{M}(T-s-\tau_i)^{p-1}\\&\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_i)^{p-q}]\boldsymbol{B}_i\Big\}^{\mathrm{T}}\mathrm{d}s.\end{aligned}\tag{10}$$

Then, the controllability of system (9) can be verified by the invertibility of Gramian matrix (10).

Corollary 1 The fractional-order damped system (9) with multiple constant delays is controllable if and only if

$$\begin{aligned}&\mathrm{rank}[\boldsymbol{B}_0,\boldsymbol{AB}_0,\boldsymbol{A}^2\boldsymbol{B}_0,\ldots,\boldsymbol{A}^{n-1}\boldsymbol{B}_0,\boldsymbol{B}_1,\boldsymbol{AB}_1,\boldsymbol{A}^2\boldsymbol{B}_1,\\&\ldots,\boldsymbol{A}^{n-1}\boldsymbol{B}_1,\ldots,\boldsymbol{B}_M,\boldsymbol{AB}_M,\boldsymbol{A}^2\boldsymbol{B}_M,\ldots,\boldsymbol{A}^{n-1}\boldsymbol{B}_M]\\&=n.\end{aligned}$$

Proof Denote

$$\begin{aligned}&\langle\boldsymbol{A}|\boldsymbol{B}_0,\boldsymbol{B}_1,\ldots,\boldsymbol{B}_M\rangle\\=&\theta_0+\boldsymbol{A}\theta_0+\boldsymbol{A}^2\theta_0+\ldots+\boldsymbol{A}^{n-1}\theta_0+\theta_1+\boldsymbol{A}\theta_1\\&+\boldsymbol{A}^2\theta_1+\ldots+\boldsymbol{A}^{n-1}\theta_1+\ldots+\theta_M+\boldsymbol{A}\theta_M\\&+\boldsymbol{A}^2\theta_M+\ldots+\boldsymbol{A}^{n-1}\theta_M,\end{aligned}$$

where n is the order of $\boldsymbol{A}$ and $\theta_i=\mathrm{Im}\boldsymbol{B}_i$. Here, we view matrices $\boldsymbol{B}_i$ $(i=0,1,\ldots,M)$ as the linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$, and denote $\mathrm{Im}\boldsymbol{B}_i$ the images of these linear transformations which are subspaces of $\mathbb{R}^n$. It is easy to see that $\langle\boldsymbol{A}|\boldsymbol{B}_0,\boldsymbol{B}_1,\ldots,\boldsymbol{B}_M\rangle$ is spanned by the columns of matrix

$$\begin{aligned}&[\boldsymbol{B}_0,\boldsymbol{AB}_0,\boldsymbol{A}^2\boldsymbol{B}_0,\ldots,\boldsymbol{A}^{n-1}\boldsymbol{B}_0,\boldsymbol{B}_1,\boldsymbol{AB}_1,\boldsymbol{A}^2\boldsymbol{B}_1,\ldots,\\&\boldsymbol{A}^{n-1}\boldsymbol{B}_1,\ldots,\boldsymbol{B}_M,\boldsymbol{AB}_M,\boldsymbol{A}^2\boldsymbol{B}_M,\ldots,\boldsymbol{A}^{n-1}\boldsymbol{B}_M].\end{aligned}$$

Note that $\boldsymbol{W}(T)$ is invertible and is equivalent to $\mathrm{Im}\boldsymbol{W}(T)=\langle\boldsymbol{A}|\boldsymbol{B}_0,\boldsymbol{B}_1,\ldots,\boldsymbol{B}_M\rangle$.

Now we prove that $\mathrm{Im}\boldsymbol{W}=\langle\boldsymbol{A}|\boldsymbol{B}_0,\boldsymbol{B}_1,\ldots,\boldsymbol{B}_M\rangle$. This is equivalent to show

$$\begin{aligned}\mathrm{Ker}\boldsymbol{W}=&\bigcap_{i_0=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_0}\bigcap_{i_1=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_1^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_1}\bigcap_{i_2=0}^{n-1}\\&\cdots\bigcap_{i_M=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_M^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_M}.\end{aligned}\tag{11}$$

First, we show that

$$\begin{aligned}\mathrm{Ker}\boldsymbol{W}\subset&\bigcap_{i_0=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_0}\bigcap_{i_1=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_1^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_1}\bigcap_{i_2=0}^{n-1}\\&\cdots\bigcap_{i_M=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_M^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_M}.\end{aligned}$$

Indeed, for any given $\boldsymbol{y}\in\mathrm{Ker}\boldsymbol{W}(T)$ and $\boldsymbol{y}\neq\mathbf{0}$,

$$\begin{aligned}\mathbf{0} =&\boldsymbol{y}^{\mathrm{T}}\boldsymbol{W}(T)\boldsymbol{y}\\ =&\int_{T-\tau_1}^{T}\left\{\{(T-s)^{p-1}\right.\\ &\left.\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\}^{\mathrm{T}}\boldsymbol{y}\right\}^2\mathrm{d}s\\ &+\int_{T-\tau_2}^{T-\tau_1}\left\|\left\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\right.\right.\\ &+(T-s-\tau_1)^{p-1}\\ &\left.\left.\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_1)^{p-q}]\boldsymbol{B}_1\right\}^{\mathrm{T}}\boldsymbol{y}\right\|^2\mathrm{d}s+\ldots\\ &+\int_{T-\tau_M}^{T-\tau_{M-1}}\left\|\left\{(T-s)^{p-1}\right.\right.\\ &\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0+\ldots\\ &+(T-s-\tau_{M-1})^{p-1}\\ &\left.\left.\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_{M-1})^{p-q}]\boldsymbol{B}_{M-1}\right\}^{\mathrm{T}}\boldsymbol{y}\right\|^2\mathrm{d}s\\ &+\int_{0}^{T-\tau_M}\left\|\left\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\right.\right.\\ &+\ldots+(T-s-\tau_{M-1})^{p-1}\\ &\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_{M-1})^{p-q}]\boldsymbol{B}_{M-1}\\ &+(T-s-\tau_M)^{p-1}\\ &\left.\left.\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_M)^{p-q}]\boldsymbol{B}_M\right\}^{\mathrm{T}}\boldsymbol{y}\right\|^2\mathrm{d}s,\end{aligned}$$

which means that

$$\begin{aligned}&\int_{T-\tau_1}^{T}\left\|\left\{(T-s)^{p-1}\right.\right.\\ &\left.\left.\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\right\}^{\mathrm{T}}\boldsymbol{y}\right\|^2\mathrm{d}s=0,\end{aligned}\qquad(12)$$

$$\begin{aligned}&\int_{T-\tau_2}^{T-\tau_1}\left\|\left\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\right.\right.\\ &+(T-s-\tau_1)^{p-1}\\ &\left.\left.\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_1)^{p-q}]\boldsymbol{B}_1\right\}^{\mathrm{T}}\boldsymbol{y}\right\|^2\mathrm{d}s=0,\end{aligned}\qquad(13)$$

$$\vdots$$

$$\begin{aligned}&\int_{T-\tau_M}^{T-\tau_{M-1}}\left\|\left\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\right.\right.\\ &+\ldots+(T-s-\tau_{M-1})^{p-1}\\ &\left.\left.\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_{M-1})^{p-q}]\boldsymbol{B}_{M-1}\right\}^{\mathrm{T}}\boldsymbol{y}\right\|^2\mathrm{d}s=0,\end{aligned}$$

$$\begin{aligned}&\int_{0}^{T-\tau_M}\left\|\left\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0+\ldots\right.\right.\\ &+(T-s-\tau_{M-1})^{p-1}\\ &\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_{M-1})^{p-q}]\boldsymbol{B}_{M-1}\\ &+(T-s-\tau_M)^{p-1}\\ &\left.\left.\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_M)^{p-q}]\boldsymbol{B}_M\right\}^{\mathrm{T}}\boldsymbol{y}\right\|^2\mathrm{d}s=0.\end{aligned}$$

By Lemma 3 and Eq. (12), for $s\in[T-\tau_1,T]$, we have

$$\begin{aligned}\mathbf{0}=&\boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]^{\mathrm{T}}\boldsymbol{y}\\ =&\boldsymbol{B}_0^{\mathrm{T}}\sum_{k=0}^{\infty}\frac{(\boldsymbol{A}^{\mathrm{T}})^k(T-s)^{k(p-q)}}{\Gamma[k(p-q)+p]}\boldsymbol{y}.\end{aligned}\qquad(14)$$

Let $s=T$ in Eq. (14). Then $\boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$. Together with Eq. (12), we have

$$\begin{aligned}&\int_{T-\tau_1}^{T}(T-s)^{p-1+p-q}\\ &\cdot\left\|\boldsymbol{B}_0^{\mathrm{T}}\sum_{k=1}^{\infty}\frac{(\boldsymbol{A}^{\mathrm{T}})^k(T-s)^{(k-1)(p-q)}}{\Gamma[k(p-q)+p]}\boldsymbol{y}\right\|^2\mathrm{d}s=0.\end{aligned}$$

It follows from Lemma 3 that for $s\in[T-\tau_1,T]$,

$$\mathbf{0}=\boldsymbol{B}_0^{\mathrm{T}}\sum_{k=1}^{\infty}\frac{(\boldsymbol{A}^{\mathrm{T}})^k(T-s)^{(k-1)(p-q)}}{\Gamma[k(p-q)+p]}\boldsymbol{y}.\qquad(15)$$

By taking $s=T$ in Eq. (15), we have $\boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$. By mathematical induction, we can have

$$\boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^k\boldsymbol{x}=\mathbf{0},\ k=2,3,\ldots,n-1.\qquad(16)$$

According to the Cayley-Hamilton theorem, there exist functions $\gamma_0(t),\gamma_1(t),\ldots,\gamma_{n-1}(t)$ defined on $[0,\infty)$ such that

$$\boldsymbol{E}_{p-q,p}(\boldsymbol{A}T^{p-q})=\sum_{i=0}^{n-1}\gamma_i(T)\boldsymbol{A}^i.$$

This together with Eq. (16) yields

$$[\boldsymbol{E}_{p-q,p}(\boldsymbol{A}T^{p-q})\boldsymbol{B}_0]^{\mathrm{T}}\boldsymbol{y}\equiv\mathbf{0}.$$

Also, combined with Eq. (13), we have

$$\begin{aligned}&\int_{T-\tau_2}^{T-\tau_1}\left\|(T-\tau_1-s)^{p-1}\right.\\ &\left.\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-\tau_1-s)^{p-q}]\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{y}\right\|^2\mathrm{d}s=0.\end{aligned}$$

Similar to Eqs. (14)–(16), we have

$$\boldsymbol{B}_1^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^k\boldsymbol{x}=\boldsymbol{0},\ k=0,1,\ldots,n-1. \tag{17}$$

Similarly, we can derive that for $i=2,3,\ldots,M$,

$$\boldsymbol{B}_i^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^k\boldsymbol{x}=\boldsymbol{0},\ k=0,1,\ldots,n-1. \tag{18}$$

From Eqs. (16)–(18), we obtain

$$\boldsymbol{y}\in\bigcap_{i_0=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_0}\bigcap_{i_1=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_1^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_1}\bigcap_{i_2=0}^{n-1}\cdots\bigcap_{i_M=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_M^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_M},$$

that is,

$$\mathrm{Ker}\boldsymbol{W}\subset\bigcap_{i_0=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_0}\bigcap_{i_1=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_1^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_1}\bigcap_{i_2=0}^{n-1}\cdots\bigcap_{i_M=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_M^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_M}. \tag{19}$$

Conversely, for any given

$$\boldsymbol{y}\in\bigcap_{i_0=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_0}\bigcap_{i_1=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_1^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_1}\bigcap_{i_2=0}^{n-1}\cdots\bigcap_{i_M=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_M^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_M},$$

Eq. (18) is true. For $T-\tau_1<s\le T$,

$$\begin{aligned}&\boldsymbol{B}_0^{\mathrm{T}}(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]^{\mathrm{T}}\boldsymbol{y}\\&=\sum_{i_0=0}^{n-1}\gamma_{i_0}(T-s)(T-s)^{p-1}\boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_0}\boldsymbol{y}=\boldsymbol{0};\end{aligned} \tag{20}$$

for $T-\tau_2<s\le T-\tau_1$,

$$\begin{aligned}&\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\\&+(T-\tau_1-s)\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-\tau_1-s)^{p-q}]\boldsymbol{B}_1\}^{\mathrm{T}}\boldsymbol{y}\\&=\sum_{i_0=0}^{n-1}\gamma_{i_0}(T-s)(T-s)^{p-1}\boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_0}\boldsymbol{y}\\&+\sum_{i_1=0}^{n-1}\gamma_{i_1}(T-\tau_1-s)\\&\cdot(T-\tau_1-s)^{p-1}\boldsymbol{B}_1^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_1}\boldsymbol{y}=\boldsymbol{0};\end{aligned} \tag{21}$$

$$\vdots$$

for $0<s\le T-\tau_M$,

$$\begin{aligned}&\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0+\ldots\\&+(T-\tau_M-s)\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-\tau_M-s)^{p-q}]\boldsymbol{B}_M\}^{\mathrm{T}}\boldsymbol{y}\\&=\sum_{i_0=0}^{n-1}\gamma_{i_0}(T-s)(T-s)^{p-1}\boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_0}\boldsymbol{y}+\ldots\\&+\sum_{i_M=0}^{n-1}\gamma_{i_M}(T-\tau_M-s)(T-\tau_M-s)^{p-1}\boldsymbol{B}_M^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_M}\boldsymbol{y}\\&=\boldsymbol{0}.\end{aligned} \tag{22}$$

Therefore, $\boldsymbol{y}\in\mathrm{Ker}\boldsymbol{W}(T)$, that is

$$\mathrm{Ker}\boldsymbol{W}\supset\bigcap_{i_0=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_0}\bigcap_{i_1=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_1^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_1}\bigcap_{i_2=0}^{n-1}\cdots\bigcap_{i_M=0}^{n-1}\mathrm{Ker}\boldsymbol{B}_M^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{i_M}. \tag{23}$$

From Eqs. (19) and (23), we know that Eq. (11) is correct and the proof is finished.

In particular, taking $M=1$, system (9) becomes

$${}_0^CD_t^p\boldsymbol{y}(t)-\boldsymbol{A}_0^CD_t^q\boldsymbol{y}(t)=\boldsymbol{B}_0\boldsymbol{u}(t)+\boldsymbol{B}_1\boldsymbol{u}(t-\tau_1), \tag{24}$$

and we can have Corollary 2 immediately, which is just Theorem 3.1 in He et al. (2016a).

Corollary 2 The controllability of the fractional damped system with control delay (24) holds if and only if

$$\begin{aligned}&\mathrm{rank}[\boldsymbol{B}_0,\boldsymbol{AB}_0,\boldsymbol{A}^2\boldsymbol{B}_0,\ldots,\boldsymbol{A}^{n-1}\boldsymbol{B}_0,\boldsymbol{B}_1,\boldsymbol{AB}_1,\\&\boldsymbol{A}^2\boldsymbol{B}_1,\ldots,\boldsymbol{A}^{n-1}\boldsymbol{B}_1]=n.\end{aligned}$$

Remark 1 By Eq. (10), the controllability Gramian matrix of system (24) can be written as

$$\begin{aligned}\boldsymbol{W}=&\int_{T-\tau_1}^{T}\left\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\right\}\\&\cdot\left\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\right\}^{\mathrm{T}}\mathrm{d}s\\&+\int_0^{T-\tau_1}\Big\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\\&+(T-s-\tau_1)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_1)^{p-q}]\boldsymbol{B}_1\Big\}\\&\cdot\Big\{(T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\boldsymbol{B}_0\\&+(T-s-\tau_1)^{p-1}\\&\cdot\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s-\tau_1)^{p-q}]\boldsymbol{B}_1\Big\}^{\mathrm{T}}\mathrm{d}s.\end{aligned}$$

This matrix is in accordance with the results in He et al. (2016a), and thus the results in this study have a broader range of applications than the results in He et al. (2016a).

5 Controllability of the nonlinear system

Consider the nonlinear fractional damped system whose linear part is the same as system (2) described in the following form:

$$\begin{cases} {}_0^C D_t^p \boldsymbol{y}(t) - \boldsymbol{A}{}_0^C D_t^q \boldsymbol{y}(t) = \sum_{i=0}^{M} \boldsymbol{B}_i \boldsymbol{u}[t-\tau_i(t)] \\ \qquad\qquad + f(t,\boldsymbol{y},\boldsymbol{u}),\ t\geq 0, \\ \boldsymbol{y}(0)=\boldsymbol{y}_0,\ \boldsymbol{y}'(0)=\boldsymbol{y}_0', \\ \boldsymbol{u}(t)=\boldsymbol{\psi}(t),\ -\tau_M(0)\leq t\leq 0, \end{cases} \tag{25}$$

where the nonlinear term $f:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is a continuous function.

Denote

$$\boldsymbol{Q}=\{(\boldsymbol{z},\boldsymbol{v}):\boldsymbol{z}\in C_n(J),\ \boldsymbol{v}\in C_m(J)\}$$

with the uniform norm $||(\boldsymbol{z},\boldsymbol{v})|| = ||\boldsymbol{z}||+||\boldsymbol{v}|| = \max_{t\in J}||\boldsymbol{z}(t)||+\max_{t\in J}||\boldsymbol{v}(t)||$, where $C_n(J)=\{f:[0,T]\to\mathbb{R}^n|f$ is continuous on $[0,T]\}$ and $C_m(J)=\{f:[0,T]\to\mathbb{R}^m|f$ is continuous on $[0,T]\}$ are Banach spaces with the sup norm. For each $(\boldsymbol{z},\boldsymbol{v})\in\boldsymbol{Q}$, consider the nonlinear fractional damped dynamical system as

$$\begin{cases} {}_0^C D_t^p \boldsymbol{y}(t) - \boldsymbol{A}{}_0^C D_t^q \boldsymbol{y}(t) = \sum_{i=0}^{M} \boldsymbol{B}_i \boldsymbol{u}[t-\tau_i(t)] \\ \qquad\qquad + f(t,\boldsymbol{z},\boldsymbol{v}),\ t\geq 0, \\ \boldsymbol{y}(0)=\boldsymbol{y}_0,\ \boldsymbol{y}'(0)=\boldsymbol{y}_0', \\ \boldsymbol{u}(t)=\boldsymbol{\psi}(t),\ -\tau_M\leq t\leq 0. \end{cases} \tag{26}$$

Then, by the properties of $\tau_i(t)$, the solution to system (26) can be written as

$$\begin{aligned} \boldsymbol{y}(t) =& \boldsymbol{E}_{p-q}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 - \boldsymbol{A}t^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 \\ &+ t\boldsymbol{E}_{p-q,2}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0' \\ &+ \int_0^t (t-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(t-s)^{p-q}]f(s,\boldsymbol{z},\boldsymbol{v})\mathrm{d}s \\ &+ \sum_{i=0}^{M}\int_{-\tau_i(0)}^{0}[t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\boldsymbol{\psi}(s)\mathrm{d}s \\ &+ \int_0^{t-\tau_M(t)}\sum_{i=0}^{M}[t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\boldsymbol{u}(s)\mathrm{d}s \\ &+ \int_{t-\tau_M(t)}^{t-\tau_{M-1}(t)}\sum_{i=0}^{M-1}[t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\boldsymbol{u}(s)\mathrm{d}s + \ldots \\ &+ \int_{t-\tau_1(t)}^{t}[t-r_0(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_0(s)]^{p-q}\}\boldsymbol{B}_0\dot{r}_0(s)\boldsymbol{u}(s)\mathrm{d}s. \end{aligned}$$

For simplification, we can introduce the notation as

$$\begin{aligned} &\eta(\boldsymbol{y}_0,\boldsymbol{y}_0',\boldsymbol{y}_1;\boldsymbol{z},\boldsymbol{v}) \\ =&\boldsymbol{y}_1 - \boldsymbol{E}_{p-q}(\boldsymbol{A}T^{p-q})\boldsymbol{y}_0 - T\boldsymbol{E}_{p-q,2}(\boldsymbol{A}T^{p-q})\boldsymbol{y}_0' \\ &+ \boldsymbol{A}T^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}T^{p-q})\boldsymbol{y}_0 \\ &- \int_0^T (T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]f(s,\boldsymbol{z},\boldsymbol{v})\mathrm{d}s \\ &- \sum_{i=0}^{M}\int_{-\tau_i(0)}^{0}[T-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[T-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\boldsymbol{\psi}(s)\mathrm{d}s. \end{aligned}$$

We define the control function as

$$\boldsymbol{u}(t)=\begin{cases} \boldsymbol{G}_M^{\mathrm{T}}(T,t)\boldsymbol{W}^{-1}\eta(\boldsymbol{y}_0,\boldsymbol{y}_0',\boldsymbol{y}_1;\boldsymbol{z},\boldsymbol{v}), \\ \qquad t\in(0,T-\tau_M(T)], \\ \boldsymbol{G}_{M-1}^{\mathrm{T}}(T,t)\boldsymbol{W}^{-1}\eta(\boldsymbol{y}_0,\boldsymbol{y}_0',\boldsymbol{y}_1;\boldsymbol{z},\boldsymbol{v}), \\ \qquad t\in(T-\tau_M(T),T-\tau_{M-1}(T)], \\ \qquad\vdots \\ \boldsymbol{G}_0^{\mathrm{T}}(T,t)\boldsymbol{W}^{-1}\eta(\boldsymbol{y}_0,\boldsymbol{y}_0',\boldsymbol{y}_1;\boldsymbol{z},\boldsymbol{v}), \\ \qquad t\in(T-\tau_1(T),T], \end{cases} \tag{27}$$

where the initial states $\boldsymbol{y}_0$ and $\boldsymbol{y}_0'$ and vector $\boldsymbol{y}_1\in\mathbb{R}^n$ are chosen arbitrarily.

To prove our main results for a nonlinear system, the following lemma is needed:

Lemma 5 (Dauer, 1976) If function f is bounded locally with respect to ν and satisfies

$$\lim_{||\nu||\to\infty}\frac{||f(w,\nu)||}{||\nu||}=0$$

uniformly in $w\in[0,T]$, then for each pair of constants c and d, there exists a constant r such that if $||\nu||\leq r$, then $c||f(w,\nu)||+d\leq r$ for $w\in[0,T]$.

With Lemma 5, we will present the main results for the fractional nonlinear system.

Theorem 2 Let f be a continuous function satisfying

$$\lim_{||(\boldsymbol{y},\boldsymbol{u})||\to\infty} \frac{||f(t,\boldsymbol{y},\boldsymbol{u})||}{||(\boldsymbol{y},\boldsymbol{u})||} = 0 \tag{28}$$

uniformly on $t \in [0,T]$. Let the linear fractional system (2) be controllable. Then, the nonlinear system (25) is controllable on $[0,T]$.

Proof By hypothesis, system (2) is controllable. According to Theorem 1, $\boldsymbol{W}$ given by Gramian matrix (5) is invertible. We define the operator $\boldsymbol{\psi}:\ Q \to Q$ as follows:

$$\boldsymbol{\psi}(\boldsymbol{z},\boldsymbol{v}) = (\boldsymbol{y},\boldsymbol{u}),$$

where $\boldsymbol{u}(t)$ is given by Eq. (27) and

$$\begin{aligned}\boldsymbol{y}(t) =& \boldsymbol{E}_{p-q}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 - \boldsymbol{A}t^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 \\ &+ t\boldsymbol{E}_{p-q,2}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0' + \int_0^t (t-s)^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}[\boldsymbol{A}(t-s)^{p-q}]\boldsymbol{f}[s,\boldsymbol{z}(s),\boldsymbol{v}(s)]\mathrm{d}s \\ &+ \sum_{i=0}^{m}\int_{-\tau_i(0)}^{0}[t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\boldsymbol{\psi}(s)\mathrm{d}s \\ &+ \int_0^{t-\tau_M(t)}\sum_{i=0}^{M}[t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\boldsymbol{u}(s)\mathrm{d}s \\ &+ \int_{t-\tau_M(t)}^{t-\tau_{M-1}(t)}\sum_{i=0}^{M-1}[t-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\boldsymbol{u}(s)\mathrm{d}s \\ &+ \ldots + \int_{t-\tau_1(t)}^{t}[t-r_0(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_0(s)]^{p-q}\}\boldsymbol{B}_0\dot{r}_0(s)\boldsymbol{u}(s)\mathrm{d}s.\end{aligned} \tag{29}$$

Denote

$$\boldsymbol{u}_i(t) = \boldsymbol{G}_i^{\mathrm{T}}(T,t)\boldsymbol{W}^{-1}\eta(\boldsymbol{y}_0,\boldsymbol{y}_0',\boldsymbol{y}_1;\boldsymbol{z},\boldsymbol{v}), i=0,1,\ldots,M.$$

Then

$$\begin{aligned}\boldsymbol{u}_i(t) =& \boldsymbol{G}_i^{\mathrm{T}}(T,t)\boldsymbol{W}^{-1}\eta[y(0,\boldsymbol{y}_1;\boldsymbol{z},\boldsymbol{v})] \\ =& \boldsymbol{G}_i^{\mathrm{T}}(T,t)\boldsymbol{W}^{-1}\Big\{\boldsymbol{y}_1 - \boldsymbol{E}_{p-q}(\boldsymbol{A}T^{p-q})\boldsymbol{y}_0 \\ &+ \boldsymbol{A}T^{p-q}\boldsymbol{E}_{p-q,p-q+1}(\boldsymbol{A}T^{p-q})\boldsymbol{y}_0 \\ &- T\boldsymbol{E}_{p-q,2}(\boldsymbol{A}T^{p-q})\boldsymbol{y}_0' \\ &- \int_0^T (T-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}] \\ &\cdot f[s,\boldsymbol{z}(s),\boldsymbol{v}(s)]\mathrm{d}s - \sum_{i=0}^{M}\int_{-\tau_i(0)}^{0}[T-r_i(s)]^{p-1} \\ &\cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[T-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\boldsymbol{\psi}(s)\mathrm{d}s\Big\}.\end{aligned} \tag{30}$$

Next, we claim that constant $r>0$ exists such that

$$\boldsymbol{\psi}(Q(r)) \subset Q(r),$$

where $Q(r) = \{(\boldsymbol{z},\boldsymbol{v}) \in Q : \|\boldsymbol{z}\| \le r/2 \text{ and } \|\boldsymbol{v}\| \le r/2\}$. For simplication, we introduce the following constants:

$$\begin{cases} a_1 = \sup\limits_{t\in[0,T]} \|\boldsymbol{E}_{p-q}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 - \boldsymbol{A}t^{p-q}\boldsymbol{E}_{p-q,p-q+1} \\ \qquad \cdot (\boldsymbol{A}t^{p-q})\boldsymbol{y}_0 + t\boldsymbol{E}_{p-q,2}(\boldsymbol{A}t^{p-q})\boldsymbol{y}_0'\|, \\ a_2 = \sup\limits_{s\in[0,T]} \|\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(T-s)^{p-q}]\|, \\ a_3 = \sup\limits_{t\in[0,T]} \Big\| \sum\limits_{i=0}^{M}\int_{-\tau_i(0)}^{0}[t-r_i(s)]^{p-1} \\ \qquad \cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\boldsymbol{\psi}(s)\mathrm{d}s\Big\|, \\ a_4 = \max\limits_{i\le m, i\in\mathbb{N}} \sup\limits_{t\in[0,T]} \|\boldsymbol{G}_i^{\mathrm{T}}(T,t)\|, \\ a_5 = \sup\limits_{t\in[0,T]} \Big\| \int_0^{t-\tau_M} \sum\limits_{i=0}^{M}[t-r_i(s)]^{p-1} \\ \qquad \cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\mathrm{d}s \\ \qquad + \sum\limits_{j=1}^{M}\int_{t-\tau_j(t)}^{t-\tau_{j-1}(t)}\sum\limits_{i=0}^{j-1}[t-r_i(s)]^{p-1} \\ \qquad \cdot \boldsymbol{E}_{p-q,p}\{\boldsymbol{A}[t-r_i(s)]^{p-q}\}\boldsymbol{B}_i\dot{r}_i(s)\mathrm{d}s\Big\|, \\ d_1 = 4a_4||\boldsymbol{W}^{-1}||(||\boldsymbol{y}_1| + a_1 + a_3), d_2 = 4(a_1+a_3), \\ c_1 = 4a_2a_4T^p||\boldsymbol{W}^{-1}||p^{-1}, c_2 = 4a_2T^pp^{-1}, \\ c = \max\{a_5c_1, c_1, c_2\}, \\ d = \max\{a_5d_1, d_1, d_2\}, \\ \sup||f|| = \sup\{||f[s,\boldsymbol{z}(s),\boldsymbol{v}(s)]||; s\in J\}. \end{cases} \tag{31}$$

By Eqs. (29) and (30), we have

$$\begin{aligned}||\boldsymbol{u}_i(t)|| \le & ||\boldsymbol{G}_i^{\mathrm{T}}(T,t)||||\boldsymbol{W}^{-1}||\big(||\boldsymbol{y}_1||+a_1+a_3 \\ & +a_2T^p p^{-1}\sup||f||\big)\le a_4||\boldsymbol{W}^{-1}||(||\boldsymbol{y}_1||+a_1 \\ & +a_3)+a_4||\boldsymbol{W}^{-1}||a_2T^p p^{-1}\sup||f|| \\ & \le \frac{d_1}{4}+\frac{c_1}{4}\sup||f||,\ i=0,1,\ldots,M,\end{aligned} \tag{32}$$

and

$$\begin{aligned}&||\boldsymbol{y}(t)|| \le a_1+a_3+a_5\max_i||\boldsymbol{u}_i(s)|| \\ &\qquad +a_2\int_0^t(t-s)^{p-1}\sup||f||\mathrm{d}s \\ &\le \frac{d_2}{4}+a_5\left(\frac{d_1}{4}+\frac{c_1}{4}\sup||f||\right)+a_2T^p p^{-1}\sup||f|| \\ &\le \frac{d}{2}+\frac{c}{2}\sup||f||.\end{aligned} \tag{33}$$

Since function f satisfies property (28), by Lemma 5, for any fixed $c,d>0$, there is a positive constant r such that if $||(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})||\le r$, then

$$c||f(t,\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})||+d\le r \text{ holds for all } t\in[0,T]. \tag{34}$$

Now we can choose c and d given by Eq. (31) and take r such that Eq. (34) holds. Thus, if $\|\boldsymbol{z}\|\le\frac{r}{2}$ and $\|\boldsymbol{v}\|\le\frac{r}{2}$, then $||\boldsymbol{z}(s)||+||\boldsymbol{v}(s)||\le r$ for all $s\in[0,T]$, which leads to $d+c\sup||f||\le r$. Therefore, by inequality (32), we derive $||\boldsymbol{u}_i(s)||\le\frac{r}{4}$ for all $s\in[0,T]$, and hence $||\boldsymbol{u}_i(s)||\le\frac{r}{4}$ by inequality (33), and $\|\boldsymbol{y}\|\le\frac{r}{2}$. Thus, $\boldsymbol{\psi}(Q(r))\subset Q(r)$.

Analogous to the proof of Theorem 4.1 in our previous work (He et al., 2016b), we can verify that $\boldsymbol{\psi}$ has a fixed point $(\boldsymbol{z},\boldsymbol{v})\in Q(r)$, which reads $\boldsymbol{\psi}(\boldsymbol{z},\boldsymbol{v})=(\boldsymbol{z},\boldsymbol{v})\equiv(\boldsymbol{y},\boldsymbol{u})$. Therefore, $\boldsymbol{y}(t)$ is the solution to system (25). We can see that $\boldsymbol{y}(T)=\boldsymbol{y}_1$ and the control function $\boldsymbol{u}(t)$ actuates the state of system (25) from the initial state $\boldsymbol{y}_0$ to $\boldsymbol{y}_1$ on $[0,T]$. That is, system (25) is controllable on $[0,T]$.

6 Examples

In this section, we use our main results to prove the controllability of the fractional delay systems.

Example 1 Consider a linear fraction damped system

$${}_0^CD_t^p\boldsymbol{y}(t)+\boldsymbol{A}{}_0^CD_t^q\boldsymbol{y}(t)=\boldsymbol{B}_0\boldsymbol{u}(t)+\boldsymbol{B}_1\boldsymbol{u}[t-\tau(t)] \tag{35}$$

with $p=1.5, q=0.5, \boldsymbol{y}(t)\in\mathbb{R}^2, \tau(t)=t/2$, and $\boldsymbol{A}$, $\boldsymbol{B}_0$, and $\boldsymbol{B}_1$ given by

$$\boldsymbol{A}=\begin{pmatrix}0&2\\-2&0\end{pmatrix},\ \boldsymbol{B}_0=\begin{pmatrix}2&0\\0&2\end{pmatrix},\ \boldsymbol{B}_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

The Mittag-Leffler matrix function is given by

$$\boldsymbol{E}_{1,1.5}(\boldsymbol{A}t)=\sum_{j=0}^{\infty}\begin{pmatrix}\dfrac{(-2)^jt^{2j}}{\Gamma(1.5+2j)} & \dfrac{(-2)^jt^{2j+1}}{\Gamma(2.5+2j)}\\ -\dfrac{(-2)^jt^{2j+1}}{\Gamma(2.5+2j)} & \dfrac{(-2)^jt^{2j}}{\Gamma(1.5+2j)}\end{pmatrix}.$$

Then

$$\begin{aligned}&(t-s)^{p-1}\boldsymbol{E}_{p-q,p}[\boldsymbol{A}(t-s)^{p-q}] \\ &\qquad=\begin{pmatrix}\cos_p(t-s) & \sin_p(t-s)\\ -\sin_p(t-s) & \cos_p(t-s)\end{pmatrix},\end{aligned}$$

where

$$\begin{cases}\cos_p(t-s)=\displaystyle\sum_{j=0}^{\infty}\frac{(-2)^j(t-s)^{2j+0.5}}{\Gamma(1.5+2j)},\\ \sin_p(t-s)=\displaystyle\sum_{j=0}^{\infty}\frac{(-2)^j(t-s)^{2j+1.5}}{\Gamma(2.5+2j)}.\end{cases}$$

$$\boldsymbol{G}_0(t,s)=\begin{pmatrix}2\cos_p(t-s) & 2\sin_p(t-s)\\ -2\sin_p(t-s) & 2\cos_p(t-s)\end{pmatrix},$$

$$\begin{aligned}\boldsymbol{G}_1(t,s)=&\begin{pmatrix}2\cos_p(t-s) & 2\sin_p(t-s)\\ -2\sin_p(t-s) & 2\cos_p(t-s)\end{pmatrix}\\ &+\begin{pmatrix}2\cos_p(t-2s) & 2\sin_p(t-2s)\\ -2\sin_p(t-2s) & 2\cos_p(t-2s)\end{pmatrix}.\end{aligned}$$

By a simple computation, one can see that the controllability matrix is

$$\begin{aligned}&\boldsymbol{W}(T)\\ &=\int_{\frac{T}{2}}^{T}\left[4\cos_p^2(T-s)+4\sin_p^2(T-s)\right]\begin{pmatrix}1&0\\0&1\end{pmatrix}\mathrm{d}s\\ &+\int_0^{\frac{T}{2}}\Big\{[2\cos_p(T-s)+2\cos_p(T-2s)]^2\\ &+\left[2\sin_p(T-s)+2\sin_p(T-2s)\right]^2\Big\}\begin{pmatrix}1&0\\0&1\end{pmatrix}\mathrm{d}s.\end{aligned}$$

We can see that $\boldsymbol{W}(T)$ is positive definite for any $T>0$. So, system (35) is controllable on $[0,T]$. Fig. 1 shows that under a suitable control input, we can transfer the state from $(0,0)^{\mathrm{T}}$ to $(9.3,7.5)^{\mathrm{T}}$. The results of the control function are shown in Fig. 2.

Example 2 Consider the nonlinear system

$$\begin{aligned}{}_0^CD_t^p\boldsymbol{y}(t)+\boldsymbol{A}{}_0^CD_t^q\boldsymbol{y}(t)=&\boldsymbol{B}_0\boldsymbol{u}(t)+\boldsymbol{B}_1\boldsymbol{u}[t-\tau(t)]\\ &+f[\boldsymbol{y}(t)]\end{aligned} \tag{36}$$

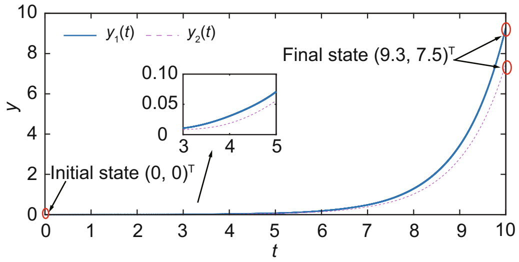

Fig. 1 State $\boldsymbol{y}(t)$ of linear system (35)

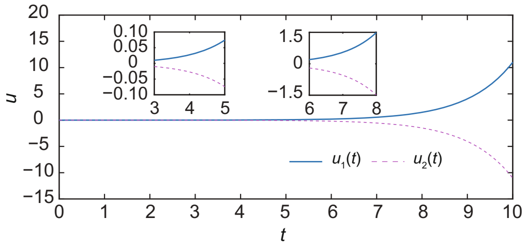

Fig. 2 Control input $\boldsymbol{u}(t)$ for linear system (35)

with $p = 1.5, q = 0.5$, $\boldsymbol{y}(t) = (y_1(t), y_2(t))^{\mathrm{T}}$, $\boldsymbol{A}$, $\boldsymbol{B}_0$, and $\boldsymbol{B}_1$ given by Example 1, and $f(\boldsymbol{y})$ given by

$$f(\boldsymbol{y}) = \left(\frac{1 + y_2}{1 + y_1^2 + y_2^2}, \frac{1 + y_1}{1 + y_1^2 + y_2^2} \right)^{\mathrm{T}}. \tag{37}$$

We can see that $f(\boldsymbol{y})$ is continuous and meets condition (28) in Theorem 2; nonlinear system (36) is thus controllable on $[0, T]$. Fig. 3 shows that under the control input as shown in Fig. 2, we can transfer the state from $(0, 0)^{\mathrm{T}}$ to $(9.5, 7.3)^{\mathrm{T}}$.

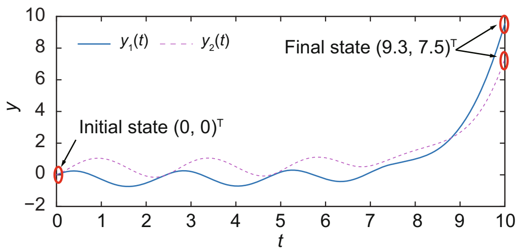

Fig. 3 State $\boldsymbol{y}(t)$ of nonlinear system (36)

7 Conclusions

We have studied the controllability problem for the fractional-order damped systems in linear and nonlinear cases with multiple time-varying delays in control. We have provided a necessary and sufficient condition of the controllability of the linear system. Based on this, we have also proposed a sufficient condition which guarantees the controllability of the nonlinear fractional-order damped system. The delays considered in the study are time-varying delays, which is a continuation of our previous research (He et al., 2016a, 2016b). Numerical simulations have been carried out to illustrate the effectiveness of the proposed theory.

Contributors

Hua-cheng ZHOU and Chun-hai KOU designed the research. Bin-bin HE and Hua-cheng ZHOU drafted the manuscript. Chun-hai KOU helped organize the manuscript. Bin-bin HE and Hua-cheng ZHOU revised and finalized the paper.

Compliance with ethics guidelines

Bin-bin HE, Hua-cheng ZHOU, and Chun-hai KOU declare that they have no conflict of interest.

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