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Constant-gain nonlinear adaptive observers revisited: an application to chemostat systems*

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Abstract: This study deals with constant-gain adaptive observers for nonlinear systems, for which relatively few solutions are available for some particular cases. We introduce an asymptotic observer of constant gain for nonlinear systems that have linear input. This allows the observer design to be formulated within the linear matrix inequality paradigm provided that a strictly positive real condition between the input disturbance and the output is fulfilled. The proposed observer is then applied to a large class of nonlinear chemostat dynamical systems that are widely used in the fermentation process, cell cultures, medicine, etc. In fact, under standard practical assumptions, the necessary change of the chemostat state coordinates exists, allowing use of the constant-gain observer. Finally, the developed theory is illustrated by estimating pollutant concentration in a *Spirulina maxima* wastewater treatment facility.

Key words: Nonlinear observers; Adaptive observers; Coordinate change; Chemostat; Pollutant observation
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1 Introduction

Adaptive observers were initially introduced for linear systems in the early 1970s. Luders and Narendra (1973) proposed the first observer for multi-input multi-output (MIMO) linear systems with a parameter adaptation algorithm. Both the state and the unknown parameter can be asymptotically estimated provided that detectability and persistent excitation (PE) properties of some observer internal signals are fulfilled. Later, Kreisselmeier (1977) proposed the adaptive observers with exponential convergence. The analysis can be extended to systems with a possibly nonlinear input and output injection term and the unknown perturbation multiplied by a constant vector at the state equation's right-hand side, which is perhaps the most general case where PE can be checked through some clear test. This observer theory was extended to the so-called state affine systems assuming the existence of some nominal exponentially stable observer and the PE property; this last PE property is not easy to test, see Zhang (2002), Besançon et al. (2006), and Liang et al. (2008). The research on observer theory for nonlinear systems was extensively developed beginning in 1980s in Bastin and Gevers (1988), and in 1990s in Diop and Fliess (1991), Gauthier et al. (1992), Raghavan and Hedrick (1994), and Marino and Tomei (1995a). An excellent and comprehensive account of fundamental results can be found in the well-known monograph (Marino and Tomei, 1995b).

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In this work, constant-gain nonlinear adaptive observers, i.e., observers with an adaptation differential equation of the same dimension as the unknown parameter vector, are considered. On the other hand, the class of dynamic nonlinear adaptive observers requires more differential equations for the time-varying adaptive gain to obtain better adaptation. The latter approach has deserved more attention and offers a variety of nice results; among them, those based on high-gain techniques have been achieved under different kinds of disturbances, time delays, and so on (Karimi et al., 2010; Wu, 2013).

The celebrated adaptive high-gain observer theory deals with uniformly observable systems or the class of systems in the strictly triangular form (Marino and Tomei, 1995b; Farza et al., 2014); this lasts as far as the dependence of the right-hand side on the state, and the input is concerned. A limiting fact is that the constant unknown parameter is supposed to be multiplied by a vector field having the same triangular dependence on the state, whereas in general, disturbance enters every row of the system's right-hand side. Farza et al. (2009) presented an adaptive high-gain observer for a class of uniformly observable single-output nonlinear systems, in which the compulsory use of the high-gain parameter makes it necessary to analyze the case where it goes to infinity, therefore leading to a resolution of the PE condition as stated in Farza et al. (2014).

Based on the extended Kalman filter, Lafont et al. (2011) proposed an adaptive observer. In Čelikovský et al. (2018), the uncertain parameter was multiplied by a more general vector field that can depend on the state. The constructive proof of the basic high-gain technique allows one to design gains with reasonable values (i.e., the gain did not need to go to infinity), and the PE property needs to be valid for limited gain values. This aspect represents a clear advantage concerning other approaches (Farza et al., 2009, 2014). In contrast to the dynamic case, there are few available results for constant-gain nonlinear adaptive observers. To the best of our knowledge, the linear case's primary achievements and a partial generalization for nonlinear systems were collected in Marino and Tomei (1995b). Roughly speaking, one can have a static adaptive observer with identification of the unknown parameter provided that the observer error dynamics is strictly positive real (SPR) from the unknown parameter to the observed error. Extending particular nonlinear systems having linear dependence on the state, the possible nonlinear input/output injection and the constant unknown disturbance are multiplied by both a constant vector and a scalar nonlinearity depending on output only. Unfortunately, the seemingly rich possibility of q unknown parameters is compressed in a single scalar quantity, which depends only on measurable quantities (output, input). The additional strong assumption enables the use of a strict positive realness property, which is a severe restriction. Moreover, one cannot directly use the results (Zhang, 2002; Liang et al., 2008), because they require that the vector field be output-dependent only. Furthermore, these results require some a priori stability assumptions that might be difficult to achieve.

Due to the success of high-gain techniques, one can be tempted to combine this with static adaptation following a strict passive approach. However, as argued later in the problem motivation section, adaptive high-gain observer stability and SPR conditions lead to opposite simultaneous goals. That is to say, one may obtain a stabilizing high gain without any possibility of satisfying the SPR condition, and vice versa.

In this study, constant-gain adaptive observers for nonlinear systems are considered. The analysis concerns nonlinear affine systems where the unknown disturbance enters linearly through the same input channel. The observer design depends on the solvability of a linear matrix inequality (LMI) subjected to an SPR-type condition between the disturbance and the system's output. Our aim is twofold: (1) to provide a combination of the LMI approach with the constant-gain adaptive observer conditions generalizing those in Marino and Tomei (1995b), and with less restrictive conditions than those in Mondal et al. (2010) or Pourgholi and Majd (2011); (2) to show the theory's applicability to the observation of the unknown states of a large class of nonlinear systems of the chemostat process with the simultaneously unknown constant component of the substrate dilution factor inlet. To this end, a suitable smooth coordinate change for chemostat's nonlinear process is given. Finally, the step-by-step observer design is illustrated in one continuous culture of *Spirulina maxima* for wastewater treatment.

To summarize, the contributions of this study are as follows:

1. We propose a constant-gain adaptive observer for a class of nonlinear systems that generalize previous achievements.

2. For the proof of concept, we consider the challenging chemostat bioprocess. A smooth state transformation leading these complex nonlinear dynamics to a linear one at the input is proposed.

3. The adaptive observer design and performance are validated on a nonlinear system that models a chemostat for pollutant removal based on a culture of *Spirulina maxima*.

The paper is organized as follows: Motivation and problem context are given in Section 2. In Section 3, a constant-gain adaptive observer for a class of nonlinear systems is introduced, for which some LMI sufficient conditions will allow the design of the proposed observers. Section 4 discusses whether it is possible to apply the proposed theory to an essential nonlinear biological process known as the chemostat. Indeed, based on some appealing biological chemostat model properties, it is possible to find a smooth change of variables leading to a transformed chemostat system that is linear at the input. In Section 5, the complete design of the constant-gain adaptive observer is illustrated by estimating the pollutants of a continuous culture of *Spirulina maxima* in a wastewater treatment facility. The efficacy of the observer design is evaluated by a numerical simulation study in Section 6. Conclusions and outlooks for future research are given in Section 7.

2 Motivation and problem context

Consider the following nonlinear system:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})(u+\delta), \\ y = h(\boldsymbol{x}) = x_1, \end{cases} \tag{1}$$

where $\boldsymbol{x}(t)$ is the n-dimensional state vector, and y, u, and δ denote the scalar output, input, and constant parameter perturbation, respectively. Here, $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{x})$ denote appropriate smooth vector fields, and $h(\boldsymbol{x})$ denotes an appropriate smooth function. Furthermore, assume that all right-side vector fields in system (1) are globally uniformly Lipschitz on a subset Ω of $\mathbb{R}^n$ that is forward invariant under the dynamics of the above system for any admissible input.

An adaptive feature is suitable because an unknown disturbance (δ) enters in a way that cannot be suppressed by standard observer techniques (e.g., high gain without adaptation). With a slight abuse of notation, adaptive observers may be classified into static ones and dynamical ones. The first group adds an adaptive differential equation for parameter estimation with the same dimension as the unknown parameter vector. The second one adds yet more dynamical equations for better adaptation. The latter approach has deserved more attention and offers a bunch of good results, some of which based on high-gain techniques have dealt with different kinds of disturbances and time delays; for instance, see Zhang (2002) and Hammouri and Nadri (2013). In contrast, to the best of our knowledge, basic achievements for the linear case, as well as a partial generalization for the nonlinear systems, were collected in Marino and Tomei (1995b). One can have a static adaptive observer with identification of the unknown parameter if the observer error dynamics is strictly passive from the unknown parameter to the observed error. The necessary and sufficient condition for some fixed gains ($\boldsymbol{l}$) is the famous SPR condition, based on a version of the KYP (Kalman, Yacubovich, and Popov) lemma. An extension to particular nonlinear systems of the form

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(y,u,t) + \boldsymbol{P}\boldsymbol{\beta}(y,u,t)\delta \\ y = \boldsymbol{c}\boldsymbol{x} \end{cases}$$

was shown in Marino and Tomei (1995b). Note that nonlinearities depend only on measured quantities (input, output, and time). There is a seemingly rich possibility of q unknown parameters, but they are all compressed into a single scalar quantity $\boldsymbol{\beta}(y,u,t)\delta$; here, $\boldsymbol{\beta}(\cdot)$ is a q-row vector. This single scalar quantity multiplies all components of the constant vector $\boldsymbol{P} = [p_1, p_2, \ldots, p_m]^{\mathrm{T}}$. If one assumes that $\boldsymbol{\beta}(y,u,t)$ is persistently exciting, then one can show asymptotic error convergence to zero (proof is not trivial and some effort is needed). Unfortunately, to the best of our knowledge, a more general case where $\boldsymbol{\beta}(\cdot,u,t)$ might depend on the state $\boldsymbol{x}(t)$ is not further available.

Due to the success of high-gain techniques, one can be tempted to combine this with static adaptation following a strict passive approach. To summarize, high-gain stability inequality

$$(\boldsymbol{A}+\boldsymbol{l}\boldsymbol{c})^{\mathrm{T}}\boldsymbol{P}(r) + \boldsymbol{P}(r)(\boldsymbol{A}+\boldsymbol{l}\boldsymbol{c}) < -\boldsymbol{Q}$$

is not straightforwardly combinable with a static adaptive controller. One can enhance gains and take some $\boldsymbol{P}(r) > 0$ for proving the stability of high gain without an unknown parameter, i.e., not guaranteeing $\boldsymbol{Pp} = \boldsymbol{c}^{\mathrm{T}}$ ($\boldsymbol{p}$ is a constant vector). Alternatively, one can guarantee $\boldsymbol{Pp} = \boldsymbol{c}^{\mathrm{T}}$, but without the possibility of taking high gains. On the other hand, it is possible to adjust $\boldsymbol{l}$ while increasing r to preserve the SPR condition, but this leads to the minimization of $\boldsymbol{l}$ such that high gain is countered.

3 An adaptive nonlinear observer based on the LMI approach

Assume that system (1) can be transformed into the following nonlinear system:

$$\left\{\begin{array}{l} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{p}(u+\delta), \\ y = \boldsymbol{cx} = c_1 x_1, \\ \boldsymbol{c} = [c_1,\ 0,\ \ldots,\ 0]. \end{array}\right. \tag{2}$$

As usual, $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, while y, u, and δ are the scalar output, input, and constant parameter perturbation, respectively.

A Luenberger-type adaptive observer for system (2) is given by

$$\left\{\begin{array}{l} \dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}) + \boldsymbol{p}(u+\phi) + \boldsymbol{lc}(\boldsymbol{z}-\boldsymbol{x}), \\ \dot{\phi} = -k\boldsymbol{ce},\ k > 0, \end{array}\right. \tag{3}$$

where k is a constant defining the adaptation law and $\boldsymbol{e} = \boldsymbol{z} - \boldsymbol{x}$ is the observation.

Note that the goal is to simultaneously satisfy asymptotic state estimation and identification of the unknown parameter δ. To this end, we propose the following:

Theorem 1 Consider system (2) and denote $\boldsymbol{A}$ as the Jacobian matrix of $\boldsymbol{f}(\boldsymbol{x})$, which is assumed to be bounded in a given domain $\Omega \subset \mathbb{R}^n$. Suppose the following LMIs are satisfied:

$$\left\{\begin{array}{l} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{PA} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{Yc} < -2\boldsymbol{P}k_{\mathrm{L}} - \boldsymbol{Q} < 0, \\ \boldsymbol{Pp} = \boldsymbol{c}^{\mathrm{T}}, \end{array}\right. \tag{4}$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are symmetric positive definite matrices, $\boldsymbol{Y}$ is an $n \times 1$ vector such that $\boldsymbol{l} = \boldsymbol{P}^{-1}\boldsymbol{Y}$, and k_{L} is a Lipschitz-type bound of $\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{Ax}$. Then system (3) is an asymptotic observer of system (2).

Proof First, notice that the observer error $\boldsymbol{e}(\cdot) = \boldsymbol{z} - \boldsymbol{x}$ has the following dynamics:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}(t) =& \boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{lc}(\boldsymbol{z}-\boldsymbol{x}) + \boldsymbol{p}(\phi-\delta) \\ =& \boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{z}-\boldsymbol{e}) + \boldsymbol{lce} + \boldsymbol{p}(\phi-\delta) \\ =& \boldsymbol{Ae} + \boldsymbol{lce} + \boldsymbol{p}(\phi-\delta) + \mathcal{O}(\boldsymbol{e}), \end{aligned}$$

where $\boldsymbol{A} = \dfrac{\partial}{\partial t}\boldsymbol{f}(\boldsymbol{z}^*)$, and $\boldsymbol{z}^*$ is some state value belonging to the bounded set Ω, for definiteness. In the following, denote the error of the unknown parameter estimate as

$$\epsilon = \phi - \delta,$$

from which it is clear that

$$\dot{\epsilon} = \dot{\phi}.$$

Let us consider now the following Lyapunov candidate function:

$$V(t) = \frac{1}{2}k\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{Pe}(t) + \frac{1}{2}\epsilon^2(t),\ k > 0,$$

whose derivative along the error dynamics is given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V(t) =& \frac{1}{2}k\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{PA} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{Yc})\boldsymbol{e} \\ & + \cdots + k\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{Pp}\epsilon + k\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\mathcal{O}(\boldsymbol{e}) + \dot{\epsilon}\epsilon \\ =& \frac{1}{2}k\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{PA} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{Yc})\boldsymbol{e} \\ & + \cdots + k\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\mathcal{O}(\boldsymbol{e}). \end{aligned}$$

From the theorem conditions, one has

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V(t) <& -\frac{1}{2}k\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{Qe}(t) - k\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}k_{\mathrm{L}}\boldsymbol{e}(t) \\ & + \cdots + k\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\mathcal{O}(\boldsymbol{e}) \\ <& -k\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}k_{\mathrm{L}}\boldsymbol{e}(t) + k\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\mathcal{O}(\boldsymbol{e}) \\ <& 0. \end{aligned}$$

Therefore, $\boldsymbol{e} \to \boldsymbol{0}$ as $t \to \infty$. To show that ϵ also converges to zero, one can use LaSalle's principle. Considering the augmented variable $[\boldsymbol{e}, \epsilon]^{\mathrm{T}}$, one can see that we have to show that, by LaSalle's principle, the set $[\boldsymbol{0}, \epsilon]^{\mathrm{T}}$ is not invariant in forward time. Dynamics on this set is obvious,

$$\left\{\begin{array}{l} \dot{\boldsymbol{e}} = \boldsymbol{p}\epsilon, \\ \epsilon = 0, \end{array}\right.$$

so that for any $\epsilon \neq 0$, the condition $\boldsymbol{e} = \boldsymbol{0}$ is immediately violated, so that the set $[\boldsymbol{0}, \epsilon]^{\mathrm{T}}$ is not invariant in forward time. Therefore, $\phi \to \delta$ as $t \to \infty$, which is called an adaptive observer with unknown parameter identification.

The adaptation gain k does not play a role in the previous LMI (4); therefore, a relatively large k will improve the speed of convergence to the unknown disturbance $\phi \to \delta$.

Remark 1 It seems that inequality (4) is not an LMI because the decision variables are $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{Y}$. Actually, inequality (4) is equivalent to

$$\begin{aligned} &\boldsymbol{P}^{-1}\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{A}\boldsymbol{P}^{-1}+\boldsymbol{p}\boldsymbol{l}^{\mathrm{T}}+\boldsymbol{l}\boldsymbol{p}^{\mathrm{T}} \\ &<-2\boldsymbol{P}^{-1}k_{\mathrm{L}}-\boldsymbol{P}^{-1}\boldsymbol{Q}\boldsymbol{P}^{-1}, \end{aligned}$$

subject to restriction $\boldsymbol{p}=\boldsymbol{P}^{-1}\boldsymbol{c}^{\mathrm{T}}$. Therefore, the LMI structure is preserved with decision variables $\boldsymbol{P}^{-1}$, $\boldsymbol{Q}$, and $\boldsymbol{l}$.

Remark 2 The stability analysis clearly holds when both $\boldsymbol{z}$ and $\boldsymbol{x}(=\boldsymbol{z}-\boldsymbol{e})$ belong to the set Ω. When $\boldsymbol{z}$ is outside Ω, from the fact that $\boldsymbol{e}(\cdot)$ is bounded and converges to zero, it follows that sooner or later $\boldsymbol{z}(\cdot)$ will enter the domain Ω.

In general, transforming system (1) into system (2) is not an easy task. However, the proposed observer can be applied to a large class of nonlinear dynamical systems that represent the behavior of the widely used chemostat biological process, as we will see in the next section.

4 A smooth change of coordinates for the chemostat bioprocess

In this section, a large class of chemostat nonlinear models is considered. The primary purpose is to introduce a change of state variables such that external disturbances will enter through a constant vector, which turns instrumental for the possible use of the constant-gain nonlinear adaptive observer developed in the previous section.

Chemostats are nonlinear processes that can be described by a system of form (1), namely,

$$\left\{\begin{array}{l} \dot{\boldsymbol{\xi}}=\boldsymbol{f}(\boldsymbol{\xi})+\boldsymbol{g}(\boldsymbol{\xi})(u+\delta), \\ y=h(\boldsymbol{\xi})=\xi_1, \end{array}\right. \tag{5}$$

with the following particular structure:

$$\left\{\begin{array}{l} \boldsymbol{f}(\boldsymbol{\xi})=\boldsymbol{K}\boldsymbol{\varphi}(\boldsymbol{\xi}), \\ \boldsymbol{g}(\boldsymbol{\xi})=\boldsymbol{\xi}_{\mathrm{in}}-\boldsymbol{\xi}, \end{array}\right. \tag{6}$$

where $\xi_1(t),\ \xi_2(t),\ldots,\ \xi_n(t)$ are the n component concentrations in the state vector $\boldsymbol{\xi}$, while $\boldsymbol{\xi}_{\mathrm{in}}$ denotes a vector of the inlet concentration component by component. In turn, $\boldsymbol{\varphi}(\boldsymbol{\xi})$ denotes the vector of the m reaction rates, while $\boldsymbol{K}$ is an $n\times m$ real matrix of yield coefficients, which are negative or positive depending on whether ξ_i is a reactant or a product of the reaction, respectively (Bastin and Dochain, 1990).

To recall some interesting properties of system (5) and without loss of generality, one may consider the following assumptions:

Assumption 1 The dilution rate is bounded as follows:

$$0<u_{\min}\leq u(t),\ \forall t.$$

Assumption 2 The feed rates are bounded as follows:

$$0\leq \xi_{\mathrm{in}_i},\ u\leq F_{\max},\ \forall i,\ \forall t.$$

Assumption 3 Each reaction involves at least one reactant that is neither a catalyst nor an autocatalyst.

Now let us present a slightly different version of a bounded input bounded state (BIBS) stability property given in Bastin and Dochain (1990).

Theorem 2 Consider the chemostat dynamical system (5) and suppose that Assumptions 1–3 are satisfied. Then, for all square-integrable input $u(t)$ and constant input disturbance δ, the state variables of the system are positive and bounded for all t's.

An explicit expression for the upper bound on the state variables can be formulated by introducing the following assumption on the initial conditions:

Assumption 4 The initial values of the state variables $\xi_i(t)$ have upper bounds as follows:

$$a_n\xi_n(0)+\sum_{i\in I}\xi_i(0)\leq \frac{(a_n+q)F_{\max}}{u_{\min}},$$

where

$$a_n=\max_{j\in J}\frac{\sum_{i\in I}\bar{k}_{ij}}{-\bar{k}_{nj}}\geq 0,$$

$\bar{k}_{ij}$ denotes entry (i,j) of matrix $\boldsymbol{K}$, and $I=\{n_1,n_2,\ldots,n_q\}$ and $J=\{m_1,m_2,\ldots,m_p\}$ are sets of indices. n_i $(i=1,2,\ldots,q)$ are the indices of the components (excluding ξ_n) involved in the reactions with an index $m_j\in J$ and m_i $(i=1,2,\ldots,p)$ are the indices of the reactions that involve ξ_n as a reactant (i.e., not as a product).

Corollary 1 Under the same conditions as in the previous theorem and assuming Assumption 4 is satisfied, then the state variables $\xi_i(t)$ of the chemostat dynamical system (5) are non-negative and bounded for all t's, as follows:

$$\xi_i \leq \max\left\{1, \frac{1}{a_n}\right\}(a_n+q)\frac{F_{\max}}{u_{\min}} = \xi_{i_{\max}}.$$

The proof can be found in Bastin and Dochain (1990) (page 53).

Now, let us introduce the main result of this section, which consists of a change of variables for system (5) that is instrumental for the analysis and design of the constant-gain adaptive observer introduced in the previous section.

Lemma 1 Consider the chemostat dynamical system (5) and suppose that Assumptions 1–3 are satisfied. Then there exists a change of variables $x_i = \alpha(\xi_i(t))$ given by

$$\alpha(\xi_i(t)) = \begin{cases} \ln(\xi_i(t)), \text{ if } \xi_{\text{in}_i} = 0, \ \forall t, \\ \ln(\xi_{\text{in}_i} - \xi_i(t)), \text{ if } \xi_{\text{in}_i} - \xi_i > 0, \ \forall t, \end{cases}$$

such that

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{p}(u+\delta), \\ y = \boldsymbol{c}\boldsymbol{x}, \end{cases} \tag{7}$$

where $\boldsymbol{f}(\boldsymbol{x}) = \text{diag}(\exp(x_1), \exp(x_2), \ldots, \exp(x_n))^{-1} \cdot \boldsymbol{K}\bar{\boldsymbol{\varphi}}(\boldsymbol{x})$, $\boldsymbol{p} = [-1, \ -1, \ \ldots, \ -1]^{\text{T}}$, and $\bar{\boldsymbol{\varphi}}(\boldsymbol{x}) = \boldsymbol{\varphi}(\boldsymbol{\xi})$ with $\xi_i = \alpha^{-1}(x_i)$.

Proof First, notice that the i^{th} state or concentration of system (5) can be written as follows:

$$\dot{\xi}_i = \sum_{j\sim i}(\pm)k_{ij}\varphi_j + (\xi_{\text{in}_i} - \xi_i)u,$$

where the notation "$j \sim i$" means that the summation is taken on the reactions with index j, which involve the component with index i; for details, see Dochain (2008).

Now, from the lemma conditions, Theorem 2, and from the system properties, one can see that both ξ_i and $\xi_{\text{in}_i} - \xi_i$ are positive quantities; therefore, the invertible function $\alpha(\xi_i(t))$ is well defined. With this, one may analyze the two possible cases.

For the case $\xi_{\text{in}_i} = 0$, the change of variables is given by $x_i = \ln(\xi_i(t))$, and as a consequence,

$$\dot{x}_i = \frac{1}{\exp(x_i)}\sum_{j\sim i}(\pm)k_{ij}\varphi_j - u.$$

For the case $\xi_{\text{in}_i} - \xi_i > 0$, the change of variables is given by $x_i = \ln(\xi_{\text{in}_i} - \xi_i(t))$ and therefore,

$$\dot{x}_i = \frac{1}{\exp(x_i)}\sum_{j\sim i}(\pm)k_{ij}\varphi_j - u.$$

One determines that all the elements of the input vector $\boldsymbol{P}$ are -1, whereas the derivative terms in the right-hand side of the previous expression can be arranged as follows:

$$\begin{aligned}\boldsymbol{f}(\boldsymbol{x}) =& \text{diag}(\exp(x_1), \exp(x_2), \ldots, \exp(x_n))^{-1} \\ & \cdot \boldsymbol{K}\bar{\boldsymbol{\varphi}}(\boldsymbol{x}),\end{aligned}$$

where $\bar{\boldsymbol{\varphi}}(\boldsymbol{x})$ is $\boldsymbol{\varphi}(\alpha^{-1}(\boldsymbol{x}))$.

Let us mention that the proposed change of coordinates can be easily applied on a number of bioprocess models cited in the literature, see for instance Dochain (2008).

5 Adaptive observer design for a wastewater treatment system

This section concerns the design of the constant-gain adaptive observer developed in Section 3, for the Monod dynamical wastewater treatment facility model. First, according to Theorem 2, the biological consistency property of forwarding state invariance in the positive orthant is recalled. Based on this, the system is transformed to tackle the design of the constant-gain adaptive observer.

Here, we consider a continuous microorganism culture where biomass (x) is, by principle, able to remove pollutants that serve as a substrate (s). Commonly, mass-balance relations lead to the following mathematical description:

$$\begin{cases} \dot{x} = x\mu(s) - x(u+\delta), \\ \dot{s} = -a_3^{-1}x\mu(s) + (a_4 - s)(u+\delta), \end{cases} \tag{8}$$

where Monod's law of microorganism growth is given by

$$\mu(s) = \frac{a_1 s}{a_2 + s}, \tag{9}$$

in which a_1 is the maximal growth rate, a_2 is the saturation constant, a_3 is the yield coefficient, and a_4 is the input substrate concentration. The control input is the rate of dilution $u(t)$ feeding the bioreactor, and the input channel is biased by the constant unknown dilution factor $\delta \neq 0$.

Observing that, without some additional time-varying dilution factor of the substrate input, it may be considered as an active control. Then constant unknown perturbation enters the system to be controlled and observed through the same channel as the known and controlled input.

Because the permissible inputs $u(t)$ are all square-integrable functions, and the external disturbance δ is a positive constant, one may state the following assertions:

1. The positive orthant $\mathcal{O}^+ := \{x > 0, s > 0\}$ is forward invariant with respect to system (8) for every integrable input signal $u(t)$ and perturbation δ.

2. Every solution of system (8) starting in the positive orthant $\mathcal{O}^+$ is bounded for every integrable input signal $u(t)$ and perturbation δ.

These assertions are a direct consequence of Theorem 2. An alternative proof can be found in Čelikovský et al. (2018).

First, let us consider the transformation of system (8). From the previous facts, one may restrict the system states to the set $x > 0$ and $0 \leq s < a_4$. Then, according to Lemma 1, there exists a state transformation given by

$$\begin{aligned} &x_1 = \ln x,\ x_2 = \ln(a_4 - s), \\ &x = \exp(x_1),\ s = a_4 - \exp(x_2). \end{aligned}$$

Thus, Eq. (8) can be expressed as Eq. (10), where input and external disturbances go through a constant input vector. Hence, it is possible to use an observer as the one in Eq. (3), that is to say,

$$\begin{cases} \dot{x}_1 = \dfrac{a_1 a_4 - a_1 \exp(x_2)}{a_2 + a_4 - \exp(x_2)} - (u + \delta), \\ \dot{x}_2 = \dfrac{\exp(x_1)}{a_3 \exp(x_2)} \dfrac{a_1 a_4 - a_1 \exp(x_2)}{a_2 + a_4 - \exp(x_2)} - (u + \delta). \end{cases} \tag{10}$$

It can be appreciated that Eq. (10) is not in the adaptive observer form. Moreover, it can be seen that $\boldsymbol{p} = [-1,\ -1]^{\mathrm{T}}$, so it does not hold that $p_1 > 0$ or that $p_1 s + p_2$ is Hurwitz. As a consequence, it is not possible to use the approach given in Marino and Tomei (1995b).

Next, let us consider the observer design. To simplify the observer design, we write the transformed system (10) in the following compact form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_2) \\ f_1(x_2) f_2(x_1, x_2) \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} (u + \delta), \tag{11}$$

where

$$f_1(x_2) = \frac{a_1 a_4 - a_1 \exp(x_2)}{a_2 + a_4 - \exp(x_2)},\ f_2(x_1, x_2) = \frac{\exp(x_1)}{a_3 \exp(x_2)}.$$

Then, according to Theorem 1, the constant-gain observer is given by

$$\begin{cases} \dot{\boldsymbol{z}} = \begin{bmatrix} f_1(z_2) \\ f_1(z_2) f_2(x_1, z_2) \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} (u + \phi) + \boldsymbol{lce}, \\ \dot{\phi} = -k\boldsymbol{ce}. \end{cases} \tag{12}$$

Now, let us compute the following partial derivatives:

$$\begin{aligned} \hat{f}_{12} &:= \frac{\partial f_1(z_2)}{\partial z_2} = \frac{-a_1 a_2 \exp(z_2)}{(a_2 + a_4 - \exp(z_2))^2}, \\ \hat{f}_{22} &:= \frac{\partial f_1(z_2) f_2(x_1, z_2)}{\partial z_2} \\ &= \exp(x_1) \frac{-a_2 a_4 + 2a_2 \exp(z_2) - (a_4 - \exp(z_2))^2}{a_3 \exp(z_2)(a_2 + a_4 - \exp(z_2))^2}. \end{aligned}$$

Note that both partial derivatives of $f_1(z_2)$ and $f_1(z_2) \cdot f_2(x_1, z_2)$ with respect to z_1 are equal to zero, which is due to the fact that $z_1 = x_1$, because this last is available for measurement.

As a consequence, the error $\boldsymbol{e}(t) = \boldsymbol{z}(t) - \boldsymbol{x}(t)$ dynamics is given by

$$\dot{\boldsymbol{e}}(t) = \begin{bmatrix} c_1 l_1 & \hat{f}_{12}(z_2) \\ c_1 l_2 & \hat{f}_{22}(x_1, z_2) \end{bmatrix} \boldsymbol{e}(t) + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \epsilon + \mathcal{O}(\boldsymbol{e}). \tag{13}$$

Now, recalling that $s < a_4$, then $s \leq a_4 - s_\Delta$ for some $s_\Delta > 0$, and one may obtain the following useful bounds:

$$-\frac{a_1 a_4}{a_2} < \hat{f}_{12}(z_2) < -\frac{a_1 a_2 s_\Delta}{(a_2 + a_4 - s_\Delta)^2}, \tag{14}$$

$$\frac{-a_1 a_4 \max\{x\}}{a_3 s_\Delta} \left(\frac{a_2 + a_4}{a_2^2} \right) < \hat{f}_{22}(x_1, z_2) < 0. \tag{15}$$

In the search for a numerical solution for the inequalities in Theorem 1, it is more convenient to use the equivalent formulation given in Remark 1, namely,

$$\begin{aligned} &\boldsymbol{P}^{-1} \left[\boldsymbol{lc} + \begin{bmatrix} 0 & \hat{f}_{12} \\ 0 & \hat{f}_{22} \end{bmatrix} \right]^{\mathrm{T}} + \left[\boldsymbol{lc} + \begin{bmatrix} 0 & \hat{f}_{12} \\ 0 & \hat{f}_{22} \end{bmatrix} \right] \boldsymbol{P}^{-1} \\ &< -2\boldsymbol{P}^{-1} k_{\mathrm{L}} - \boldsymbol{P}^{-1} \boldsymbol{Q} \boldsymbol{P}^{-1} < 0 \end{aligned} \tag{16}$$

with

$$\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} > 0,\ \boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} > 0,$$

such that

$$\boldsymbol{Pp} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} = \boldsymbol{c}^{\mathrm{T}}.$$

As mentioned before, the numerical search can be restricted to the following domain:

$$\hat{f}_{12} \in \left\{ -\frac{a_1 a_4}{a_2}, -\frac{\alpha a_1 a_2}{(a_2 + a_4 - \alpha)^2} \right\},$$

$$\hat{f}_{22} \in \left\{ \frac{-a_1 a_4 \max\{x\}}{a_3 \alpha} \left(\frac{a_2 + a_4}{a_2^2} \right), 0 \right\}.$$

Therefore, if there exists any solution to LMI (16), then a constant-gain adaptive observer will exist for system (10), or equivalently, system (8).

6 Numerical simulation

The model's nominal parameters were adjusted from the experimental data of a batch culture of blue-green microalgae *Spirulina maxima* (Čelikovský et al., 2018). To be precise, nominal parameters are given in Table 1, initial conditions of the system and observer are given in Table 2, and the input and perturbation values are given in Table 3. In addition, the results of LMI (4) for $s_\Delta = 0.1$ are as follows:

$$\begin{cases} \boldsymbol{A} = \begin{bmatrix} 0 & -1.4 \times 10^{-6} \\ 0 & -3.985 \end{bmatrix}, \\ \boldsymbol{P} = \begin{bmatrix} 500.7501 & -499.7501 \\ -499.7501 & 499.7501 \end{bmatrix}, \\ \boldsymbol{Q} = \begin{bmatrix} 2.5048 & -2.5023 \\ -2.5023 & 2.4998 \end{bmatrix} \times 10^{11}, \\ k_{\mathrm{L}} = 3.984. \end{cases} \quad (17)$$

The observer dynamics is given by

$$\begin{cases} \dot{\boldsymbol{z}} = \begin{bmatrix} f_1(z_2) \\ f_1(z_2) f_2(x_1, z_2) \end{bmatrix} + \boldsymbol{p}(u + \phi) + \boldsymbol{lce}, \\ \dot{\phi} = -k\boldsymbol{ce}, \end{cases} \quad (18)$$

where

$$\boldsymbol{c} = [-1,\ 0],\ \boldsymbol{l} = [4.6774,\ 0.6924]^{\mathrm{T}},$$
$$\boldsymbol{p} = [-1,\ -1]^{\mathrm{T}},\ k = 1.$$

The software used in the numerical simulation was MATLAB® with ode113, a relative tolerance of 1×10^{-6}, and an absolute tolerance of 1×10^{-3}, whereas Yalmip and Sedumi were used to solve LMI (4).

The observer gains were computed depending on the Jacobian matrix of the vector field $\boldsymbol{f}(\boldsymbol{x})$ evaluated at $z_1 = x_1 = 13.2312$ and $z_2 = 13.0869$. The measured output was corrupted with a signal issued from a function Gaussian noise generator wgn of MATLAB at 0 dBW of magnitude ±0.03 mg/L.

The state of the bioreactor, formed by the biomass concentration (x) and substrate concentration (s), with input/disturbance values $u+\delta$ given in Table 3, is depicted in Fig. 1. In turn, the bioreactor substrate and the constant-gain observer substrate affected by the output noise are shown in Fig. 2, where it is possible to see that after some initial time, the adaptive observer converges to the substrate of the system. Note that the observed state is robust against the input perturbations of 10% ($u + \delta = 0.0015 \pm 10\%u$) at $t = 400$ and 800 h and that the output noise effects are negligible.

Table 1 Nominal parameters

a_1 (1/h)	a_2 (mg/L)	a_3	a_4 (mg/L)	max(x)
0.027	25	0.2899	205	534.75

Table 2 Initial conditions of the system and observer

$x_1(0) = z_1(0)$ (mg/L)	$x_2(0)$ (mg/L)	$z_2(0)$ (mg/L)
3.5553	3.9703	4.2195

Table 3 Input and perturbation values

t (h)	$u(t)$ (1/h)	$\delta(t)$ (1/h)
0–400	0.015	0
400–800	0.015	0.0015
800–1200	0.015	−0.0015

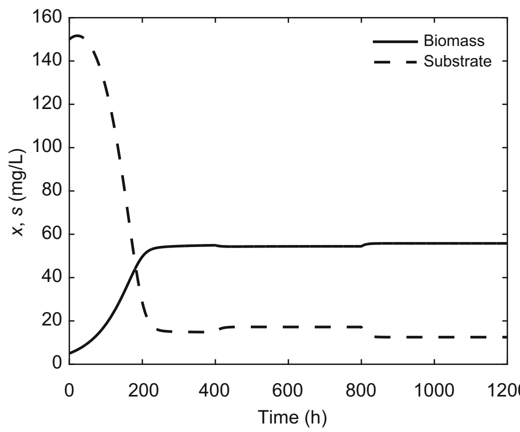

Fig. 1 Bioreactor state responses for input/disturbance stimulus

To illustrate the advantages of the proposed adaptation, we have chosen the classical high-gain observer given in Gauthier et al. (1992). In Fig. 3, one can see that the high-gain observer is not able to give a reasonable estimate of the substrate in the presence of input disturbances or output noise. As is known, the high-gain observer shows peaking phenomena, even without noisy output signals; note that the estimated substrate also shows a gap relative to the real substrate concentration values; see the white dashed line in the zoom of Fig. 3.

The observed perturbation is displayed in Fig. 4. In the absence of perturbations, the observed value converges to zero in around 300 h. When the perturbation is present at time $t = 400$ and 800 h, the observed perturbation follows the real perturbation after a relatively short convergence time, despite the presence of output noise.

In Fig. 5, one can appreciate the different errors between real and observed states. Note that the error in biomass is due to the observer's correction output injection terms (see Eq. (12)). While the substrate and perturbation errors are relatively small, they converge to zero in a short time concerning the system dynamics.

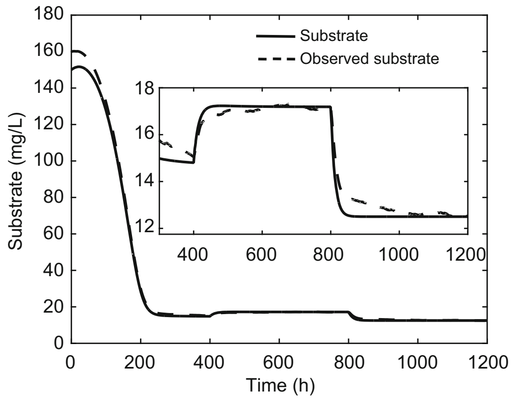

Fig. 2 Dynamics of the observed substrate by the constant-gain adaptive observer

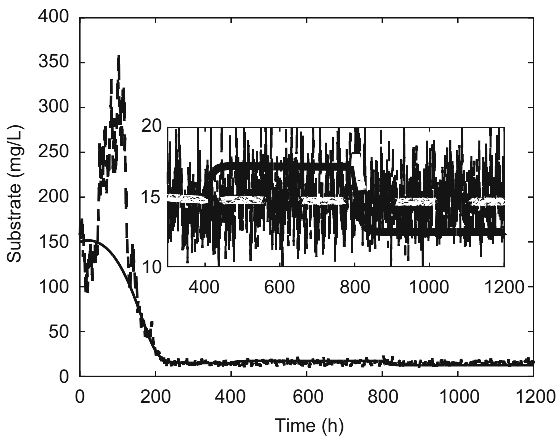

Fig. 3 High-gain substrate estimation. The full line is the real state, the black dashed line is the high-gain state with output noise, and the white dashed line is the high-gain state without output noise

7 Conclusions

Adaptive observers of constant gain for nonlinear systems are revisited, which is the main issue from the theoretical perspective. A solution has been given for nonlinear systems that are linear in the input. The critical point is to find a change of variables to obtain a transformed system with the desired structure, i.e., nonlinear state but linear input. If so, it is possible to reformulate the observation problem within the linear matrix inequality

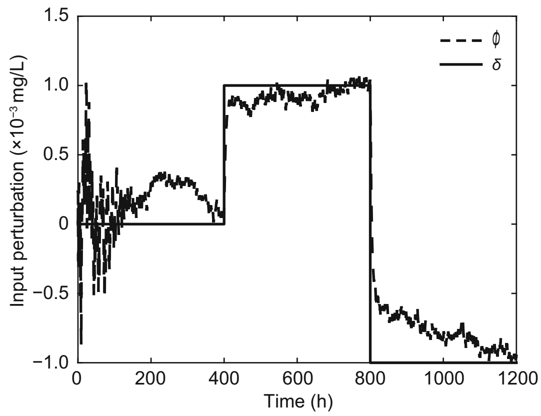

Fig. 4 Estimation of the input perturbation δ by the constant-gain adaptive observer

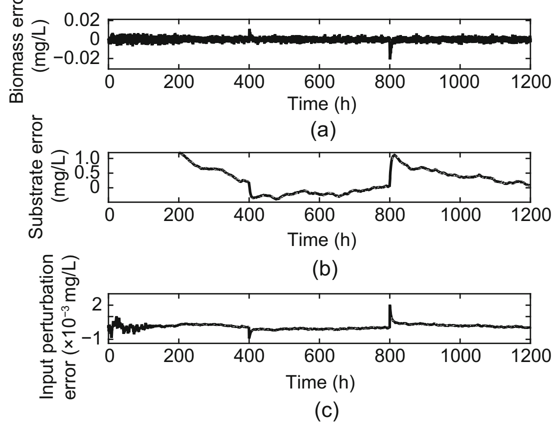

Fig. 5 Estimation errors by the constant-gain adaptive observer: (a) biomass error; (b) substrate error; (c) input perturbation error

(LMI) paradigm. In this vein, one may determine the observer output injection gains provided that an SPR-type condition from the input disturbance to the system output holds.

From the practical point of view, the main aim was to apply this theory to chemostat nonlinear dynamical systems. Such a process has been widely used in biological processes and has received a lot of attention from researchers in biology, medicine, control engineering, etc. Therefore, based on appealing natural properties, a suitable change of variables has been proposed for the chemostat nonlinear dynamic model. It is worth noting that its positive and bounded state properties increase the possibility of finding good observer gains such that the SPR property holds in a vast area around some equilibrium point.

The proposed observer efficacy has been illustrated using a model (issued from experimental data) of a wastewater facility for pollutant removal by the green microalgae *Spirulina maxima*.

Finally, based on the observer theory and the design developed in this work, we aim to consider disturbance rejection as part of our future work.

Contributors

Jorge A. TORRES and Sergej ČELIKOVSKÝ conceptualized the research. Arno SONCK developed the theoretical findings of the work. Arno SONCK and Alma R. DOMINGUEZ contributed to the design and simulation of the experiment results. Arno SONCK and Jorge A. TORRES drafted the manuscript. Sergej ČELIKOVSKÝ revised and finalized the paper.

Compliance with ethics guidelines

Jorge A. TORRES, Arno SONCK, Sergej ČELIKOVSKÝ, and Alma R. DOMINGUEZ declare that they have no conflict of interest.

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