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Convergence of time-varying networks and its applications*

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Abstract: In this study, we present the convergence of time-varying networks. Then, we apply the convergence property to cooperative control of nonlinear multiagent systems (MASs) with unknown control directions (UCDs), and illustrate a new kind of Nussbaum-type function based control algorithms. It is proven that if the time-varying networks are cut-balance, the convergence of nonlinear MASs with nonidentical UCDs is achieved using the presented algorithms. A critical feature of this application is that the designed algorithms can deal with nonidentical UCDs by employing conventional Nussbaum-type functions. Finally, one simulation example is given to illustrate the effectiveness of the presented algorithms.

Key words: Time-varying networks; Unknown control directions; Nussbaum-type function; Cut-balance condition
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1 Introduction

Multiagent systems (MASs), also called the "undirected/directed network," generally consist of multiple interacting agents to cope with problems that are challenging for a single agent, if each node of the network is described by first-order dynamics. The cooperative control of undirected/directed networks has received much attention in the last several decades, and the design of distributed control algorithms is a fundamental task because of their considerable applications (Fax and Murray, 2004; Hong et al., 2006; Li et al., 2010; Yu WW et al., 2010; Zheng Y et al., 2011; Zheng YS and Wang, 2012; Wen et al., 2014; Wen and Zheng, 2019). Consensus seeking is one of the fundamental distributed control algorithms in MASs such that the agents can cooperate to achieve some common limits (values).

Existing results of MASs show that the connectivity condition is related to the coupling strength of the undirected/directed networks, and usually exponential convergence is achieved to a common value (Olfati-Saber and Murray, 2004). In Moreau (2005), the exponential convergence of consensus was achieved under weak conditions, where the coupling strength of the undirected/directed networks was uniformly bounded. For time-varying networks to satisfy the cut-balance condition, the problem of convergence (not necessarily consensus) was investigated in Hendrickx and Tsitsiklis (2013), where the coupling strength can be unknown in priori or possibly random. Note that the above results are based on the consensus-seeking problem of linear MASs. However, it is unclear how to extend the results in the mentioned literature to the convergence of nonlinear MASs.

Recently, the design of distributed control algorithms for consensus of MASs with unknown

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control directions (UCDs) has attracted a lot of research interest (Wang QL and Sun, 2020), since some practical systems do not possess the prior knowledge of control directions. Hence, the consideration of UCDs when designing consensus control algorithms is necessary. An effective method to cope with UCDs is to employ Nussbaum-type functions, which were first proposed by Nussbaum (1983), and then this method is extended to deal with various single dynamics (Zhang Y et al., 2000; Ge et al., 2004; Wang CL et al., 2017; Yu ZX et al., 2018). However, when multiple control inputs with UCDs are considered, for instance, in MASs or interconnected systems, one critical challenge is how to deal with the problem of multiple Nussbaum-type functions. To completely solve the problem of multiple control inputs with nonidentical UCDs, Peng and Ye (2014) suggested constructing a partial Lyapunov function for each agent, where only one Nussbaum-type function exists. Inspired by the idea of constructing partial Lyapunov functions, Wang QL et al. (2019a) achieved an adaptive leaderless consensus of MASs with nonidentical UCDs for linear MASs under switching topologies with unbalanced subgraphs, and Wang G (2019) achieved an adaptive consensus for nonlinear MASs under fixed and directed graphs having a spanning tree. Furthermore, by introducing nonlinear proportional-integral (PI) functions, the consensus of linear MASs with nonidentical UCDs was investigated in Psillakis (2017) and Wang QL et al. (2019b, 2019c) under switching topologies with balanced/unbalanced subgraphs.

Based on the analysis of the literature, in this work, we present the convergence of time-varying networks and apply the convergence property to nonlinear MASs with UCDs under time-varying networks, where the UCDs are completely nonidentical and the time-varying networks satisfy the cut-balance condition. By employing Nussbaum-type functions, we can design new control algorithms to achieve the convergence of high-order nonlinear MASs. The contributions of this study are presented as follows:

1. The convergence property is derived for time-varying networks, and results are generalized to time-varying networks with disturbances. Compared with the results in Hendrickx and Tsitsiklis (2013), where disturbances were not considered, the network topology in this study is considered in a general form.

2. We apply the convergence property to the cooperative control of nonlinear MASs with nonidentical UCDs under time-varying networks satisfying the cut-balance condition. The communication network in this study is the mild condition for nonlinear MASs with nonidentical UCDs. Furthermore, one critical feature is that conventional Nussbaum-type functions are sufficient to deal with nonidentical UCDs.

Notations: A signal η satisfying $\int_0^\infty |\eta(\tau)|\,\mathrm{d}\tau < \infty$ belongs to the space of $\mathcal{L}_1$; i.e., $\eta \in \mathcal{L}_1$. For set T, T^{o} denotes its interior. The space of square integrable signals is denoted by $\mathcal{L}_2$, while $\mathcal{L}_\infty$ denotes the space of bounded signals. Notations $\sup(\cdot)$ and $\inf(\cdot)$ denote the least upper bound and the greatest lower bound, respectively. A vector $\boldsymbol{\xi} = [\xi_1, \xi_2, \ldots, \xi_N] \in \mathbb{R}^N$ is said to be sorted if $\xi_1 \le \xi_2 \le \ldots \le \xi_N$.

2 Preliminaries and problem formulation

2.1 Preliminaries

We omit the basic concepts of directed graphs in this study, and the detailed definition was introduced in Wang QL et al. (2019a). The concept of directed graphs is revisited as follows: We define a directed graph as $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \boldsymbol{A}(t))$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ is the set of nodes, $\mathcal{E}(t) \subseteq \mathcal{V}\times\mathcal{V}$ the set of edges, and $\boldsymbol{A}(t) = [a_{ik}(t)] \in \mathbb{R}^{N\times N}$ the adjacency matrix. The definition of the unbounded interaction graph is given as follows:

Definition 1 The graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ is the unbounded interaction graph if it satisfies: (1) $a_{ik}(t)$ is nonnegative and measurable; (2) $a_{ii}(t) = 0$ for each i and $\int_0^\infty a_{ik}(\tau)\mathrm{d}\tau = \infty$ if $(k, i) \in \mathcal{E}(t)$.

In the following, the cut-balance assumption will be given on the coupling strength of $\mathcal{A}(t)$ for time-varying networks:

Assumption 1 (Hendrickx and Tsitsiklis, 2013) For a constant $K \ge 1$ and the non-empty subset S of $\{1, 2, \ldots, N\}$, the cut-balance condition is

$$K^{-1} \sum_{i\in S, k\notin S} a_{ki}(t) \le \sum_{i\in S, k\notin S} a_{ik}(t) \le K \sum_{i\in S, k\notin S} a_{ki}(t)$$

for $t \ge 0$.

Remark 1 (Hendrickx and Tsitsiklis, 2013) The cut-balance condition indicates that a group of agents influence the remaining agents while they are influenced by a proportional contribution. Moreover, the cut-balance condition is rather weak, and, in general, it seems difficult to verify. However, it has been verified that several types of graphs satisfy the cut-balance condition, such as symmetric graphs ($a_{ik}(t) = a_{ki}(t)$) and type-symmetric graphs ($a_{ik}(t) \leq K a_{ki}(t)$).

Lemma 1 (Hendrickx and Tsitsiklis, 2013) For every $i, k = 1, 2, \ldots, N$, $i \neq k$, and the non-empty subset S, let $b_{ik} \geq 0$ and satisfy the cut-balance condition as

$$K^{-1} \sum_{i \in S, k \notin S} b_{ki} \leq \sum_{i \in S, k \notin S} b_{ik} \leq K \sum_{i \in S, k \notin S} b_{ki}$$

when $K \geq 1$. Then,

$$\sum_{i=1}^{\nu} K^{-i} \sum_{k=1}^{N} b_{ik}(t) \Big[y_k(t) - y_i(t) \Big] \geq 0$$

for every sorted vector $\boldsymbol{y} \in \mathbb{R}^N$ and each $\nu \leq N$.

2.2 Problem formulation

Let $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ be a time-varying network with a collection of N nodes. Each node v_i has first-order dynamics described by

$$\dot{\xi}_i(t) = \sum_{k=1}^{N} a_{ik}(t) \Big[\xi_k(t) - \xi_i(t) \Big] + \varrho_i(t), \quad (1)$$

where $i, k = 1, 2, \ldots, N$, $a_{ik}(t)$ is the coupling strength, $\xi_i(t) \in \mathbb{R}$ is the state information, and the function $\varrho_i \in \mathcal{L}_1$. For arbitrary initial states $\xi_i(0)$, $i = 1, 2, \ldots, N$, we say that the time-varying networks achieve convergence if

$$\lim_{t \to \infty} \xi_i(t) = \xi_i^* \in \mathcal{L}_\infty, \quad (2)$$

where $i = 1, 2, \ldots, N$.

3 Convergence of time-varying networks

In this section, we will show that the convergence of time-varying networks is guaranteed. To obtain the convergence property, a key lemma will be first introduced:

Lemma 2 Suppose the function $f : [0, \infty) \to \mathbb{R}$ is continuously differentiable. If $\mathrm{d}f/\mathrm{d}t \in \mathcal{L}_1$ is satisfied, then $\lim_{t \to \infty} f(t)$ exists.

Proof The proof is omitted because it is a direct consequence of Barbalat's lemma.

With the above lemma, we have the following result:

Theorem 1 Let Assumption 1 hold, and assume that $\mathcal{G}(t)$ is the unbounded interactions graph. The time-varying network (1) can achieve the convergence objective (2). Furthermore, we have: (1) $\xi_i(t)$ ($i = 1, 2, \ldots, N$) is bounded, and for every k and i, $\int_0^\infty a_{ik}(\tau) \left| \xi_k(\tau) - \xi_i(\tau) \right| \mathrm{d}\tau < \infty$; (2) If i and k are in the same connected component, $\lim_{t \to \infty} \xi_i(t) = \xi_i^* = \lim_{t \to \infty} \xi_k(t) = \xi_k^*$.

Proof Let $\xi(t) : \mathbb{R}^+ \to \mathbb{R}^N$ be a solution to the time-varying network (1), expressed as

$$\begin{aligned} \xi_i(t) =& \xi_i(0) + \int_0^t \sum_{k=1}^{N} a_{ik}(\tau) \Big[\xi_k(\tau) - \xi_i(\tau) \Big] \mathrm{d}\tau \\ &+ \int_0^t \varrho_i(\tau) \mathrm{d}\tau, \end{aligned} \quad (3)$$

where $i = 1, 2, \ldots, N$.

We define $y_i(t) = \xi_{p_i(t)}(t)$, where $p_i(t)$ is the index of the i^{th} smallest component of $\xi(t)$. This implies that $\boldsymbol{y}(t)$ is sorted. Furthermore, we assume $b_{ik}(t) = a_{p_i(t) p_k(t)}(t)$. Let the set $T_{ij}(t) := \{\tau \in [0, t] : y_i(\tau) = \xi_j(\tau)\}$ describe the times at which the state value of the j^{th} agent is smallest at the i^{th} moment. Consequently, we have $\cup_{i=1}^{N} T_{ij}(t) = \cup_{j=1}^{N} T_{ij}(t) = [0, t]$ and $T_{ij}^{\circ}(t) \cap T_{ik}^{\circ}(t) = \varnothing$ with $j \neq k$. Recalling the same analysis in Hendrickx and Tsitsiklis (2013), we have

$$\begin{aligned} y_i(t) =& y_i(0) + \int_0^t \sum_{k=1}^{N} b_{ik}(\tau) \Big[y_k(\tau) - y_i(\tau) \Big] \mathrm{d}\tau \\ &+ \sum_{k=1}^{N} \int_{T_{ik}(t)} \varrho_k(\tau) \mathrm{d}\tau, \end{aligned} \quad (4)$$

where b_{ik} ($i, k = 1, 2, \ldots, N$) satisfies the cut-balance condition in Assumption 1. Since $\varrho_i \in \mathcal{L}_1$, we have the term $\sum_{k=1}^{N} \int_{T_{ik}(t)} \varrho_k(\tau) \mathrm{d}\tau \in \mathcal{L}_\infty$ as

$$\begin{aligned} \left| \sum_{k=1}^{N} \int_{T_{ik}(t)} \varrho_k(\tau) \mathrm{d}\tau \right| &\leq \sum_{k=1}^{N} \int_{T_{ik}(t)} \left| \varrho_k(\tau) \right| \mathrm{d}\tau \\ &\leq \sum_{k=1}^{N} \int_0^\infty \left| \varrho_k(\tau) \right| \mathrm{d}\tau. \end{aligned} \quad (5)$$

Considering the weighted sum of the first ν

($\nu \le N$) agents, we have

$$S_\nu(t) = \sum_{i=1}^{\nu} K^{-i}\left[y_i(t) - \sum_{k=1}^{N}\int_{T_{ik}(t)} \varrho_k(\tau)\mathrm{d}\tau\right] = \int_0^t \sum_{i=1}^{\nu} K^{-i}\sum_{k=1}^{N} b_{ik}(\tau)\left[y_k(\tau) - y_i(\tau)\right]\mathrm{d}\tau + S_\nu(0). \tag{6}$$

According to Lemma 1, the integrand of Eq. (6) is always nonnegative. Therefore, $S_\nu(t)$ is nondecreasing. Since $b_{ik}(t)$ $(i, k = 1, 2, \ldots, N)$ is nonnegative, it can be seen from Eq. (4) that

$$y_1(0)+\sum_{k=1}^{N}\int_{T_{1k}(t)} \varrho_k(\tau)\mathrm{d}\tau \le y_i(t) \le y_N(0) + \sum_{k=1}^{N}\int_{T_{Nk}(t)} \varrho_k(\tau)\mathrm{d}\tau \tag{7}$$

for each i. Then, because of the $\mathcal{L}_1$ property of ϱ_k, y_i is bounded. Equivalently, $S_\nu(t)$ is bounded, and therefore it converges. Now, using a similar reasoning to the proof of item (b) of Theorem 1 in Hendrickx and Tsitsiklis (2013), we have $\int_0^\infty a_{ik}(\tau)\left|\xi_k(\tau) - \xi_i(\tau)\right|\mathrm{d}\tau < \infty$ for $i, k = 1, 2, \ldots, N$. Furthermore, the boundedness of y_i yields the boundedness of ξ_i. We will now prove that the boundedness of $\int_0^\infty a_{ik}(\tau)\left|\xi_k(\tau) - \xi_i(\tau)\right|\mathrm{d}\tau$ implies that $\lim_{t\to\infty} I_{ik}(t)$ exists and is finite where $I_{ik}(t) := \int_0^t a_{ik}(\tau)\left(\xi_k(\tau) - \xi_i(\tau)\right)\mathrm{d}\tau$. Otherwise, there exists a sequence of times $\{\tau_\nu^{ik}\}_{\nu=1}^{\infty}$ with $\lim_{\nu\to\infty}\tau_\nu^{ik} = +\infty$ such that

$$\left|I_{ik}(\tau_{\nu+1}^{ik}) - I_{ik}(\tau_\nu^{ik})\right| = \left|\int_{\tau_\nu^{ik}}^{\tau_{\nu+1}^{ik}} a_{ik}(\tau)\left[\xi_k(\tau) - \xi_i(\tau)\right]\mathrm{d}\tau\right| > \epsilon \tag{8}$$

for $\epsilon > 0$. Since $\int_0^\infty a_{ik}(\tau)\left|\xi_k(\tau) - \xi_i(\tau)\right|\mathrm{d}\tau < \infty$, it holds that

$$\lim_{t\to\infty}\int_t^\infty a_{ik}(\tau)\left|\xi_k(\tau) - \xi_i(\tau)\right|\mathrm{d}\tau = 0. \tag{9}$$

Therefore, it has $\nu_0 \in \mathbb{N}$ such that

$$\int_{\tau_\nu^{ik}}^\infty a_{ik}(\tau)\left|\xi_k(\tau) - \xi_i(\tau)\right|\mathrm{d}\tau < \epsilon,\ \forall \nu \ge \nu_0. \tag{10}$$

Inequality (10) yields

$$\begin{aligned}&\left|I_{ik}(\tau_{\nu+1}^{ik}) - I_{ik}(\tau_\nu^{ik})\right| \\ &= \left|\int_{\tau_\nu^{ik}}^{\tau_{\nu+1}^{ik}} a_{ik}(\tau)\left[\xi_k(\tau) - \xi_i(\tau)\right]\mathrm{d}\tau\right| \\ &\le \int_{\tau_\nu^{ik}}^{\infty} a_{ik}(\tau)\left|\xi_k(\tau) - \xi_i(\tau)\right|\mathrm{d}\tau < \epsilon,\end{aligned} \tag{11}$$

which contradicts inequality (8). Taking the limit $t \to \infty$ in Eq. (3), we obtain

$$\lim_{t\to\infty}\xi_i(t) = \xi_i(0) + \sum_{k=1}^{N}\lim_{t\to\infty} I_{ik}(t) + \lim_{t\to\infty}\int_0^t \varrho_i(\tau)\mathrm{d}\tau. \tag{12}$$

From Lemma 2, the property $\varrho_i \in \mathcal{L}_1$ implies that $\lim_{t\to\infty}\int_0^t \varrho_i(\tau)\mathrm{d}\tau$ exists. Since both the limits in the right side of Eq. (12) exist and are finite, ξ_i converges to some constant values; i.e., $\lim_{t\to\infty}\xi_i(t) = \xi_i^* < \infty$. If there is an edge (k, i) in the graph, i.e., $\int_t^\infty a_{ik}(\tau)\mathrm{d}\tau = \infty$ for all t, then for every $a_{ik}^* > 0$, there exists a function $\tilde{T}_{ik}: [0, \infty) \to \mathbb{R}_+$ such that $\int_t^{t+\tilde{T}_{ik}(t)} a_{ik}(\tau)\mathrm{d}\tau = a_{ik}^*$ for $t \ge 0$. Then, using the triangle inequality, we obtain

$$\begin{aligned}&\int_t^{t+\tilde{T}_{ik}(t)} a_{ik}(\tau)\mathrm{d}\tau\left|\xi_i^* - \xi_k^*\right| \\ &\le \int_t^{t+\tilde{T}_{ik}(t)} a_{ik}(\tau)\left|\xi_i(\tau) - \xi_k(\tau)\right|\mathrm{d}\tau \\ &\quad + \int_t^{t+\tilde{T}_{ik}(t)} a_{ik}(\tau)\left|\xi_i(\tau) - \xi_i^*\right|\mathrm{d}\tau \\ &\quad + \int_t^{t+\tilde{T}_{ik}(t)} a_{ik}(\tau)\left|\xi_k(\tau) - \xi_k^*\right|\mathrm{d}\tau,\end{aligned} \tag{13}$$

which yields

$$\begin{aligned}\left|\xi_i^* - \xi_k^*\right| \le& \frac{1}{a_{ik}^*}\int_t^{t+\tilde{T}_{ik}(t)} a_{ik}(\tau)\left|\xi_i(\tau) - \xi_k(\tau)\right|\mathrm{d}\tau \\ &+ \max_{\tau\in[t, t+\tilde{T}_{ik}(t)]}\left|\xi_i(\tau) - \xi_i^*\right| \\ &+ \max_{\tau\in[t, t+\tilde{T}_{ik}(t)]}\left|\xi_k(\tau) - \xi_k^*\right|.\end{aligned} \tag{14}$$

Taking the limit $t \to \infty$ in the right side of inequality (14), we obtain $\lim_{t\to\infty}\xi_i(t) = \xi_i^*$ and $\lim_{t\to\infty}\xi_k(t) = \xi_k^*$ from Eq. (9). If i and k are in the same connected component, there exists a finite sequence of edges that connects agents i and k, and thus we have $\xi_i^* = \xi_k^*$.

4 Application to cooperative control of high-order nonlinear agents

In this section, we will apply the convergence property to the cooperative control of nonlinear MASs with non-identical UCDs. The related definition and lemmas will be introduced to address the main results.

Definition 2 (Nussbaum, 1983) Notation $\chi(\cdot)$ represents the Nussbaum-type function if it satisfies

$$\begin{cases} \lim_{\kappa\to\infty}\sup\left[\frac{1}{\kappa}\int_0^{\kappa}\chi(\tau)\,\mathrm{d}\tau\right]=+\infty, \\ \lim_{\kappa\to\infty}\inf\left[\frac{1}{\kappa}\int_0^{\kappa}\chi(\tau)\,\mathrm{d}\tau\right]=-\infty. \end{cases} \tag{15}$$

In general, $\mathrm{e}^{k^2}\cos(\pi k/2)$, $k^2\sin k$, and $k^2\cos k$ are Nussbaum-type functions. For a Nussbaum-type function $\chi(p)$, if $s(\cdot)$ is a bounded function with $I_v=[g^-,g^+]$, where g^- and g^+ are unknown parameters with $0\notin I_v$, we say that the function $\chi(\cdot,p)=s(\cdot)\chi(p)+h(\cdot)$ satisfies Definition 2, where the continuous function $h(\cdot)\in\mathcal{L}_\infty$ (Shahnazi and Wang, 2018). The property of $\chi(\cdot,p)$ is summarized as follows:

Lemma 3 (Zheng YF et al., 2013) Suppose that the Nussbaum-type function $\chi(\cdot,p)$ satisfies $\chi(\cdot,p)\in\mathcal{L}_\infty$, that $V(t)\ge 0$, and that the smooth function $p(t)\in\mathcal{L}_\infty$ is within $[0,\infty)$. For all $t\in[0,\infty)$, if

$$\dot V(t)\le\chi(\cdot,p)\dot p+\epsilon(t) \tag{16}$$

is satisfied, where $\epsilon(t)$ satisfies $\int_0^t\epsilon(\tau)\mathrm{d}\tau<\infty$, it is concluded that $V(t)\in\mathcal{L}_\infty$, $p(t)\in\mathcal{L}_\infty$, and $\int_0^t[\chi(\cdot,p(\tau))\dot p(\tau)]\,\mathrm{d}\tau\in\mathcal{L}_\infty$.

Lemma 4 (Su and Lin, 2016; Song et al., 2017) Suppose that the function $x(t)$ is smooth and that the initial conditions $x(0)$ and $x^{(m)}(0)$ ($m=1,2,\ldots,n-1$) are bounded. Let

$$s(x,t)=\left(\gamma+\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-1}x(t), \tag{17}$$

where $\gamma>0$ is a constant. If there is a scalar $k>0$ such that $|s(x,t)|\le k$ with $t\ge 0$, then we have

$$\left\|x^{(m)}(t)\right\|\le\frac{2^m k}{\gamma^{n-m-1}},\ t\ge T_0, \tag{18}$$

where $m=1,2,\ldots,n-1$ and T_0 is the finite time depending on the values of $x(0)$ and $x^{(m)}(0)$. Furthermore, if $\lim_{t\to\infty}s(x,t)=s^*\in\mathcal{L}_\infty$, then $\lim_{t\to\infty}x(t)=x^*\in\mathcal{L}_\infty$ and its derivatives $\lim_{t\to\infty}x^{(m)}(t)=x_m^*\in\mathcal{L}_\infty$. Particularly, if $s(x,t)\to 0$ as $t\to\infty$, then $x(t)\to 0$ and $x^{(m)}(t)\to 0$ for $m=1,2,\ldots,n-1$ as $t\to\infty$.

Consider a group of agents with n^{th}-order, where the dynamics of the i^{th} agent is described by

$$x_i^{(n)}(t)=\boldsymbol{\varphi}_i(\overline{\boldsymbol{x}}_i(t))^{\mathrm{T}}\boldsymbol{\theta}_i+g_i(t)u_i(t)+d_i(t), \tag{19}$$

where $i=1,2,\ldots,N$, and $\overline{\boldsymbol{x}}_i(t)=[x_i(t),\dot x_i(t),\ldots,x_i^{(n-1)}(t)]^{\mathrm{T}}$ with $x_i^{(m)}(t)\in\mathbb{R}$ being the m^{th} element of the state. Furthermore, $\boldsymbol{\theta}_i\in\mathbb{R}^{\ell_i}$ is the unknown and constant parameter, $\boldsymbol{\varphi}_i(t):\mathbb{R}\to\in\mathbb{R}^{\ell_i}$ the known and smooth function, $g_i(t)\in\mathbb{R}$ the high-frequency gain, $d_i(t)\in\mathbb{R}$ the bounded unknown disturbance, and $u_i(t)\in\mathbb{R}$ the control input.

Assumption 2 $g_i(t)\ne 0$ is unknown, bounded with constant sign; that is, $|g_i(t)|\in[g_{\min},g_{\max}]$ with $0<g_{\min}\le g_{\max}$.

The purpose of this section is to design distributed control laws for agents (19) under Assumptions 1 and 2 such that the following objective is guaranteed:

$$\begin{cases} \lim_{t\to\infty}x_i(t)=x_i^*\in\mathcal{L}_\infty, \\ \lim_{t\to\infty}x_i^{(m)}(t)=0, \end{cases} \tag{20}$$

where $m=1,2,\ldots,n-1$, and $i=1,2,\ldots,N$.

To facilitate the design approach, we define the states as follows:

$$\begin{aligned} z_i(t)=&\left(\gamma+\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-1}x_i(t) \\ =&C_{n-1}^0\gamma^{n-1}x_i(t)+C_{n-1}^1\gamma^{n-2}\dot x_i(t)+\ldots \\ &+C_{n-1}^{n-2}\gamma x_i^{(n-2)}(t)+C_{n-1}^{n-1}x_i^{(n-1)}(t), \end{aligned} \tag{21}$$

$$\begin{aligned} q_i(t)=&C_{n-1}^0\gamma^{n-1}\dot x_i(t)+C_{n-1}^1\gamma^{n-2}\ddot x_i(t)+\ldots \\ &+C_{n-1}^{n-3}\gamma^2 x_i^{(n-2)}(t)+C_{n-1}^{n-2}\gamma x_i^{(n-1)}(t), \end{aligned} \tag{22}$$

where $\gamma>0$ and C_i^j's are coefficients of the binomial expansion. For nonlinear MAS (19), the result is given in the following:

Theorem 2 Let Assumptions 1 and 2 hold, and assume that $\mathcal{G}(t)$ is the unbounded interaction graph. Consider the nonlinear agent (19). The convergence objective (20) is achieved if the distributed control algorithms are designed by

$$\begin{aligned} u_i(t)=&\chi(p_i(t))\Big[\lambda_1\phi_i(t)+\dot e_i(t)+\boldsymbol{\varphi}_i(\overline{\boldsymbol{x}}_i(t))^{\mathrm{T}}\hat{\boldsymbol{\theta}}_i(t) \\ &+q_i(t)+\hat D_i(t)\tanh(\phi_i(t)\lambda_t)\Big], \end{aligned} \tag{23}$$

with

$$\begin{cases} \dot{p}_i(t) = \phi_i(t)\Big[\lambda_1\phi_i(t) + \dot{e}_i(t) + \boldsymbol{\varphi}_i(\overline{\boldsymbol{x}}_i(t))^{\mathrm{T}}\hat{\boldsymbol{\theta}}_i(t) \\ \qquad + q_i(t) + \hat{D}_i(t)\tanh(\phi_i(t)\lambda_t)\Big], \\ \dot{\hat{\boldsymbol{\theta}}}_i(t) = \phi_i(t)\boldsymbol{\varphi}_i(\overline{\boldsymbol{x}}_i(t)), \\ \dot{\hat{D}}_i(t) = \phi_i(t)\tanh\left(\phi_i(t)\lambda_t\right), \end{cases} \tag{24}$$

$$\begin{cases} \phi_i(t) = z_i(t) + e_i(t), \\ \dot{e}_i(t) = \sum_{k=1}^{N} a_{ik}\left[z_i(t) - z_k(t)\right], \end{cases} \tag{25}$$

where $\lambda_1 > 0$, $\lambda_t = 1 + t^2$, and $\tanh(\cdot)$ is the hyperbolic tangent function.

Proof Consider the following Lyapunov function:

$$V_i(t) = \frac{1}{2}\phi_i^2(t) + \frac{1}{2}\tilde{\boldsymbol{\theta}}_i(t)^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_i(t) + \frac{1}{2}\tilde{D}_i^2(t), \tag{26}$$

where $\tilde{\boldsymbol{\theta}}_i(t) := \hat{\boldsymbol{\theta}}_i(t) - \boldsymbol{\theta}_i$, $\tilde{D}_i(t) := \hat{D}_i(t) - D_i$, and $D_i > 0$ is assumed to be the upper bound of $|d_i(t)|$. According to Eqs. (23), (24), and (26), the time derivative of $V_i(t)$ is

$$\begin{aligned} \dot{V}_i(t) =& \phi_i(t)\big[g_i(t)u_i(t) + \boldsymbol{\varphi}_i(\boldsymbol{x}_i(t))^{\mathrm{T}}\hat{\boldsymbol{\theta}}_i(t) + q_i(t) \\ & + \dot{e}_i(t) + \hat{D}_i(t)\tanh(\phi_i(t)\lambda_t)\big] + \lambda_1\phi_i^2(t) \\ & - \lambda_1\phi_i^2(t) + \phi_i(t)d_i(t) \\ & - \phi_i(t)\tanh(\phi_i(t)\lambda_t)D_i \\ \leq& \left[g_i(t)\chi(p_i(t)) + 1\right]\dot{p}_i(t) - \lambda_1\phi_i^2(t) \\ & + |\phi_i(t)|\, D_i - \phi_i(t)\tanh\left(\phi_i(t)\lambda_t\right) D_i \\ \leq& \chi(\cdot, p_i(t))\dot{p}_i(t) + \frac{0.2785}{\lambda_t}D_i, \end{aligned} \tag{27}$$

where $\chi(\cdot, p_i(t)) = g_i(t)\chi(p_i(t)) + 1$, and the property

$$|\phi_i(t)| - \phi_i(t)\tanh\left(\phi_i(t)\lambda_t\right) \leq \frac{0.2785}{\lambda_t} \tag{28}$$

introduced in Polycarpou and Ioannou (1996) is used. Note that for switching topologies the coupling strength $a_{ik}(t)$ may not be continuous, but its integral is continuous and smooth. Therefore, we know from Eq. (25) that $e_i(t)$ is continuous, which means that $\phi_i(t)$ is continuous. Furthermore, $\hat{\boldsymbol{\theta}}_i(t)$ and $p_i(t)$ are continuous with the similar reason, which means that $V_i(t)$ and $p_i(t)$ are smooth. Since $V_i(t)$ and $p_i(t)$ are smooth in the interval of $[0, \infty)$ with $V_i(t) \geq 0$, using Lemma 3, we obtain

$$V_i(t),\ p_i(t), \int_0^t \chi(p_i(\tau))\dot{p}_i(\tau)\mathrm{d}\tau \in \mathcal{L}_\infty. \tag{29}$$

Due to the boundedness of $V_i(t)$ and $\int_0^t \chi[p_i(\tau)]\dot{p}_i(\tau)\mathrm{d}\tau$, we know from Eq. (26) that $\phi_i(t)$, $\hat{\boldsymbol{\theta}}_i(t)$, and $\hat{D}_i(t) \in \mathcal{L}_\infty$. Meanwhile, it can be seen from inequality (27) that $\int_0^t \phi_i^2(\tau)\mathrm{d}\tau \in \mathcal{L}_\infty$. Integrating inequality (28) over $[0, t]$, we obtain $\int_0^t |\phi_i(\tau)|\mathrm{d}\tau \leq \hat{D}_i(t) - \hat{D}_i(0) + 0.2785$, which results in $\phi_i(t) \in \mathcal{L}_1$ due to the boundedness of $\hat{D}_i(t)$.

It can be seen from Eq. (25) that

$$\dot{e}_i(t) = \sum_{k=1}^{N} a_{ik}(t)\left[e_k(t) - e_i(t)\right] + \varrho_i^{\phi}(t), \tag{30}$$

with

$$\varrho_i^{\phi}(t) = \sum_{k=1}^{N} a_{ik}(t)\left[\phi_i(t) - \phi_k(t)\right], \tag{31}$$

where $i = 1, 2, \ldots, N$. Due to the fact that $\phi_i(t) \in \mathcal{L}_1$, we have $\varrho_i^{\phi}(t) \in \mathcal{L}_1$ since the number N and $a_{ik}(t)$ in Eq. (31) are limited. It is easy to see that Eq. (30) is regarded as Eq. (1) by letting $\xi_i(t) = e_i(t)$ and $\varrho_i(t) = \varrho_i^{\phi}(t)$. In view of Theorem 1 with $\varrho_i^{\phi}(t) \in \mathcal{L}_1$, one has that all $e_i(t)$ are bounded and $\lim_{t\to\infty} e_i(t) = e_i^*$, where e_i^* is finite. Then, from Eq. (25) and the boundedness of $\phi_i(t)$, the boundedness of $z_i(t)$ is obtained directly.

According to Lemma 4, the boundedness of $z_i(t)$ yields the boundedness of $x_i^{(m)}(t)$ and $q_i(t)$. Therefore, from Eq. (23) we have $u_i(t) \in \mathcal{L}_\infty$, which implies $x_i^{(n)}(t) = \boldsymbol{\varphi}_i(\overline{\boldsymbol{x}}_i(t))^{\mathrm{T}}\boldsymbol{\theta}_i + g_i(t)u_i(t) + d_i(t) \in \mathcal{L}_\infty$ and $\dot{z}_i(t) \in \mathcal{L}_\infty$. Therefore, we have $\dot{\phi}_i(t) = \dot{z}_i(t) + \dot{e}_i(t) \in \mathcal{L}_\infty$. Using Barbalat's lemma, $\phi_i(t)$, $\dot{\phi}_i(t)$, and $\int_0^t \phi_i^2(\tau)\mathrm{d}\tau \in \mathcal{L}_\infty$, we have $\lim_{t\to\infty} \phi_i(t) = 0$. Furthermore, we have proven that $\lim_{t\to\infty} e_i(t) = e_i^* \in \mathcal{L}_\infty$. Then from Eq. (25) we know that $\lim_{t\to\infty} z_i(t) = z_i^* \in \mathcal{L}_\infty$.

Since $\lim_{t\to\infty} z_i(t) = z_i^* \in \mathcal{L}_\infty$, according to Lemma 4, from Eq. (21) we have $\lim_{t\to\infty} x_i^{(m)}(t) = x_{im}^* \in \mathcal{L}_\infty$ and $\lim_{t\to\infty} x_i(t) = x_i^* \in \mathcal{L}_\infty$. We have proved that $x_i^{(m)}(t) \in \mathcal{L}_\infty$. Thus, due to $\lim_{t\to\infty} x_i(t) = x_i^* \in \mathcal{L}_\infty$, and $\dot{x}_i(t)$ and $\ddot{x}_i(t) \in \mathcal{L}_\infty$, according to Barbalat's lemma, we have $\lim_{t\to\infty} \dot{x}_i(t) = 0$. Consequently, with the similar reason for $x_i^{(m)}(t)$ $(m = 2, 3, \ldots, n-1)$, we have $\lim_{t\to\infty} x_i^{(m)}(t) = 0$ $(m = 2, 3, \ldots, n-1)$.

Remark 2 Explanations of Eq. (23) are given as follows: $\chi(\cdot)$ is the Nussbaum-type function to deal with UCDs. $\phi_i(t) = z_i(t) + e_i(t)$ is a key design variable to cope with high-order dynamics, by which we can invoke Theorem 1 with Eq. (25) and avoid dealing with multiple non-identical Nussbaum-type functions. Furthermore, $q_i(t)$ presents the redundant part of $\dot{z}_i(t)$, $\boldsymbol{\varphi}_i(\overline{\boldsymbol{x}}_i(t))^{\mathrm{T}}\hat{\boldsymbol{\theta}}_i(t)$ is the estimate of the uncertainty, and $\hat{D}_i(t)\tanh(\phi_i(t)\lambda_t)$ is a smooth term that copes with the bounded disturbance.

5 Simulation examples

In this section, a group of nonlinear agents (Park et al., 2008) are presented, where the dynamics is given as $x_i^{(3)}(t) = \theta_{1i}x_i^2(t) + \theta_{2i}x_i(t) + \theta_{3i}\dot{x}_i(t) + \theta_{4i}\ddot{x}_{i(t)} + g_iu_i(t) + d_i(t)$, where $\boldsymbol{\theta}_i = [\theta_{1i}, \theta_{2i}, \theta_{3i}, \theta_{4i}]^{\mathrm{T}} = [1, -6, -2.92, -1.2]^{\mathrm{T}}$, $g_1 = 10$, $g_2 = -5$, $g_3 = 5$, $g_4 = -10$, and $d_i(t) = 0.2\cos(2it)$. Assume that the initial conditions $[x_i(0), \dot{x}_i(0), \ddot{x}_i(0)]^{\mathrm{T}}$ for i=1, 2, 3, and 4 are $[3, -1, 1]^{\mathrm{T}}$, $[-4, 1, -0.5]^{\mathrm{T}}$, $[1, 4, -2]^{\mathrm{T}}$, and $[-3, 2, 5]^{\mathrm{T}}$, respectively. We choose Eqs. (23)–(25) with $\chi(p(t)) = p(t)^2\sin(p(t))$, $\lambda_1 = 5$, and $\gamma_3 = 1$. Let the initial states $p_i(t)$, $\hat{\boldsymbol{\theta}}_i(t)$, and $\hat{D}_i(t)$ be zero.

We consider a group of four agents under Assumption 1 with two cases, where the initial conditions are the same but the time-varying networks are different, as shown in Figs. 1 and 2 (for cases 1 and 2, respectively). For case 1, simulation results are shown in Figs. 3–5. It is seen that the convergence is reached for states of nonlinear MASs with non-identical UCDs, and that the signals of closed-loop systems are all bounded. For case 2, it is known from Figs. 6–8 that the consensus of nonlinear MASs can be guaranteed.

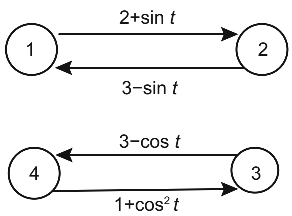

Fig. 1 Graph $\mathcal{G}(t)$ satisfying the cut-balance condition for case 1

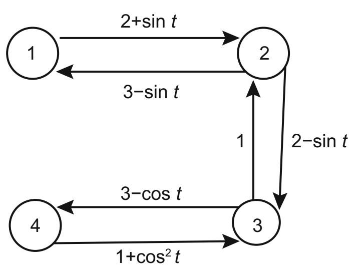

Fig. 2 Graph $\mathcal{G}(t)$ satisfying the cut-balance condition for case 2

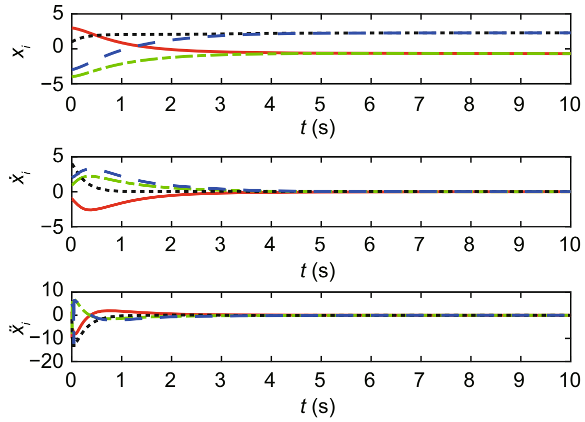

Fig. 3 State trajectories of agents under time-varying networks for case 1

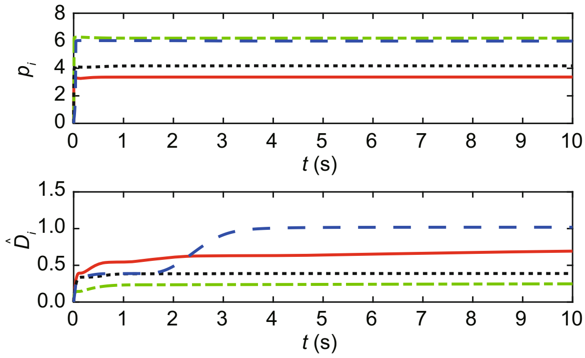

Fig. 4 $p_i(t)$ and $\hat{D}_i(t)$ of agents under time-varying networks for case 1

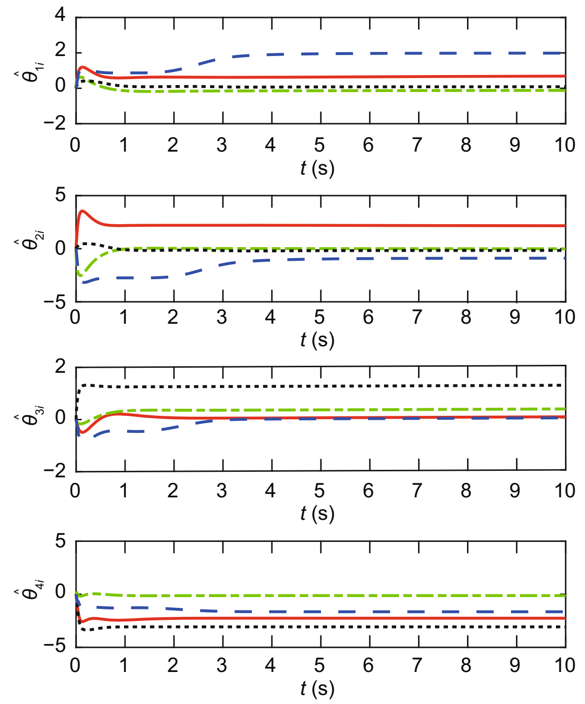

Fig. 5 Estimated $\hat{\boldsymbol{\theta}}_i(t) = [\hat{\theta}_{1i}(t), \hat{\theta}_{2i}(t), \hat{\theta}_{3i}(t), \hat{\theta}_{4i}(t)]^{\mathrm{T}}$ of agents under time-varying networks for case 1

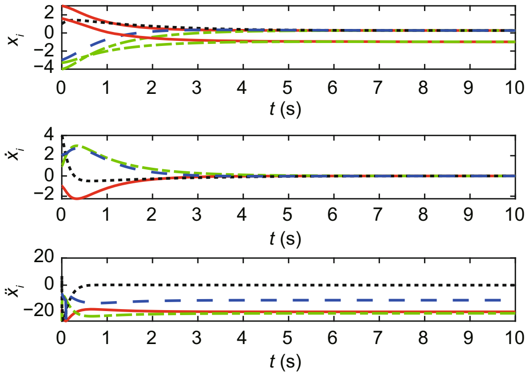

Fig. 6 State trajectories of agents under time-varying networks for case 2

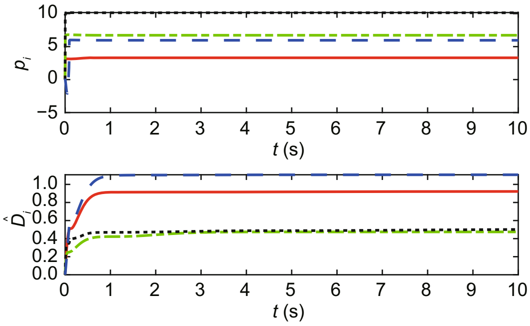

Fig. 7 $p_i(t)$ and $\hat{D}_i(t)$ of agents under time-varying networks for case 2

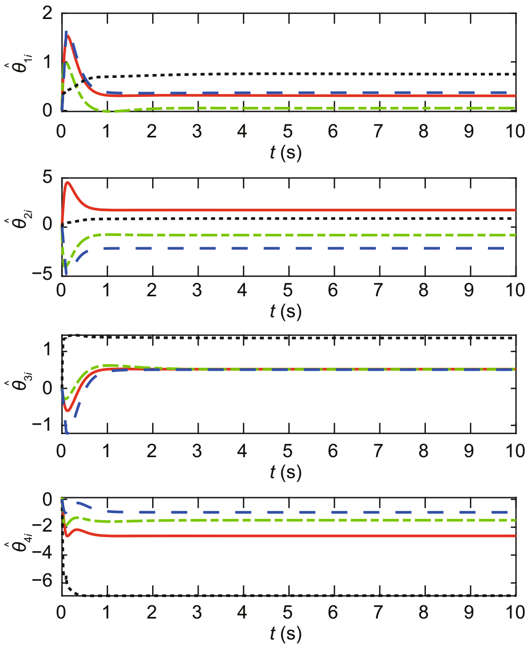

Fig. 8 Estimated $\hat{\theta}_i(t) = [\hat{\theta}_{1i}(t), \hat{\theta}_{2i}(t), \hat{\theta}_{3i}(t), \hat{\theta}_{4i}(t)]^{\mathrm{T}}$ of agents under time-varying networks for case 2

6 Conclusions

In this study, we addressed the convergence property of time-varying networks, and then investigated the cooperative control of nonlinear MASs with UCDs under time-varying networks, where the control algorithms with classical Nussbaum-type functions were presented. It was shown that if the time-varying networks are cut-balance, the convergence of nonlinear MASs can be achieved with the proposed algorithms of nonlinear MASs subject to non-identical UCDs. Further research may focus on robust convergence of more general time-varying networks and cooperative control of nonlinear agents with complex dynamics.

Compliance with ethics guidelines

Qingling WANG declares that he has no conflict of interest.

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