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Sampling formulas for 2D quaternionic signals associated with various quaternion Fourier and linear canonical transforms*#

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Abstract: The main purpose of this paper is to study different types of sampling formulas of quaternionic functions, which are bandlimited under various quaternion Fourier and linear canonical transforms. We show that the quaternionic bandlimited functions can be reconstructed from their samples as well as the samples of their derivatives and Hilbert transforms. In addition, the relationships among different types of sampling formulas under various transforms are discussed. First, if the quaternionic function is bandlimited to a rectangle that is symmetric about the origin, then the sampling formulas under various quaternion Fourier transforms are identical. If this rectangle is not symmetric about the origin, then the sampling formulas under various quaternion Fourier transforms are different from each other. Second, using the relationship between the two-sided quaternion Fourier transform and the linear canonical transform, we derive sampling formulas under various quaternion linear canonical transforms. Third, truncation errors of these sampling formulas are estimated. Finally, some simulations are provided to show how the sampling formulas can be used in applications.

Key words: Quaternion Fourier transforms; Quaternion linear canonical transforms; Sampling theorem; Quaternion partial and total Hilbert transforms; Generalized quaternion partial and total Hilbert transforms; Truncation errors

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1 Introduction

Sampling theory is one of the most important mathematical techniques used in communication engineering, information theory, signal analysis, image processing, and so on (Zayed, 1993; Cheng and Kou, 2019, 2020). Sampling theories of multi-dimensional real signals in $\mathbb{R}^N$ appeared in Zayed (1993). Sampling theories of high-dimensional signals in several complex variable settings (Kou and Qian, 2005a) and the Clifford analysis setting (Kou and Qian, 2005b)

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were obtained. Recently, the quaternions, which are hyper-complex numbers, have been proved to be an effective tool in quite a few applications of multi-dimensional signal processing analysis (Jiang et al., 2016; Zou et al., 2016; Hu and Kou, 2018; Bahia and Sacchi, 2020; Alon and Paran, 2021), and account for correlated nature of the signal components in a natural way. Meanwhile, numerous novel tools in multi-dimensional signal and image processing have been developed using the quaternion modeling technique. The representative tools are quaternion Fourier transforms (QFTs), quaternion fractional Fourier transforms (QFrFTs), and quaternion linear canonical transforms (QLCTs) (Ell et al., 2014; Kou and Morais, 2014; Chen et al., 2015; Kou et al., 2017; Lian, 2021). Some important theorems have been studied, such as the inverse theorems associated with QFTs and QLCTs (Hu and Kou, 2017) and convolution theorems associated with QFTs and QLCTs (Pei et al., 2001; Hitzer E, 2017).

The sampling expansions associated with the right-sided QFT were studied in Cheng and Kou (2018), but the main shortcoming of this approach is that only the right-sided QFT case was considered. In this study, we study not only the sampling formula associated with the right-sided QFT, but also the sampling formulas associated with the left- and two-sided QFTs. Therefore, sampling formulas associated with various QLCTs are obtained.

There are five main points in this study: (1) to study the quaternionic function sampling formulas bandlimited (BL) to a rectangle that is symmetric about the origin under various QFTs; (2) to study the quaternionic function sampling formulas BL to a rectangle that is not symmetric about the origin under various QFTs; (3) to explore not only the sampling formulas using samples of themselves, but also samples of the partial derivatives and quaternion partial and total Hilbert transforms; (4) to obtain the sampling formulas associated with various QLCTs using relationships between QFTs and QLCTs; (5) to estimate truncation errors of these sampling formulas.

2 Preliminaries of quaternion, QFTs, and QLCTs

In this section, we are devoted to the exposition of basic preliminary materials that are used extensively throughout this paper.

Let $\mathbb{H}$ denote the Hamiltonian skew field of quaternions, which has been proved to provide a natural framework for a unified treatment of three- and four-dimensional signals.

A quaternionic number takes the form of

$$q = q_0 + \mathrm{i}q_1 + \mathrm{j}q_2 + \mathrm{k}q_3, \tag{1}$$

where $q_0 - q_3 \in \mathbb{R}$, and $\mathrm{i}, \mathrm{j}, \mathrm{k}$ are orthogonal imaginary parts obeying the following rules: $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$. In this way, the quaternionic algebra can be regarded as a non-commutative extension of complex numbers $\mathbb{C}$. When $q_0 = 0$, q becomes a pure quaternion. Let $\mu := \mathrm{i}\mu_1 + \mathrm{j}\mu_2 + \mathrm{k}\mu_3$ denote the unit pure quaternion such that $\mu^2 = -1$. Let $\mathbb{H}_\mu$ be the field spanned by $\{1, \mu\}$, which is the sub-field of $\mathbb{H}$. That is, $\mathbb{H}_\mu := \{q | q = q_0 + \mu q_\mu, q_0, q_\mu \in \mathbb{R},\ \mu^2 = -1\}$. The quaternion conjugate of a quaternion q is defined by $q^* = q_0 - \mathrm{i}q_1 - \mathrm{j}q_2 - \mathrm{k}q_3$, which implies the modulus of $q \in \mathbb{H}$ defined as $|q| = \sqrt{qq^*} = \sqrt{q^*q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. From Eq. (1), it follows that a quaternionic function $f : \mathbb{R}^2 \to \mathbb{H}$ can be expressed as $f(x, y) = f_0(x, y) + \mathrm{i}f_1(x, y) + \mathrm{j}f_2(x, y) + \mathrm{k}f_3(x, y)$, where $f_n \in \mathbb{R}, n = 0, 1, 2, 3$.

Let $L^p(\mathbb{R}^2, \mathbb{H})$ (integer $p \geq 1$) be the linear space of all quaternionic functions in $\mathbb{R}^2$, whose quaternion modulus is $L^p(\mathbb{R}^2, \mathbb{H}) := \{f | f : \mathbb{R}^2 \to \mathbb{H}, \|f\|_p := \left(\int_{\mathbb{R}} |f(x, y)|^p \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} < \infty\}$.

Based on the quaternion concept, various QFTs (Ell et al., 2014) and QLCTs (Kou et al., 2013) have been introduced. For $f \in L^1(\mathbb{R}^2, \mathbb{H})$, the two-sided QFT is as follows:

$$\mathcal{F}_{\mathrm{T}}[f](v, u) := \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}vx} f(x, y) \mathrm{e}^{-\mathrm{j}uy} \mathrm{d}x\mathrm{d}y. \tag{2}$$

The right-sided QFT is as follows:

$$\mathcal{F}_{\mathrm{R}}[f](v, u) := \int_{\mathbb{R}^2} f(x, y) \mathrm{e}^{-\mathrm{i}vx} \mathrm{e}^{-\mathrm{j}uy} \mathrm{d}x\mathrm{d}y.$$

The left-sided QFT is as follows:

$$\mathcal{F}_{\mathrm{L}}[f](v, u) := \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}vx} \mathrm{e}^{-\mathrm{j}uy} f(x, y) \mathrm{d}x\mathrm{d}y.$$

QLCTs are the generalizations of QFTs. Let $\boldsymbol{A}_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be real matrix parameters with unit determinant, i.e., $\det(\boldsymbol{A}_i) = a_i d_i - c_i b_i = 1$, for $i = 1, 2$.

The two-sided QLCT is as follows:

$$\mathcal{L}_{\mathrm{T}}^{\mathrm{i,j}}(f)(v,u) := \begin{cases} \int_{\mathbb{R}^2} K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v) f(x,y) K_{\boldsymbol{A}_2}^{\mathrm{j}}(y,u)\mathrm{d}x\mathrm{d}y, \\ \quad b_1, b_2 \neq 0, \\ \int_{\mathbb{R}^2} \sqrt{d_1}\mathrm{e}^{\mathrm{i}\frac{c_1 d_1 v^2}{2}} f(\mathrm{d}v, y) K_{\boldsymbol{A}_2}^{\mathrm{j}}(y,u)\mathrm{d}y, \\ \quad b_1 = 0, b_2 \neq 0, \\ \int_{\mathbb{R}^2} K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v) f(x, \mathrm{d}u)\sqrt{d_2}\mathrm{e}^{\mathrm{j}\frac{c_2 d_2 v^2}{2}}\mathrm{d}x, \\ \quad b_1 \neq 0, b_2 = 0, \\ \mathrm{e}^{\mathrm{i}\frac{c_1 d_1 u^2}{2}} f(\mathrm{d}v, \mathrm{d}u)\sqrt{d_1}\sqrt{d_2}\mathrm{e}^{\mathrm{j}\frac{c_2 d_2 v^2}{2}}, \\ \quad b_1 = 0, b_2 = 0. \end{cases}$$

The right-sided QLCT is as follows:

$$\mathcal{L}_{\mathrm{R}}^{\mathrm{i,j}}(f)(v,u) := \int_{\mathbb{R}^2} f(x,y) K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v) K_{\boldsymbol{A}_2}^{\mathrm{j}}(y,u)\mathrm{d}x\mathrm{d}y.$$

The left-sided QLCT is as follows:

$$\mathcal{L}_{\mathrm{L}}^{\mathrm{i,j}}(f)(v,u) := \int_{\mathbb{R}^2} K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v) K_{\boldsymbol{A}_2}^{\mathrm{j}}(y,u) f(x,y)\mathrm{d}x\mathrm{d}y.$$

Herein kernels $K_{\boldsymbol{A}_1}^{\mathrm{i}}$ and $K_{\boldsymbol{A}_2}^{\mathrm{j}}$ of QLCTs are given by $K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v) := \frac{1}{\sqrt{\mathrm{i}2\pi b_1}}\mathrm{e}^{\mathrm{i}(\frac{a_1}{2b_1}x^2 - \frac{1}{b_1}vx + \frac{d_1}{2b_1}v^2)}$ and $K_{\boldsymbol{A}_2}^{\mathrm{j}}(y,u) := \frac{1}{\sqrt{\mathrm{j}2\pi b_2}}\mathrm{e}^{\mathrm{j}(\frac{a_2}{2b_2}y^2 - \frac{1}{b_2}yu + \frac{d_2}{2b_2}u^2)}$. Note that when $b_i = 0$ $(i = 1, 2)$, QLCT of a function is essentially a chirp multiplication and is of no particular interest to us. Hence, without loss of generality, we set $b_i \neq 0$ $(i = 1, 2)$ throughout the paper. It is significant to note that QLCT converts to its special cases when we take different matrices $\boldsymbol{A}_i$ $(i = 1, 2)$. For example, when $\boldsymbol{A}_1 = \boldsymbol{A}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, QLCT is reduced to QFT, where $\sqrt{-\mathrm{i}} = \mathrm{e}^{-\mathrm{i}\pi/4}$ and $\sqrt{-\mathrm{j}} = \mathrm{e}^{-\mathrm{j}\pi/4}$. If $\boldsymbol{A}_1 = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ and $\boldsymbol{A}_2 = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$, QLCT becomes QFrFT multiplied by the fixed phase factors $\mathrm{e}^{-\mathrm{i}\alpha/2}$ and $\mathrm{e}^{-\mathrm{j}\beta/2}$. QFTs are invertible, and for their inversion theorems, readers can refer to Hu and Kou (2017).

Lemma 1 (Hu and Kou, 2017) Suppose that the quaternionic function $f(x,y)$ satisfies one of the following conditions:

(1) f and the corresponding QFT both belong to $L^1(\mathbb{R}^2, \mathbb{H})$.

(2) $f \in L^2(\mathbb{R}^2, \mathbb{H})$, if $\mathrm{supp}\,(\mathcal{F}_n[f]) \subset [\sigma_1, \sigma_2] \times [\sigma_3, \sigma_4]$, where $n = \mathrm{R}$, L, or T, and σ_i $(i = 1, 2, 3, 4)$ are real constants, and can be chosen at will.

Then

$$f(x,y) = \frac{1}{4\pi^2}\int_D \mathrm{e}^{\mathrm{i}vx}\mathcal{F}_{\mathrm{T}}[f](v,u)\mathrm{e}^{\mathrm{j}uy}\mathrm{d}u\mathrm{d}v.$$

$$f(x,y) = \frac{1}{4\pi^2}\int_D \mathcal{F}_{\mathrm{R}}[f](v,u)\mathrm{e}^{\mathrm{j}uy}\mathrm{e}^{\mathrm{i}vx}\mathrm{d}u\mathrm{d}v.$$

$$f(x,y) = \frac{1}{4\pi^2}\int_D \mathrm{e}^{\mathrm{j}uy}\mathrm{e}^{\mathrm{i}vx}\mathcal{F}_{\mathrm{L}}[f](v,u)\mathrm{d}u\mathrm{d}v,$$

where $D = \mathbb{R}^2$ for condition (1) and $D = [\sigma_1, \sigma_2] \times [\sigma_3, \sigma_4]$ for condition (2).

Lemma 2 (Quaternion Cauchy-Schwarz inequality) (Kou and Qian, 2005a) If $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then Hölder's inequality yields

$$\left|\int_{\mathbb{R}^2} f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y\right| \leq \int_{\mathbb{R}^2} |f(x,y)g(x,y)|\,\mathrm{d}x\mathrm{d}y \leq \|f(x,y)\|_2\|g(x,y)\|_2. \tag{3}$$

Lemma 3 (Parseval equality) (Hitzer EMS, 2007) Set $f \in L^2(\mathbb{R}^2, \mathbb{H})$. Then we have

$$\mathbb{E} := \|f\|_2 = \frac{1}{4\pi^2}\|\mathcal{F}_n\|_2, \quad n = \mathrm{T}, \mathrm{R}, \mathrm{L}. \tag{4}$$

3 Sampling theorems for bandlimited (BL) quaternionic functions

3.1 Sampling theorems for BL quaternionic functions in the QFT sense

Let f be a quaternionic function defined on $\mathbb{T}^2$, where $\mathbb{T} = [0, 2\pi]$ and $\mathbb{T}^2$ is the Cartesian product of $\mathbb{T} \times \mathbb{T}$. The space $L^p(\mathbb{T}^2, \mathbb{H})$ consists of all quaternionic functions such that $\int_{\mathbb{T}^2} |f(x,y)|^p\mathrm{d}x\mathrm{d}y < \infty$. For a function $f \in L^2(\mathbb{T}^2, \mathbb{H})$, we can define the right-sided quaternion Fourier coefficients as $c_{n,m} = \frac{1}{4\pi^2}\int_{\mathbb{T}^2} f(x,y)\mathrm{e}^{-\mathrm{i}nx}\mathrm{e}^{-\mathrm{j}my}\mathrm{d}x\mathrm{d}y$. Then the right-sided quaternion Fourier series can be written as $f(x,y) \sim \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_{n,m}\mathrm{e}^{\mathrm{j}my}\mathrm{e}^{\mathrm{i}nx}$.

Lemma 4 (Bessel's inequality) Let $f \in L^2(\mathbb{T}^2, \mathbb{H})$. Then Bessel's inequality holds: $\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |c_{n,m}|^2 \leq \frac{1}{4\pi^2}\int_{\mathbb{T}^2} |f(x,y)|^2\mathrm{d}x\mathrm{d}y < \infty$.

The proof of Lemma 4 is available in the supplementary materials.

Due to Bessel's inequality, the right-sided quaternion Fourier series $\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_{n,m}\mathrm{e}^{\mathrm{i}nx}\mathrm{e}^{\mathrm{j}my}$ will converge in the sense of L^2. Indeed, it can be proved that the system $\{\mathrm{e}^{\mathrm{i}nx}\mathrm{e}^{\mathrm{j}my} | (n,m) \in \mathbb{Z}^2\}$ is complete in $L^2(\mathbb{T}^2, \mathbb{H})$. Then with an argument similar to that in Pan (2000), we have the following lemma:

Lemma 5 Let $f \in L^2(\mathbb{T}^2, \mathbb{H})$, $f(x,y) \sim \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_{n,m}\mathrm{e}^{\mathrm{j}my}\mathrm{e}^{\mathrm{i}nx}$, and $S_{N,M} = \sum_{n=-N}^{N}\sum_{m=-M}^{M} c_{n,m}\mathrm{e}^{\mathrm{j}my}\mathrm{e}^{\mathrm{i}nx}$. Then $\lim_{N\to\infty,M\to\infty}\|f - S_{N,M}\|_2 = \lim_{N\to\infty,M\to\infty}\int_{\mathbb{T}^2}|f(x,y)-S_{N,M}(x,y)|^2\mathrm{d}x\mathrm{d}y = 0$.

The Parseval equality holds as

$$\|f(x,y)\|_2 = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|c_{n,m}|^2.$$

Definition 1 $f(x,y)$ is said to be a BL signal (function) of $[-\sigma_1,\sigma_1]\times[-\sigma_2,\sigma_2]$ in the right-sided QFT sense, i.e., $\mathcal{F}_{\mathrm{R}}[f](v,u)=0$, for $|v|>\sigma_1$ or $|u|>\sigma_2$.

In the following, for simplicity, we use the following abbreviated notations: $(\sigma_1,\sigma_2) := [-\sigma_1,\sigma_1]\times[-\sigma_2,\sigma_2]$. To formulate our sampling formulas, we need a very important result:

Theorem 1 Suppose $f(x,y) = f_0(x,y) + \mathrm{i}f_1(x,y) + \mathrm{j}f_2(x,y) + \mathrm{k}f_3(x,y)$, $f \in L^2\cup L^1(\mathbb{R}^2,\mathbb{H})$. Then the following four statements are equivalent:

(1) $\mathcal{F}_{\mathrm{R}}[f](v,u)=0$, for $|v|>\sigma_1$ or $|u|>\sigma_2$.

(2) $\mathcal{F}_{\mathrm{R}}[f_n](v,u)=0$, for $|v|>\sigma_1$ or $|u|>\sigma_2$, $n=0,1,2,3$.

(3) $\mathcal{F}_{\mathrm{L}}[f](v,u)=0$, for $|v|>\sigma_1$ or $|u|>\sigma_2$.

(4) $\mathcal{F}_{\mathrm{T}}[f](v,u)=0$, for $|v|>\sigma_1$ or $|u|>\sigma_2$.

Proof It is sufficient to prove the equivalence between the first and second statements because $\mathcal{F}_{\mathrm{R}}[f_n](v,u) = \mathcal{F}_{\mathrm{L}}[f_n](v,u) = \mathcal{F}_{\mathrm{T}}[f_n](v,u)$, where f_n are real functions, $n=0,1,2,3$.

(2) $\Longrightarrow$ (1). As $f \in L^2\cup L^1(\mathbb{R}^2,\mathbb{H})$, then $f_n \in L^2\cup L^1(\mathbb{R}^2,\mathbb{R})$ and $\mathcal{F}_{\mathrm{R}}[f](v,u) = \mathcal{F}_{\mathrm{R}}[f_0](v,u) + \mathrm{i}\mathcal{F}_{\mathrm{R}}[f_1](v,u) + \mathrm{j}\mathcal{F}_{\mathrm{R}}[f_2](v,u) + \mathrm{k}\mathcal{F}_{\mathrm{R}}[f_3](v,u)$, so the second statement can imply the first statement.

(1) $\Longrightarrow$ (2). Because

$$\begin{cases} f = f_0 + \mathrm{i}f_1 + \mathrm{j}f_2 + \mathrm{k}f_3, \\ -\mathrm{i}f_i = f_0 + \mathrm{i}f_1 - \mathrm{j}f_2 - \mathrm{k}f_3, \\ -\mathrm{j}f_j = f_0 - \mathrm{i}f_1 + \mathrm{j}f_2 - \mathrm{k}f_3, \\ -\mathrm{k}f_k = f_0 - \mathrm{i}f_1 - \mathrm{j}f_2 + \mathrm{k}f_3, \end{cases}$$

it follows that

$$\begin{cases} f_0 = \frac{1}{4}(f - \mathrm{i}f_i - \mathrm{j}f_j - \mathrm{k}f_k), \\ f_1 = \frac{-\mathrm{i}}{4}(f - \mathrm{i}f_i + \mathrm{j}f_j + \mathrm{k}f_k), \\ f_2 = \frac{-\mathrm{j}}{4}(f + \mathrm{i}f_i - \mathrm{j}f_j + \mathrm{k}f_k), \\ f_3 = \frac{-\mathrm{k}}{4}(f + \mathrm{i}f_i + \mathrm{j}f_j - \mathrm{k}f_k). \end{cases}$$

Taking the right-sided QFT on both sides, we obtain

$$\begin{aligned} \mathcal{F}_{\mathrm{R}}[f_0](v,u) =& \frac{1}{4}\Big(\mathcal{F}_{\mathrm{R}}[f](v,u) - \mathrm{i}\mathcal{F}_{\mathrm{R}}[f](v,-u)\mathrm{i} \\ &- \mathrm{j}\mathcal{F}_{\mathrm{R}}[f](-v,u)\mathrm{j} - \mathrm{k}\mathcal{F}_{\mathrm{R}}[f](-v,-u)\mathrm{k}\Big), \\ \mathcal{F}_{\mathrm{R}}[f_1](v,u) =& \frac{-\mathrm{i}}{4}\Big(\mathcal{F}_{\mathrm{R}}[f](v,u) - \mathrm{i}\mathcal{F}_{\mathrm{R}}[f](v,-u)\mathrm{i} \\ &+ \mathrm{j}\mathcal{F}_{\mathrm{R}}[f](-v,u)\mathrm{j} + \mathrm{k}\mathcal{F}_{\mathrm{R}}[f](-v,-u)\mathrm{k}\Big), \\ \mathcal{F}_{\mathrm{R}}[f_2](v,u) =& \frac{-\mathrm{j}}{4}\Big(\mathcal{F}_{\mathrm{R}}[f](v,u) + \mathrm{i}\mathcal{F}_{\mathrm{R}}[f](v,-u)\mathrm{i} \\ &- \mathrm{j}\mathcal{F}_{\mathrm{R}}[f](-v,u)\mathrm{j} + \mathrm{k}\mathcal{F}_{\mathrm{R}}[f](-v,-u)\mathrm{k}\Big), \\ \mathcal{F}_{\mathrm{R}}[f_3](v,u) =& \frac{-\mathrm{k}}{4}\Big(\mathcal{F}_{\mathrm{R}}[f](v,u) + \mathrm{i}\mathcal{F}_{\mathrm{R}}[f](v,-u)\mathrm{i} \\ &+ \mathrm{j}\mathcal{F}_{\mathrm{R}}[f](-v,u)\mathrm{j} - \mathrm{k}\mathcal{F}_{\mathrm{R}}[f](-v,-u)\mathrm{k}\Big). \end{aligned}$$

Hence, the first statement implies the second statement.

Remark (1) We can see from Theorem 1 that if $f \in L^2\cup L^1(\mathbb{R}^2,\mathbb{H})$ is BL to (σ_1,σ_2) in the right-sided QFT sense, then it is also BL to (σ_1,σ_2) in the two- and left-sided QFT senses, and vice versa. In this case, f is said to be BL to (σ_1,σ_2) in the QFT sense. (2) If $f(x,y)$ is BL to a rectangle that is not symmetric about the origin, then the sampling formulas of f under the various types of QFTs will be different. We will show them in Theorems 3–5.

Theorem 2 (Sampling theorem for QFTs) Suppose that $f \in L^2(\mathbb{R}^2,\mathbb{H})$ is BL to (σ_1,σ_2) in the QFT sense. Then $f(x,y)$ can be reconstructed from its sampled values at the points $(\frac{n\pi}{\sigma_1}, \frac{m\pi}{\sigma_2})$, $(n,m)\in\mathbb{Z}^2$, via the following formula:

$$f(x,y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Bigg[f(x_n,y_m) \cdot \frac{\sin(\sigma_1(x-x_n))}{\sigma_1(x-x_n)}\frac{\sin(\sigma_2(y-y_m))}{\sigma_2(y-y_m)}\Bigg], \tag{5}$$

where $x_n = \frac{n\pi}{\sigma_1}$ and $y_m = \frac{m\pi}{\sigma_2}$. The series converges in the L^2 norm, and it is absolutely and uniformly convergent on any compact subset of $\mathbb{R}^2$.

Proof From Hitzer EMS (2007), we have $\mathcal{F}_{\mathrm{R}}[f](v,u) \in L^2([-\sigma_1,\sigma_1]\times[-\sigma_2,\sigma_2],\mathbb{H})$. Therefore, by Lemma 5, we have

$$\begin{cases} \mathcal{F}_{\mathrm{R}}[f](v,u) = \sum\limits_{n=-\infty}^{\infty}\sum\limits_{m=-\infty}^{\infty}\tilde{C}_{n,m}\mathrm{e}^{\mathrm{i}v\frac{n\pi}{\sigma_1}}\mathrm{e}^{\mathrm{j}u\frac{m\pi}{\sigma_2}}, \\ \mathrm{e}^{\mathrm{j}uy}\mathrm{e}^{\mathrm{i}vx} = \sum\limits_{n'=-\infty}^{\infty}\sum\limits_{m'=-\infty}^{\infty} d_{n',m'}\mathrm{e}^{\mathrm{j}u\frac{m'\pi}{\sigma_2}}\mathrm{e}^{\mathrm{i}v\frac{n'\pi}{\sigma_1}}, \end{cases}$$

where

$$\begin{aligned}&\tilde{C}_{n,m}\\=&\frac{1}{4\sigma_1\sigma_2}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\mathcal{F}_{\rm R}[f](v,u)\mathrm{e}^{-\mathrm{j}u\frac{m\pi}{\sigma_2}}\mathrm{e}^{-\mathrm{i}v\frac{n\pi}{\sigma_1}}\mathrm{d}v\mathrm{d}u\\=&\frac{4\pi^2}{4\sigma_1\sigma_2}f\Big(-\frac{n\pi}{\sigma_1},-\frac{m\pi}{\sigma_2}\Big),\end{aligned}\tag{6}$$

$$\begin{aligned}&d_{n',m'}\\=&\frac{1}{4\sigma_1\sigma_2}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\mathrm{e}^{\mathrm{j}uy}\mathrm{e}^{\mathrm{i}vx}\mathrm{e}^{-\mathrm{i}v\frac{n'\pi}{\sigma_1}}\mathrm{e}^{-\mathrm{j}u\frac{m'\pi}{\sigma_2}}\mathrm{d}v\mathrm{d}u\\=&\frac{\sin(\sigma_1(x-\frac{n'\pi}{\sigma_1}))}{\sigma_1(x-\frac{n'\pi}{\sigma_1})}\frac{\sin(\sigma_2(y-\frac{m'\pi}{\sigma_2}))}{\sigma_2(y-\frac{m'\pi}{\sigma_2})}.\end{aligned}$$

From Lemma 1, we have

$$\begin{aligned}&f(x,y)\\=&\frac{1}{4\pi^2}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\mathcal{F}_{\rm R}[f](v,u)\mathrm{e}^{\mathrm{j}uy}\mathrm{e}^{\mathrm{i}vx}\mathrm{d}v\mathrm{d}u\\=&\frac{1}{4\pi^2}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}\tilde{C}_{n,m}\\&\cdot\mathrm{e}^{\mathrm{i}v\frac{n\pi}{\sigma_1}}\mathrm{e}^{\mathrm{j}u\frac{m\pi}{\sigma_2}}d_{n',m'}\mathrm{e}^{\mathrm{j}u\frac{m'\pi}{\sigma_2}}\mathrm{e}^{\mathrm{i}v\frac{n'\pi}{\sigma_1}}\mathrm{d}v\mathrm{d}u\\=&\frac{1}{4\pi^2}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}\tilde{C}_{n,m}\\&\cdot\mathrm{e}^{\mathrm{i}v\frac{n\pi}{\sigma_1}}\mathrm{e}^{\mathrm{j}u\frac{m\pi}{\sigma_2}}\mathrm{e}^{\mathrm{j}u\frac{m'\pi}{\sigma_2}}\mathrm{e}^{\mathrm{i}v\frac{n'\pi}{\sigma_1}}\\&\cdot\frac{\sin(\sigma_1(x-\frac{n'\pi}{\sigma_1}))}{\sigma_1(x-\frac{n'\pi}{\sigma_1})}\frac{\sin(\sigma_2(y-\frac{m'\pi}{\sigma_2}))}{\sigma_2(y-\frac{m'\pi}{\sigma_2})}\mathrm{d}v\mathrm{d}u\\=&\frac{4\sigma_1\sigma_2}{4\pi^2}\sum_{n'=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}\tilde{C}_{-n',-m'}\\&\cdot\frac{\sin(\sigma_1(x-\frac{n'\pi}{\sigma_1}))}{\sigma_1(x-\frac{n'\pi}{\sigma_1})}\frac{\sin(\sigma_2(y-\frac{m'\pi}{\sigma_2}))}{\sigma_2(y-\frac{m'\pi}{\sigma_2})},\end{aligned}\tag{7}$$

which, in view of Eq. (6), implies Eq. (5). The convergence in Eq. (7) is understood to be in the sense of L^2. However, Eq. (5) is readily seen to converge absolutely and uniformly on any compact subset of $\mathbb{R}^2$ when we apply the quaternion Cauchy-Schwarz inequality (3) and Eqs. (4) and (5).

If $f(x,y)$ is BL to a rectangle that is not symmetric about the origin, then the sampling formula is as follows:

Theorem 3 (Sampling theorem for two-sided QFT) If $f\in L^2(\mathbb{R}^2,\mathbb{H})$ and $\mathcal{F}_{\rm T}[f](v,u)=0$, for $|v-v_0|>\sigma_1$ or $|u-u_0|>\sigma_2$, we have

$$\begin{aligned}f(x,y)=&\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Bigg[\mathrm{e}^{\mathrm{i}v_0(x-x_n)}f(x_n,y_m)\mathrm{e}^{\mathrm{j}u_0(y-y_m)}\\&\cdot\mathrm{sinc}\frac{\sigma_1(x-x_n)}{\pi}\mathrm{sinc}\frac{\sigma_2(y-y_m)}{\pi}\Bigg].\end{aligned}\tag{8}$$

The series converges in the L^2 norm, and it is absolutely and uniformly convergent on any compact subset of $\mathbb{R}^2$.

Proof Because $\mathcal{F}_{\rm T}[f](v,u)=0$, for $|v-v_0|>\sigma_1$ or $|u-u_0|>\sigma_2$, then $\mathrm{e}^{-\mathrm{i}v_0x}f(x,y)\mathrm{e}^{-\mathrm{j}u_0y}$ is BL to (σ_1,σ_2). Hence from Theorem 2, by substituting $\mathrm{e}^{-\mathrm{i}v_0x}f(x,y)\mathrm{e}^{-\mathrm{j}u_0y}$ for $f(x,y)$ in sampling series (5), we obtain sampling formula (8).

Theorem 4 (Sampling theorem for right-sided QFT) If $f\in L^2(\mathbb{R}^2,\mathbb{H})$ and $\mathcal{F}_{\rm R}[f](v,u)=0$, for $|v-v_0|>\sigma_1$ or $|u-u_0|>\sigma_2$, we have

$$\begin{aligned}&f(x,y)\\=&\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\big[f(x_n,y_m)\mathrm{e}^{\mathrm{i}v_0(x-x_n)}\cos(u_0(y-y_m))\\&+f(-x_n,y_m)\mathrm{e}^{-\mathrm{i}v_0(x-x_n)}\sin(u_0(y-y_m))j\big]\\&\cdot\mathrm{sinc}\frac{\sigma_1(x-x_n)}{\pi}\mathrm{sinc}\frac{\sigma_2(y-y_m)}{\pi}.\end{aligned}\tag{9}$$

The series converges in the L^2 norm. Moreover, it converges absolutely and uniformly on any compact subset of $\mathbb{R}^2$.

Proof From the assumption of f, we have

$$\begin{aligned}&f(x,y)\\=&\frac{1}{4\pi^2}\int_{-\sigma_1+v_0}^{\sigma_1+v_0}\left(\int_{-\sigma_2+u_0}^{\sigma_2+u_0}\mathcal{F}_{\rm R}[f](v,u)\mathrm{e}^{\mathrm{j}uy}\mathrm{d}u\right)\mathrm{e}^{\mathrm{i}vx}\mathrm{d}v.\end{aligned}$$

Because $\mathcal{F}_{\rm R}^{\rm i}[f](v,y):=\int_{\mathbb{R}}f(x,y)\mathrm{e}^{-\mathrm{i}xv}\mathrm{d}x=\frac{1}{2\pi}\int_{-\sigma_2+u_0}^{\sigma_2+u_0}\mathcal{F}_{\rm R}[f](v,u)\mathrm{e}^{\mathrm{j}uy}\mathrm{d}u$, for $|v-v_0|\le\sigma_1$, we have

$$f(x,y)\mathrm{e}^{-\mathrm{i}v_0x}=\sum_{n=-\infty}^{\infty}f(x_n,y)\mathrm{e}^{-\mathrm{i}v_0x_n}\mathrm{sinc}\frac{\sigma_1(x-x_n)}{\pi}.\tag{10}$$

Because $\int_{\mathbb{R}}\big(\int_{\mathbb{R}}f(x,y)\mathrm{e}^{-\mathrm{i}xv_0}\mathrm{e}^{-\mathrm{i}xv}\mathrm{d}x\big)\mathrm{e}^{-\mathrm{j}yu}\mathrm{d}y=\mathcal{F}_{\rm R}[f](v+v_0,u)$, assume $g(y)=\int_{\mathbb{R}}f(x,y)\mathrm{e}^{-\mathrm{i}xv_0}\mathrm{e}^{-\mathrm{i}xv}\mathrm{d}x$. Then $\mathcal{F}_{\rm R}^{\rm i}[g](u)=0$.

For $|u-u_0|>\sigma_2$, we have

$$\begin{aligned}&\int_{\mathbb{R}} f(x,y)\mathrm{e}^{-\mathrm{i}xv_0}\mathrm{e}^{-\mathrm{i}vx}\mathrm{d}x\\ &=\sum_{m=-\infty}^{\infty}\int_{\mathbb{R}} f(x,y_m)\mathrm{e}^{-\mathrm{i}xv_0}\mathrm{e}^{-\mathrm{i}xv}\mathrm{d}x\\ &\quad\cdot\operatorname{sinc}\left(\frac{\sigma_2(y-y_m)}{\pi}\right)\mathrm{e}^{\mathrm{j}u_0(y-y_m)}.\end{aligned}\tag{11}$$

Applying $\mathcal{F}_{\mathrm{R}}^{-\mathrm{i}}$ to both sides of Eq. (11) at point x_n, we have

$$\begin{aligned}&f(x_n,y)\mathrm{e}^{-\mathrm{i}x_nv_0}\\ &=\sum_{m=-\infty}^{\infty}\Big(f(x_n,y_m)\mathrm{e}^{-\mathrm{i}x_nv_0}\cos(u_0(y-y_m))\\ &\quad+f(-x_n,y_m)\mathrm{e}^{\mathrm{i}x_nv_0}\sin(u_0(y-y_m))\mathrm{j}\Big)\\ &\quad\cdot\operatorname{sinc}\frac{\sigma_2(y-y_m)}{\pi}.\end{aligned}\tag{12}$$

By substituting $f(x_n,y)\mathrm{e}^{-\mathrm{i}x_nv_0}$ in the above equation into Eq. (10), we obtain Eq. (9).

Theorem 5 (Sampling theorem for left-sided QFT) If $f\in L^2(\mathbb{R}^2,\mathbb{H})$ and $\mathcal{F}_{\mathrm{L}}[f](v,u)=0$, for $|v-v_0|>\sigma_1$ or $|u-u_0|>\sigma_2$, we have

$$\begin{aligned}&f(x,y)\\ &=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Big(\mathrm{e}^{\mathrm{j}u_0(y-y_m)}f(x_n,y_m)\cos(v_0(x-x_n))\\ &\quad+\mathrm{e}^{-\mathrm{j}u_0(y-y_m)}\sin(v_0(x-x_n))\mathrm{i}f(x_n,-y_m)\Big)\\ &\quad\cdot\operatorname{sinc}\frac{\sigma_1(x-x_n)}{\pi}\operatorname{sinc}\frac{\sigma_2(y-y_m)}{\pi}.\end{aligned}\tag{13}$$

The series converges in the L^2 norm. Moreover, it converges absolutely and uniformly on any compact subset of $\mathbb{R}^2$.

Proof Using an argument similar to the proof of Theorem 4, we can easily carry out the proof of this theorem.

Next, we will introduce the quaternion partial and total Hilbert transforms (Bulow and Sommer, 2001; Kou et al., 2017) associated with QFTs, and derive the sampling formula using samples of its quaternion partial and total Hilbert transforms. In addition, the sampling formula using the samples of its partial derivatives is derived. Then, the sampling rate can be reduced by taking multiple types of samples simultaneously.

Definition 2 Let $f\in L^2\cup L^1(\mathbb{R}^2,\mathbb{H})$. Then the quaternion partial Hilbert transform $\mathcal{H}_1$ of f along the x and y axes and the quaternion total Hilbert transform $\mathcal{H}_2$ along the x and y axes of f are given by

$$\begin{aligned}\mathcal{H}_1[f(\cdot,y)](x)&:=\text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(t,y)}{x-t}\mathrm{d}t,\ \text{a.e.},\\ \mathcal{H}_1[f](x,\cdot)(y)&:=\text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(x,t)}{y-t}\mathrm{d}t,\ \text{a.e.},\\ \mathcal{H}_2[f(\cdot,\cdot)](x,y)&:=\text{p.v.}\frac{1}{\pi^2}\int_{\mathbb{R}^2}\frac{f(t,s)}{(x-t)(y-s)}\mathrm{d}t\mathrm{d}s,\ \text{a.e.}.\end{aligned}$$

Let

$$\begin{aligned}\mathcal{H}_x[f](x,y)&:=\mathcal{H}_1[f(\cdot,y)](x),\\ \mathcal{H}_y[f](x,y)&:=\mathcal{H}_1[f](x,\cdot)(y),\\ \mathcal{H}_{xy}[f](x,y)&=\mathcal{H}_2[f(\cdot,\cdot)](x,y).\end{aligned}$$

Using the analogous definition of the quaternion analytic signal given in Bulow and Sommer (2001) and Hahn and Snopek (2005), we have the following quaternion analytic signal associated with $\mathcal{F}_{\mathrm{R}}$:

Lemma 6 Let $f\in L^2(\mathbb{R}^2,\mathbb{R})$. Then we have

$$\begin{aligned}\mathcal{F}_{\mathrm{R}}[\mathcal{H}_x[f]](v,u)&=\mathcal{F}_{\mathrm{R}}[f](v,-u)(-\mathrm{i})\operatorname{sgn}(v),\\ \mathcal{F}_{\mathrm{R}}[\mathcal{H}_y[f]](v,u)&=\mathcal{F}_{\mathrm{R}}[f](v,u)(-\mathrm{j})\operatorname{sgn}(u),\\ \mathcal{F}_{\mathrm{R}}[\mathcal{H}_{xy}[f]](v,u)&=\mathcal{F}_{\mathrm{R}}[f](v,-u)\mathrm{ij}\operatorname{sgn}(u)\operatorname{sgn}(v).\end{aligned}$$

The quaternion analytic signal is defined by

$$\begin{aligned}f^{\mathrm{q}}=&f+\mathcal{H}_x[f](x,y)\mathrm{i}+\mathcal{H}_y[f](-x,y)\mathrm{j}\\ &+\mathcal{H}_{xy}[f](-x,y)\mathrm{k}.\end{aligned}$$

Moreover,

$$\mathcal{F}_{\mathrm{R}}[f^{\mathrm{q}}](v,u)=(1+\operatorname{sgn}(u))(1+\operatorname{sgn}(v))\mathcal{F}_{\mathrm{R}}[f](v,u).$$

Theorem 6 If $f\in L^2(\mathbb{R}^2,\mathbb{H})$ and $\mathcal{F}_{\mathrm{R}}[f](v,u)=0$, for $|v|>\sigma_1$ or $|u|>\sigma_2$, we have

$$\begin{aligned}f(x,y)=&\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Bigg[f(\widetilde{x}_n,\widetilde{y}_m)\cos\frac{\sigma_2\widetilde{Y}_m}{2}\cos\frac{\sigma_1\widetilde{X}_n}{2}\\ &-\mathcal{H}_x[f](\widetilde{x}_n,\widetilde{y}_m)\cos\frac{\sigma_2\widetilde{Y}_m}{2}\sin\frac{\sigma_1\widetilde{X}_n}{2}\\ &-\mathcal{H}_y[f](\widetilde{x}_n,\widetilde{y}_m)\sin\frac{\sigma_2\widetilde{Y}_m}{2}\cos\frac{\sigma_1\widetilde{X}_n}{2}\\ &+\mathcal{H}_{xy}[f](\widetilde{x}_n,\widetilde{y}_m)\sin\frac{\sigma_2\widetilde{Y}_m}{2}\sin\frac{\sigma_1\widetilde{X}_n}{2}\Bigg]\\ &\cdot\operatorname{sinc}\frac{\sigma_1\widetilde{X}_n}{2\pi}\operatorname{sinc}\frac{\sigma_2\widetilde{Y}_m}{2\pi},\end{aligned}\tag{14}$$

where $\widetilde{x}_n:=\frac{2n\pi}{\sigma_1}$, $\widetilde{y}_m:=\frac{2m\pi}{\sigma_2}$, $\widetilde{X}_n:=x-\widetilde{x}_n$, and $\widetilde{Y}_m:=y-\widetilde{y}_m$.

Proof If $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then $f_k \in L^2(\mathbb{R}^2, \mathbb{R})$, $k = 0, 1, 2, 3$. It follows that the quaternion analytic signal f_k^{q} takes the form of

$$\begin{aligned} f_k^{\mathrm{q}} =& f_k + \mathcal{H}_x[f_k](x,y)\mathrm{i} + \mathcal{H}_y[f_k](-x,y)\mathrm{j} \\ &+ \mathcal{H}_{xy}[f_k](-x,y)\mathrm{k}. \end{aligned}$$

From Lemma 6, we have $\mathcal{F}_{\mathrm{R}}[f_k^{\mathrm{q}}](v,u) = (1 + \mathrm{sgn}(v))(1+\mathrm{sgn}(u))\mathcal{F}_{\mathrm{R}}[f_k](v,u)$; that is to say, f_k^{q} is BL to $[0,\sigma_1] \times [0, \sigma_2]$. It follows from Theorem 4 that

$$\begin{aligned} & f_k^{\mathrm{q}}(x,y) \\ =& \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Bigg[f_k^{\mathrm{q}}(\widetilde{x}_n, \widetilde{y}_m) \mathrm{e}^{\mathrm{i}\frac{\sigma_1 \widetilde{X}_n}{2}} \cos \frac{\sigma_2 \widetilde{Y}_m}{2} \\ &+ f_k^{\mathrm{q}}(-\widetilde{x}_n, \widetilde{y}_m) \mathrm{e}^{-\mathrm{i}\frac{\sigma_1 \widetilde{X}_n}{2}} \sin \frac{\sigma_2 \widetilde{Y}_m}{2} \mathrm{j} \Bigg] \\ & \cdot \mathrm{sinc} \frac{\sigma_1 \widetilde{X}_n}{2\pi} \mathrm{sinc} \frac{\sigma_2 \widetilde{Y}_m}{2\pi}. \end{aligned} \tag{15}$$

Noting that $f_k(x,y)$ is the real part of $f_k^{\mathrm{q}}(x,y)$, with some straightforward calculations, we can derive the sampling formula of f_k as

$$\begin{aligned} & f_k(x,y) \\ =& \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Bigg[f_k(\widetilde{x}_n, \widetilde{y}_m) \cos \frac{\sigma_1 \widetilde{X}_n}{2} \cos \frac{\sigma_2 \widetilde{Y}_m}{2} \\ &- \mathcal{H}_x[f_k](\widetilde{x}_n, \widetilde{y}_m) \sin \frac{\sigma_1 \widetilde{X}_n}{2} \cos \frac{\sigma_2 \widetilde{T}_m}{2} \\ &- \mathcal{H}_y[f_k](\widetilde{x}_n, \widetilde{y}_m) \sin \frac{\sigma_2 \widetilde{Y}_m}{2} \cos \frac{\sigma_1 \widetilde{X}_n}{2} \\ &+ \mathcal{H}_{xy}[f_k](\widetilde{x}_n, \widetilde{y}_m) \sin \frac{\sigma_2 \widetilde{Y}_m}{2} \sin \frac{\sigma_1 \widetilde{X}_n}{2} \Bigg] \\ & \cdot \mathrm{sinc} \frac{\sigma_1 \widetilde{X}_n}{2\pi} \mathrm{sinc} \frac{\sigma_2 \widetilde{Y}_m}{2\pi}. \end{aligned}$$

Because $f = f_0 + \mathrm{i} f_1 + \mathrm{j} f_2 + \mathrm{k} f_3$, then sampling formula (14) holds for f.

From Theorems 1 and 6, we have the following corollary:

Corollary 1 If $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then $f(x,y)$ can be reconstructed from the samples of its quaternion partial and toltal Hilbert transforms, if one of the following conditions holds:

(1) $\mathcal{F}_{\mathrm{R}}[f](v,u) = 0$, for $|v| > \sigma_1$ or $|u| > \sigma_2$,

(2) $\mathcal{F}_{\mathrm{T}}[f](v,u) = 0$, for $|v| > \sigma_1$ or $|u| > \sigma_2$,

(3) $\mathcal{F}_{\mathrm{L}}[f](v,u) = 0$, for $|v| > \sigma_1$ or $|u| > \sigma_2$.

$$\begin{aligned} & f(x,y) \\ =& \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Bigg[f(\widetilde{x}_n, \widetilde{y}_m) \cos \frac{\sigma_2 \widetilde{Y}_m}{2} \cos \frac{\sigma_1 \widetilde{X}_n}{2} \\ &- \mathcal{H}_x[f](\widetilde{x}_n, \widetilde{y}_m) \cos \frac{\sigma_2 \widetilde{Y}_m}{2} \sin \frac{\sigma_1 \widetilde{X}_n}{2} \\ &- \mathcal{H}_y[f](\widetilde{x}_n, \widetilde{y}_m) \sin \frac{\sigma_2 \widetilde{Y}_m}{2} \cos \frac{\sigma_1 \widetilde{X}_n}{2} \\ &+ \mathcal{H}_{xy}[f](\widetilde{x}_n, \widetilde{y}_m) \sin \frac{\sigma_2 \widetilde{Y}_m}{2} \sin \frac{\sigma_1 \widetilde{X}_n}{2} \Bigg] \\ & \cdot \mathrm{sinc} \frac{\sigma_1 \widetilde{X}_n}{2\pi} \mathrm{sinc} \frac{\sigma_2 \widetilde{Y}_m}{2\pi}. \end{aligned} \tag{16}$$

In the following, we derive the sampling formula involving the samples of the original function and its partial derivatives. Some lemmas are needed first:

Lemma 7 (Hitzer EMS, 2007) If $f \in L^2 \cap C^{m+n}(\mathbb{R}^2, \mathbb{H})$ and $g(x,y) := \frac{\partial^{n+m} f}{\partial^n x \partial^m y} \in L^2(\mathbb{R}^2, \mathbb{H})$, we have

$$\mathcal{F}_{\mathrm{T}}[g](v,u) = (\mathrm{i}v)^n \mathcal{F}_{\mathrm{T}}[f](v,u)(\mathrm{j}u)^m,$$

where $n, m \in \mathbb{Z}$.

Lemma 8 (Marvasti, 2001)

$$\mathrm{e}^{\mu v t} = \epsilon_1^{\mu}(v,t) + \mu v \epsilon_2^{\mu}(v,t), \quad v \in (-\sigma, \sigma),$$

where $\mu = \mathrm{i}\mu_1 + \mathrm{j}\mu_2 + \mathrm{k}\mu_3$ is the unit pure quaternion such that $\mu^2 = -1$, for special cases of $\mu = \mathrm{i}$, j, or k.

$$\epsilon_1^{\mu}(v,t) = \left(1 - \frac{|v|}{\sigma}\left(1 - \mathrm{e}^{-\mu\sigma t \mathrm{sgn}(v)}\right)\right)\mathrm{e}^{\mu t v}.$$

$$\epsilon_2^{\mu}(v,t) = \frac{\mu \mathrm{sgn}(v)}{\sigma}\left(\mathrm{e}^{-\mu\sigma t \mathrm{sgn}(v)} - 1\right)\mathrm{e}^{\mu t v}.$$

Here, $\epsilon_k(v,t) = \epsilon_k(v-\sigma,t)$, for $v \in (0,\sigma)$ and $k = 1, 2$. Both functions may be expanded in their σ-periodic boundedly converging Fourier series on $(-\sigma,\sigma)/\{0\}$; that is to say,

$$\epsilon_1^{\mu}(v,t) = \sum_{k=-\infty}^{\infty} \left(\mathrm{sinc}\frac{\sigma t - 2k\pi}{2\pi}\right)^2 \mathrm{e}^{\frac{\mu 2k\pi v}{\sigma}}.$$

$$\epsilon_2^{\mu}(v,t) = \sum_{k=-\infty}^{\infty} \left(\mathrm{sinc}\frac{\sigma t - 2k\pi}{2\pi}\right)^2 \left(t - \frac{2k\pi}{\sigma}\right) \mathrm{e}^{\frac{\mu 2k\pi v}{\sigma}}.$$

Theorem 7 If $f \in L^2 \cap C^2(\mathbb{R}^2, \mathbb{H})$ is BL to (σ_1, σ_2), and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y} \in L^2(\mathbb{R}^2, \mathbb{H})$, then the following sampling formula holds:

$$
\begin{aligned}
&f(x,y)\\
&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[f(\widetilde{x}_n,\widetilde{y}_m)+(x-\widetilde{x}_n)\frac{\partial f}{\partial x}(\widetilde{x}_n,\widetilde{y}_m)\right.\\
&+(y-\widetilde{y}_m)\frac{\partial f}{\partial y}(\widetilde{x}_n,\widetilde{y}_m)+(x-\widetilde{x}_n)(y-\widetilde{y}_m)\\
&\left.\cdot\frac{\partial^2 f}{\partial x\partial y}(\widetilde{x}_n,\widetilde{y}_m)\right]\left(\operatorname{sinc}\left(\frac{\sigma_1 x}{2\pi}-n\right)\right)^2\\
&\cdot\left(\operatorname{sinc}\left(\frac{\sigma_2 y}{2\pi}-m\right)\right)^2.
\end{aligned}
\tag{17}
$$

Proof From the assumption of f, we can recover f from its QFT domain as follows:

$$f(x,y)=\frac{1}{4\pi^2}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\mathrm{e}^{\mathrm{i}vx}\mathcal{F}_{\mathrm{T}}(v,u)\mathrm{e}^{\mathrm{j}uy}\mathrm{d}v\mathrm{d}u. \tag{18}$$

From Lemma 8, assume $\mathrm{e}^{\mathrm{i}vx}=\epsilon_1^{\mathrm{i}}(v,x)+\mathrm{i}v\epsilon_2^{\mathrm{i}}(v,x)$ and $\mathrm{e}^{\mathrm{j}uy}=\epsilon_1^{\mathrm{j}}(u,y)+\mathrm{j}\epsilon_2^{\mathrm{j}}(u,y)$. After substituting these equations into Eq. (18), we obtain

$$
\begin{aligned}
&4\pi^2 f(x,y)\\
&=\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\epsilon_1^{\mathrm{i}}(v,x)\mathcal{F}_{\mathrm{T}}(v,u)\epsilon_1^{\mathrm{j}}(u,y)\mathrm{d}v\mathrm{d}u\\
&+\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\epsilon_1^{\mathrm{i}}(v,x)\mathcal{F}_{\mathrm{T}}(v,u)\mathrm{j}\epsilon_2^{\mathrm{j}}(u,y)\mathrm{d}v\mathrm{d}u\\
&+\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\mathrm{i}\epsilon_2^{\mathrm{i}}(v,x)\mathcal{F}_{\mathrm{T}}(v,u)\epsilon_1^{\mathrm{j}}(u,y)\mathrm{d}v\mathrm{d}u\\
&+\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\mathrm{i}\epsilon_2^{\mathrm{i}}(v,x)\mathcal{F}_{\mathrm{T}}(v,u)\mathrm{j}\epsilon_2^{\mathrm{j}}(u,y)\mathrm{d}v\mathrm{d}u.
\end{aligned}
$$

Using Lemmas 7 and 8, with straightforward calculations, we have sampling formula (17).

3.2 Sampling theorems for BL quaternion functions in the QLCT sense

$f\in L^2(\mathbb{R}^2,\mathbb{H})$ is said to be a BL signal (function) to $[-\sigma_1,\sigma_1]\times[-\sigma_2,\sigma_2]$ (short for (σ_1,σ_2)) in the two-sided QLCT sense, if $\mathcal{L}_{\mathrm{T}}^{\mathrm{i,j}}[f](v,u)=0$, for $|v|>\sigma_1$ or $|u|>\sigma_2$.

Theorem 8 Suppose $f\in L^2(\mathbb{R}^2,\mathbb{H})$ and $\mathcal{L}_{\mathrm{T}}^{\mathrm{i,j}}[f](v,u)=0$, for $|v|>\sigma_1$ or $|u|>\sigma_2$. Assume $g=\mathrm{e}^{\mathrm{i}a_1x^2/(2b_1)}f(x,y)\mathrm{e}^{\mathrm{j}a_2y^2/(2b_2)}$. Then $\mathcal{F}_{\mathrm{T}}[g](v,u)=0$, for $|v|>\frac{\sigma_1}{b_1}$ or $|u|>\frac{\sigma_2}{b_2}$.

Proof From the definition of the two-sided QLCT, we obtain

$$
\begin{aligned}
&\mathcal{F}_{\mathrm{T}}[g]\left(\frac{v}{b_1},\frac{u}{b_2}\right)\\
&=\frac{1}{\sqrt{\mathrm{i}2\pi b_1}}\mathrm{e}^{\mathrm{i}(\frac{d_1}{2b_1}v^2)}\mathcal{L}_{\mathrm{T}}^{\mathrm{i,j}}[f](v,u)\frac{1}{\sqrt{\mathrm{j}2\pi b_2}}\mathrm{e}^{\mathrm{j}(\frac{d_2}{2b_2}u^2)}.
\end{aligned}
$$

Theorem 9 If $f\in L^2(\mathbb{R}^2,\mathbb{H})$ is BL to (σ_1,σ_2) in the two-sided QLCT sense, then the following sampling formula for f holds:

$$
\begin{aligned}
&f(x,y)\\
&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{a_1s_n^2-a_1x^2}{2b_1}}f(s_n,t_m)\mathrm{e}^{\mathrm{j}\frac{a_2t_m^2-a_2y^2}{2b_2}}\\
&\cdot\operatorname{sinc}\frac{\sigma_1(x-s_n)}{b_1\pi}\operatorname{sinc}\frac{\sigma_2(y-t_m)}{b_2\pi},
\end{aligned}
$$

where $s_n=\frac{nb_1\pi}{\sigma_1}$ and $t_m=\frac{mb_2\pi}{\sigma_2}$. The series converges in the L^2 norm. Moreover, it converges absolutely and uniformly on any compact subset of $\mathbb{R}^2$.

Proof Theorem 8 implies that $\mathrm{e}^{\frac{\mathrm{i}a_1x^2}{2b_1}}f(x,y)\mathrm{e}^{\frac{\mathrm{j}a_2y^2}{2b_2}}$ is BL to $(\frac{\sigma_1}{b_1},\frac{\sigma_2}{b_2})$ in the QFT sense. So, applying Theorem 2 to $\mathrm{e}^{\frac{\mathrm{i}a_1x^2}{2b_1}}f(x,y)\mathrm{e}^{\frac{\mathrm{j}a_2y^2}{2b_2}}$, we obtain the final results.

Similarly, we obtain the following sampling series of f involving the function and its partial derivatives:

Theorem 10 If $f\in L^2\cap C^2(\mathbb{R}^2,\mathbb{H})$ is BL to (σ_1,σ_2) in the two-sided QLCT sense, and xf, yf, $\frac{\partial f}{\partial x}$, $y\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $x\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x\partial y}\in L^2(\mathbb{R}^2,\mathbb{H})$, then the following sampling series for f holds:

$$
\begin{aligned}
&f(x,y)\\
&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}+(x-\widetilde{s}_n)\right.\\
&\cdot\left[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial x}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}+\mathrm{i}\frac{a_1\widetilde{s}_n}{b_1}\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\right.\\
&\left.\cdot f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\right]+(y-\widetilde{t}_m)\left[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial y}(\widetilde{s}_n,\widetilde{t}_m)\right.\\
&\left.\cdot\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}+\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\mathrm{j}\frac{a_2\widetilde{t}_m}{b_2}\right]\\
&+(x-\widetilde{s}_n)(y-\widetilde{t}_m)\left[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial^2 f}{\partial x\partial y}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\right.\\
&+\mathrm{i}\frac{a_1\widetilde{s}_n}{b_1}\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial y}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\\
&+\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial x}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\mathrm{j}\frac{a_2\widetilde{t}_m}{b_2}\\
&\left.\left.+\mathrm{i}\frac{a_1\widetilde{s}_n}{b_1}\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\mathrm{j}\frac{a_2\widetilde{t}_m}{b_2}\right]\right]\\
&\cdot\left(\operatorname{sinc}\left(\frac{\sigma_1 x}{2b_1\pi}-n\right)\right)^2\left(\operatorname{sinc}\left(\frac{\sigma_2 y}{b_2 2\pi}-m\right)\right)^2,
\end{aligned}
$$

where $\widetilde{s}_n=\frac{2nb_1\pi}{\sigma_1}$ and $\widetilde{t}_m=\frac{2mb_2\pi}{\sigma_2}$. The series converges in the L^2 norm. Moreover, it converges absolutely and uniformly on any compact subset of $\mathbb{R}^2$.

In the following, to derive the sampling series for f using the samples of its Hilbert transform associated with QLCT, we introduce the generalized partial and total Hilbert transforms for the two-sided QLCT in Kou et al. (2017).

Definition 3 Let $f \in L^2 \cup L^1(\mathbb{R}^2, \mathbb{H})$. Then f along the x and y axes of the generalized quaternion partial Hilbert transforms $\mathcal{H}^x_{\boldsymbol{A}_1}$ and $\mathcal{H}^y_{\boldsymbol{A}_2}$ and along the x and y axes of the generalized quaternion total Hilbert transform $\mathcal{H}^{xy}_{\boldsymbol{A}_1\boldsymbol{A}_2}$ are given by

$$\mathcal{H}^x_{\boldsymbol{A}_1}[f](x,y) := \text{p.v.}\frac{\mathrm{e}^{-\mathrm{i}\frac{a_1x^2}{2b_1}}}{\pi}\int_{\mathbb{R}}\frac{\mathrm{e}^{\mathrm{i}\frac{a_1t^2}{2b_1}}f(t,y)}{x-t}\mathrm{d}t, \text{a.e.},$$

$$\mathcal{H}^y_{\boldsymbol{A}_2}[f](x,y) := \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(x,t)\mathrm{e}^{\mathrm{j}\frac{a_2t^2}{2b_2}}}{y-t}\mathrm{d}t\mathrm{e}^{-\mathrm{j}\frac{a_2y^2}{2b_2}}, \text{a.e.},$$

$$\mathcal{H}^{xy}_{\boldsymbol{A}_1\boldsymbol{A}_2}[f](x,y) := \text{p.v.}\frac{\mathrm{e}^{-\mathrm{i}\frac{a_1x^2}{2b_1}}}{\pi^2} \cdot\int_{\mathbb{R}^2}\frac{\mathrm{e}^{\mathrm{i}\frac{a_1t^2}{2b_1}}f(t,s)\mathrm{e}^{\mathrm{j}\frac{a_2s^2}{2b_2}}}{(x-t)(y-s)}\mathrm{d}t\mathrm{d}s\mathrm{e}^{-\mathrm{j}\frac{a_2y^2}{2b_2}}, \text{a.e.}.$$

Lemma 9 If $f \in L^2(\mathbb{R}^2, \mathbb{H})$, assume $g(x,y) = \mathrm{e}^{\frac{\mathrm{i}a_1x^2}{2b_1}}f(x,y)\mathrm{e}^{\frac{\mathrm{j}a_2y^2}{2b_2}}$. Then we have

$$\mathcal{H}_x[g](x,y) = \mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}\mathcal{H}^x_{\boldsymbol{A}_1}[f](x,y)\mathrm{e}^{\mathrm{j}\frac{a_2y^2}{2b_2}},$$
$$\mathcal{H}_y[g](x,y) = \mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}\mathcal{H}^y_{\boldsymbol{A}_2}[y](x,y)\mathrm{e}^{\mathrm{j}\frac{a_2y^2}{2b_2}},$$
$$\mathcal{H}_{xy}[g](x,y) = \mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}\mathcal{H}^{xy}_{\boldsymbol{A}_1\boldsymbol{A}_2}[f](x,y)\mathrm{e}^{\mathrm{j}\frac{a_2y^2}{2b_2}}.$$

Proof We give only the proof of the first relationship; the other two relationships can be proved in a similar manner. By definition, we have

$$\begin{aligned}&\mathcal{H}_x[g](x,y)\\&= \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{\mathrm{e}^{\mathrm{i}\frac{a_1t^2}{2b_1}}f(t,y)}{x-t}\mathrm{e}^{\mathrm{j}\frac{a_2y^2}{2b_2}}\mathrm{d}t\\&= \text{p.v.}\frac{1}{\pi}\mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}\int_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}\frac{a_1x^2}{2b_1}}\frac{\mathrm{e}^{\mathrm{i}\frac{a_1t^2}{2b_1}}f(t,y)}{x-t}\mathrm{e}^{\mathrm{j}\frac{a_2y^2}{2b_2}}\mathrm{d}t\\&= \mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}\mathcal{H}^x_{\boldsymbol{A}_1}[f](x,y)\mathrm{e}^{\mathrm{j}\frac{a_2y^2}{2b_2}},\end{aligned}$$

which completes the proof.

Theorem 11 If $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is BL to (σ_1, σ_2) in the two-sided QLCT sense, then the following sampling series for f holds:

$$\begin{aligned}&f(x,y)\\&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1(\widetilde{s}_n^2-x^2)}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2(\widetilde{t}_m^2-y^2)}{2b_2}}\\&\cdot\cos\frac{\sigma_2\widetilde{T}_m}{2b_2}\cos\frac{\sigma_1\widetilde{s}_n}{2b_1} - \mathrm{e}^{\mathrm{i}\frac{a_1(\widetilde{s}_n^2-x^2)}{2b_1}}\mathcal{H}^x_{\boldsymbol{A}_1}[f](\widetilde{s}_n,\widetilde{t}_m)\\&\cdot\mathrm{e}^{\mathrm{j}\frac{a_2(\widetilde{t}_m^2-y^2)}{2b_2}}\cos\frac{\sigma_2\widetilde{T}_m}{2b_2}\sin\frac{\sigma_1\widetilde{S}_n}{2b_1} - \mathrm{e}^{\mathrm{i}\frac{a_1(\widetilde{s}_n^2-x^2)}{2b_1}}\\&\cdot\mathcal{H}^y_{\boldsymbol{A}_2}[f](\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2(\widetilde{t}_m^2-y^2)}{2b_2}}\sin\frac{\sigma_2\widetilde{T}_m}{2b_2}\cos\frac{\sigma_1\widetilde{S}_n}{2b_1}\\&+\mathrm{e}^{\mathrm{i}\frac{a_1(\widetilde{s}_n^2-x^2)}{2b_1}}\mathcal{H}^{xy}_{\boldsymbol{A}_1\boldsymbol{A}_2}[f](\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2(\widetilde{t}_m^2-y^2)}{2b_2}}\\&\cdot\sin\frac{\sigma_2\widetilde{T}_m}{2b_2}\sin\frac{\sigma_1\widetilde{S}_n}{2b_1}\Bigg]\text{sinc}\frac{\sigma_1\widetilde{S}_n}{2b_1\pi}\text{sinc}\frac{\sigma_2\widetilde{T}_m}{2b_2\pi},\end{aligned}\tag{19}$$

where $\widetilde{S}_n = x - \widetilde{s}_n$ and $\widetilde{T}_m = y - \widetilde{t}_m$.

Proof From Theorem 8, $g(x,y) = \mathrm{e}^{\frac{\mathrm{j}a_2y^2}{2b_2}}f(x,y)\mathrm{e}^{\frac{\mathrm{i}a_1x^2}{2b_1}}$ is BL to $[-\frac{\sigma_1}{b_1}, \frac{\sigma_1}{b_1}] \times [-\frac{\sigma_2}{b_2}, \frac{\sigma_2}{b_2}]$ in the QFT sense. Applying sampling series (16) to $g(x,y)$, we obtain

$$\begin{aligned}&g(x,y)\\&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Bigg[g(\widetilde{x}_n,\widetilde{y}_m)\cos\frac{\sigma_2\widetilde{T}_m}{2}\cos\frac{\sigma_1\widetilde{S}_n}{2}\\&- \mathcal{H}_x[g](\widetilde{x}_n,\widetilde{y}_m)\cos\frac{\sigma_2\widetilde{T}_m}{2}\sin\frac{\sigma_1\widetilde{S}_n}{2}\\&- \mathcal{H}_y[g](\widetilde{x}_n,\widetilde{y}_m)\sin\frac{\sigma_2\widetilde{T}_m}{2}\cos\frac{\sigma_1\widetilde{S}_n}{2}\\&+\mathcal{H}_{xy}[g](\widetilde{x}_n,\widetilde{y}_m)\sin\frac{\sigma_2\widetilde{T}_m}{2}\sin\frac{\sigma_1\widetilde{S}_n}{2}\Bigg]\\&\cdot\text{sinc}\frac{\sigma_1\widetilde{S}_n}{2\pi}\text{sinc}\frac{\sigma_2\widetilde{T}_m}{2\pi}.\end{aligned}$$

Substituting the relationships in Lemma 9 and $g(x,y) = \mathrm{e}^{\frac{\mathrm{j}a_2y^2}{2b_2}}f(x,y)\mathrm{e}^{\frac{\mathrm{i}a_1x^2}{2b_1}}$ in the above sampling series of g, we obtain sampling series (19).

$f \in L^2(\mathbb{R}^2, \mathbb{H})$ is a BL function to (σ_1, σ_2) in the two-sided QLCT sense, which does not mean that f is a BL function to (σ_1, σ_2) in the right- or left-sided QLCT. It is natural to ask, if f is a BL function to (σ_1, σ_2) in the right-sided QLCT sense, how could we reconstruct it from its sampled values at the point $(\frac{n\pi}{\sigma_1}, \frac{m\pi}{\sigma_2})$, $(n,m) \in \mathbb{Z}^2$. This problem is carried out by the following theorem:

Theorem 12 If $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is BL to (σ_1, σ_2)

in the right-sided QLCT sense, i.e.,

$$\mathcal{L}_{\mathrm{R}}^{\mathrm{i,j}}[f](v,u)=0, \quad \text{for } |v|>\sigma_1 \text{ or } |u|>\sigma_2,$$

then the following sampling series for f holds:

$$\begin{aligned}&f(x,y)\\&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[\frac{f(s_n,t_m)}{\sqrt{2\pi b_1\mathrm{i}}}\mathrm{e}^{\mathrm{i}\frac{a_1s_n^2}{2b_1}}\cos\frac{a_2(t_m^2-y^2)}{2b_2}\right.\\&\quad+\int_{\mathbb{R}}\frac{f(-s_n+x,t_m)}{\sqrt{2\pi b_1\mathrm{i}}}\mathrm{e}^{\mathrm{i}\frac{a_1(s_n-x)^2}{2b_2}}l(x)\mathrm{d}x\\&\quad\left.\cdot\sin\frac{a_2(t_m^2-y^2)}{2b_2}\mathrm{j}\right]\mathrm{sinc}\frac{\sigma_1(x-s_n)}{b_1\pi}\mathrm{sinc}\frac{\sigma_2(y-t_n)}{b_2\pi},\end{aligned}\tag{20}$$

where $l(x):=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\frac{a_1v^2}{b_1}}\mathrm{e}^{-\mathrm{i}vx}\mathrm{d}v=\frac{1}{2\pi}\left(\sqrt{\frac{\pi b_1}{a_1}}\cdot\cos(\frac{b_1x^2}{4a_1}-\frac{1}{4\pi})+\mathrm{i}\sqrt{\frac{\pi b_1}{a_1}}\cos(\frac{b_1x^2}{4a_1}+\frac{1}{4\pi})\right)$ $(a_i,b_i>0, i=1,2)$. The series converges in the L^2 norm. Moreover, it converges absolutely and uniformly on any compact subset of $\mathbb{R}^2$.

Proof Because

$$\begin{aligned}&f(x,y)\\&=\int_{-\sigma_1}^{\sigma_1}\left[\int_{-\sigma_2}^{\sigma_2}\mathcal{L}_{\mathrm{R}}^{\mathrm{i,j}}(v,u)K_{\boldsymbol{A}_2^{-1}}^{\mathrm{i}}(u,y)\mathrm{d}u\right]K_{\boldsymbol{A}_1^{-1}}^{\mathrm{i}}(v,x)\mathrm{d}u,\end{aligned}$$

then we have

$$\begin{aligned}&f(x,y)\mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}\\&=\int_{-\sigma_1}^{\sigma_1}\left[\int_{-\sigma_2}^{\sigma_2}\mathcal{L}_{\mathrm{R}}^{\mathrm{i,i}}(v,u)K_{\boldsymbol{A}_2^{-1}}^{\mathrm{j}}(u,y)\mathrm{d}u\frac{\mathrm{e}^{-\mathrm{i}\frac{d_1v^2}{2b_1}}}{\sqrt{2\pi b_1 i}}\right]\mathrm{e}^{\mathrm{i}\frac{vx}{b_1}}\mathrm{d}v,\end{aligned}$$

which implies that $h_y(x)=f(x,y)\mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}$ is a $\frac{\sigma_1}{b_1}$ BL function with respect to x in the QFT sense. Hence, we can recover $h_y(x)$ from the samples at point t_m via the following formula:

$$f(x,y)\mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}=\sum_{n=-\infty}^{\infty}f(s_n,y)\mathrm{e}^{\mathrm{i}\frac{a_1s_n^2}{2b_1}}\mathrm{sinc}\frac{\sigma_1(x-s_n)}{b_1\pi}.\tag{21}$$

Because $f\in L^2(\mathbb{R}^2,\mathbb{H})$ and $\mathcal{L}_{\mathrm{R}}^{\mathrm{i,j}}[f](v,u)=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}f(x,y)K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v)\mathrm{d}x\right)K_{\boldsymbol{A}_2}^{\mathrm{j}}(y,u)\mathrm{d}y$, using the relationship between kernels of QFT and QLCT, we obtain

$$\begin{aligned}&\mathcal{L}_{\mathrm{R}}^{\mathrm{i,j}}[f](v,u)\\&=\mathcal{F}_{\mathrm{R}}^{\mathrm{j}}\int_{\mathbb{R}}f(x,y)K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v)\mathrm{d}x\frac{\mathrm{e}^{\mathrm{j}\frac{a_2y^2}{2b_2}}}{\sqrt{2\pi b_2\mathrm{j}}}\frac{u\mathrm{e}^{\mathrm{j}\frac{d_2u^2}{2b_2}}}{b_2}.\end{aligned}$$

Then, from the assumption of $\mathcal{L}_{\mathrm{R}}^{\mathrm{i,j}}[f]$, $g_v(y)=\int_{\mathbb{R}}f(x,y)K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v)\mathrm{d}x\frac{1}{\sqrt{2\pi b_2\mathrm{j}}}\mathrm{e}^{\mathrm{j}\frac{a_2y^2}{2b_2}}$ is a $\frac{\sigma_2}{b_2}$ BL function with respect to y in the QFT sense. Therefore, we can recover $g_v(y)$ from the samples at point t_m via the following formula:

$$\begin{aligned}&\int_{\mathbb{R}}f(x,y)K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v)\mathrm{d}x\\&=\sum_{m=-\infty}^{\infty}\int_{\mathbb{R}}f(x,t_m)K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v)\mathrm{d}x\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\\&\quad\cdot\mathrm{sinc}\frac{\sigma_2(y-t_m)}{b_2\pi}.\end{aligned}$$

Multiplying both sides of the above equality by $\mathrm{e}^{-\mathrm{i}\frac{a_1v^2}{2b_1}}$, we have

$$\begin{aligned}&\int_{\mathbb{R}}f(x,y)\frac{\mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}\mathrm{e}^{-\mathrm{i}\frac{xv}{b_1}}}{\sqrt{2\pi b_1\mathrm{i}}}\mathrm{d}x\\&=\sum_{m=-\infty}^{\infty}\int_{\mathbb{R}}f(x,t_m)K_{\boldsymbol{A}_1}^{\mathrm{i}}(x,v)\mathrm{d}x\\&\quad\cdot\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\mathrm{e}^{-\mathrm{i}\frac{a_1v^2}{2b_1}}\mathrm{sinc}\frac{\sigma_2(y-t_m)}{b_2\pi},\end{aligned}$$

$$\begin{aligned}&\int_{\mathbb{R}}f(x,y)\frac{\mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}\mathrm{e}^{-\mathrm{i}\frac{xv}{b_1}}}{\sqrt{2\pi b_1\mathrm{i}}}\mathrm{d}x\\&=\sum_{m=-\infty}^{\infty}\left[\int_{\mathbb{R}}f(x,t_m)\frac{\mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}\mathrm{e}^{-i\frac{xv}{b_1}}}{\sqrt{2\pi b_1\mathrm{i}}}\mathrm{d}x\cos\frac{a_2(t_m^2-y^2)}{2b_2}\right.\\&\quad+\int_{\mathbb{R}}f(x,t_m)\frac{\mathrm{e}^{\mathrm{i}\frac{a_1x^2}{2b_1}}\mathrm{e}^{-\mathrm{i}\frac{xv}{b_1}}\mathrm{e}^{\mathrm{i}\frac{a_1v^2}{b_1}}}{\sqrt{2\pi b_1\mathrm{i}}}\mathrm{d}x\\&\quad\left.\cdot\sin\left(\frac{a_2(t_m^2-y^2)}{2b_2}\right)\mathrm{j}\right]\mathrm{sinc}\frac{\sigma_2(y-t_m)}{b_2\pi}.\end{aligned}$$

Taking the inverse quaternion Fourier transform $\frac{1}{2\pi}\mathcal{F}_{\mathrm{R}}^{-\mathrm{i}}$ on both sides at point $x_n=\frac{\pi n}{\sigma_1}$ and using the fact that $l(x):=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\frac{a_1v^2}{b_1}}\mathrm{e}^{-\mathrm{i}vx}\mathrm{d}v=\frac{1}{2\pi}\left(\sqrt{\frac{\pi b_1}{a_1}}\cdot\cos\left(\frac{b_1x^2}{4a_1}-\frac{1}{4\pi}\right)+\mathrm{i}\sqrt{\frac{\pi b_1}{a_1}}\cos\left(\frac{b_1x^2}{4a_1}+\frac{1}{4\pi}\right)\right)$ $(a_i,b_i>0,\ i=1,2)$, we obtain

$$\begin{aligned}&f(s_n,y)\frac{\mathrm{e}^{\mathrm{i}\frac{a_1s_n^2}{2b_1}}}{\sqrt{2\pi b_1\mathrm{i}}}\\&=\sum_{m=-\infty}^{\infty}\left[f(s_n,t_m)\frac{\mathrm{e}^{\mathrm{i}\frac{a_1s_n^2}{2b_1}}}{\sqrt{2\pi b_1\mathrm{i}}}\cos\frac{a_2(t_m^2-y^2)}{2b_2}\right.\\&\quad+\int_{\mathbb{R}}f(-s_n+x,t_m)\frac{\mathrm{e}^{\mathrm{i}\frac{a_1(s_n-x)^2}{2b_1}}}{\sqrt{2\pi b_2\mathrm{j}}}l(x)\mathrm{d}x\\&\quad\left.\cdot\sin\left(\frac{a_2(t_m^2-y^2)}{2b_2}\right)\mathrm{j}\right]\mathrm{sinc}\frac{\sigma_2(y-t_m)}{b_2\pi}.\end{aligned}$$

Substituting it into Eq. (21), we obtain sampling series (20).

4 Error analysis

4.1 Truncation errors associated with QFT

Truncation error occurs naturally in applications, because only a finite number of samples are given in practice. For $f \in L^2(\mathbb{R}^2, \mathbb{H})$ which is BL to (σ_1, σ_2) in the QFT sense, let $\mathrm{R1}_{N,M}, \mathrm{R2}_{N,M}$, and $\mathrm{R3}_{N,M}$ denote the truncation errors of $f(x,y)$:

$$\begin{aligned}&\mathrm{R1}_{N,M}(x,y)\\ :=&f(x,y)-\sum_{n=-N}^{N}\sum_{m=-M}^{M}f(nh_1,mh_2)\\ &\cdot\frac{\sin\left(\sigma_1(x-nh_1)\right)}{\sigma_1\left(x-nh_1\right)}\frac{\sin\left(\sigma_2(y-mh_2)\right)}{\sigma_2\left(y-mh_2\right)},\end{aligned} \tag{22}$$

where $h_1 := \pi/\sigma_1$ and $h_2 := \pi/\sigma_2$,

$$\begin{aligned}&\mathrm{R2}_{N,M}(x,y)\\ :=&f(x,y)-\sum_{n=-N}^{N}\sum_{m=-M}^{M}\Bigg[f(\widetilde{x}_n,\widetilde{y}_m)\cos\frac{\sigma_2\widetilde{Y}_m}{2}\\ &\cdot\cos\frac{\sigma_1\widetilde{X}_n}{2}-\mathcal{H}_x[f](\widetilde{x}_n,\widetilde{y}_m)\cos\frac{\sigma_2\widetilde{Y}_m}{2}\sin\frac{\sigma_1\widetilde{X}_n}{2}\\ &-\mathcal{H}_y[f](\widetilde{x}_n,\widetilde{y}_m)\sin\frac{\sigma_2\widetilde{Y}_m}{2}\cos\frac{\sigma_1\widetilde{X}_n}{2}\\ &+\mathcal{H}_{xy}[f](\widetilde{x}_n,\widetilde{y}_m)\sin\frac{\sigma_2\widetilde{Y}_m}{2}\sin\frac{\sigma_1\widetilde{X}_n}{2}\Bigg]\\ &\cdot\mathrm{sinc}\frac{\sigma_1\widetilde{X}_n}{2\pi}\mathrm{sinc}\frac{\sigma_2\widetilde{Y}_m}{2\pi},\end{aligned} \tag{23}$$

$$\begin{aligned}&\mathrm{R3}_{N,M}(x,y)\\ :=&f(x,y)-\sum_{n=K_1(x)-N}^{K_1(x)+N}\sum_{m=K_2(y)-M}^{K_2(y)+M}\Bigg[f(n_2h_1,m_2h_2)\\ &+(x-n_2h_1)\frac{\partial f}{\partial x}(n_2h_1,m_2h_2)\\ &+(y-m_2h_2)\frac{\partial f}{\partial y}(n_2h_1,m_2h_2)\\ &+(x-n_2h_1)(y-\widetilde{y}_m)\frac{\partial^2 f}{\partial x\partial y}(n_2h_1,m_2h_2)\Bigg]\\ &\cdot\left(\mathrm{sinc}\left(\frac{x}{2h_1}-n\right)\right)^2\left(\mathrm{sinc}\left(\frac{y}{2h_2}-m\right)\right)^2,\end{aligned} \tag{24}$$

where $\widetilde{x}_n = n_2h_1, \widetilde{y}_m = m_2h_2, \widetilde{X}_n = x - n_2h_1, \widetilde{Y}_m = y - m_2h_2, \frac{1}{2h_1} - \frac{1}{2} < K_1(x) \le \frac{1}{2h_1} + \frac{1}{2}, \frac{1}{2h_2} - \frac{1}{2} < K_2(y) \le \frac{1}{2h_2} + \frac{1}{2}$, and $(K_1(x), K_2(y))$ are integer points nearest to the truncation error observation point (x, y). The following lemmas are used to prove the estimation of truncation errors $\mathrm{R1}_{N,M}$ and $\mathrm{R2}_{N,M}$.

Lemma 10 (Splettstösser et al., 1981)

$$\left(\sum_{n=-\infty}^{+\infty}\left|\frac{\sin(\sigma(x-\frac{n\pi}{\sigma}))}{\sigma(x-\frac{n\pi}{\sigma})}\right|^2\right)^{\frac{1}{2}} \le 2,$$

where $\sigma > 0$.

Lemma 11 (Jagerman, 1966)

$$\begin{aligned}&\left(\sum_{|n|>N}\left|\frac{\sin(\sigma_1(x-nh_1))}{\sigma_1(x-nh_1)}\right|^2\right)^{\frac{1}{2}}\\ &\le\frac{|\sin(\sigma_1 x)|}{\sigma_1}\left(\frac{2N}{(Nh_1)^2-x^2}\right)^{\frac{1}{2}},\end{aligned}$$

where $|x| < Nh_1$.

Theorem 13 For $f \in L^2(\mathbb{R}^2, \mathbb{H})$ which is BL to (σ_1, σ_2) in the QFT sense, let $\mathrm{R1}_{N,M}$ be defined by Eq. (22), where $|x| < Nh_1$, $|y| < Mh_2$, $N \ge 1$, $M \ge 1$,

$$K_N=\left(h_1^2\sum_{|n|>N}\sum_{m=-\infty}^{+\infty}|f(nh_1,mh_2)|^2\right)^{\frac{1}{2}},$$

$$L_M=\left(h_2^2\sum_{|m|>M}\sum_{n=-\infty}^{+\infty}|f(nh_1,mh_2)|^2\right)^{\frac{1}{2}},$$

$$J_{N,M}=\left(h_1^2h_2^2\sum_{|m|>M}\sum_{|n|>N}|f(nh_1,mh_2)|^2\right)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}|\mathrm{R1}_{N,M}(x,y)|\le\ &I_1(x,y,N)+I_2(x,y,M)\\ &+\frac{2J_{N,M}\sqrt{MN}}{\pi^2\sqrt{((Nh_1)^2-x^2)((Mh_2)^2-y^2)}},\end{aligned}$$

where

$$I_1(x,y,N):=\frac{2\sqrt{2N}}{\pi}|\sin(\sigma_1 x)|\frac{K_N}{\sqrt{(Nh_1)^2-x^2}},$$

$$I_2(x,y,M):=\frac{2\sqrt{2M}}{\pi}|\sin(\sigma_2 y)|\frac{L_M}{\sqrt{(Mh_2)^2-y^2}}.$$

The proof of Theorem 13 is available in the supplementary materials.

Further estimates of $|\mathrm{R1}_{N,M}(x,y)|$ may be obtained by estimating quantities K_N, L_M, and $J_{N,M}$.

An immediate consequence of $\mathrm{R2}_{N,M}(x,y)$ is the following theorem which can be obtained by some arguments similar to Theorem 13:

Theorem 14 Let $\mathrm{R2}_{N,M}$ be defined by Eq. (24), $|x|<Nh_1, |y|<Mh_2, N\ge 1, M\ge 1$,

$$\mathrm{K}i_N=\frac{1}{2}\left(h_1^2\sum_{|n|>N}\sum_{m=-\infty}^{+\infty}|f_i(2nh_1,2mh_2)|^2\right)^{\frac{1}{2}},$$

$$\mathrm{L}i_M=\frac{1}{2}\left(h_2^2\sum_{|m|>M}\sum_{n=-\infty}^{+\infty}|f_i(2nh_1,2mh_2)|^2\right)^{\frac{1}{2}},$$

$$\mathrm{J}i_{N,M}=\frac{1}{4}\left(h_1^2h_2^2\sum_{|m|>M}\sum_{|n|>N}|f_i(2nh_1,2mh_2)|^2\right)^{\frac{1}{2}},$$

where $i=2,3,4,5, f_2=f, f_3=\mathcal{H}_x[f], f_4=\mathcal{H}_y[f]$, and $f_5=\mathcal{H}_{xy}[f]$. Then

$$|\mathrm{R2}_{N,M}(x,y)| \le S_2(x,y)+S_3(x,y)+S_4(x,y)+S_5(x,y),$$

where

$$\begin{aligned} &S_j(x,y,N)\\ =&\frac{2\sqrt{2N}}{\pi}\left|\sin\left(\frac{\sigma_1 x}{2}\right)\right|\frac{\mathrm{K}j_N}{\sqrt{(\frac{Nh_1}{2})^2-x^2}}\\ &+\frac{2\sqrt{2M}}{\pi}\left|\sin\left(\frac{\sigma_2 y}{2}\right)\right|\frac{\mathrm{L}j_M}{\sqrt{(Mh_2)^2-y^2}}\\ &+\frac{2\sqrt{MN}}{\pi^2}\frac{\mathrm{J}j_{N,M}}{\sqrt{\left((\frac{Nh_1}{2})^2-x^2\right)\left((\frac{Mh_2}{2})^2-y^2\right)}}. \end{aligned}$$

Herein, $j=2,3,4,5$.

Before estimating $\mathrm{R3}_{N,M}(x,y)$, there is a need to point out that BL quaternionic function $f(z_1,z_2)$ has been proved to be a quaternion holomorphic function (Hu and Kou, 2018), which is holomorphic in two variables (z_1,z_2):

$$\frac{\partial}{\partial\bar{z}_1}f(z_1,z_2)=0, f(z_1,z_2)\frac{\partial}{\partial\bar{z}_2}=0,$$

where $z_1\in\mathbb{H}_\mathrm{i}$, $\mathbb{H}_\mathrm{i}=\{z_1|z_1=x_1+\mathrm{i}x_2, x_1,x_2\in\mathbb{R}\}$, $z_2\in\mathbb{H}_\mathrm{j}$, $\mathbb{H}_\mathrm{j}=\{z_2|z_2=y_1+\mathrm{j}y_2, y_1,y_2\in\mathbb{R}\}$, $\frac{\partial}{\partial\bar{z}_1}=\frac{\partial}{\partial x_1}+\mathrm{i}\frac{\partial}{\partial x_2}$, and $\frac{\partial}{\partial\bar{z}_2}=\frac{\partial}{\partial y_2}+\mathrm{j}\frac{\partial}{\partial y_2}$. Then truncation error $\mathrm{R3}_{N,M}(x,y)$ is given by

$$\begin{aligned} &\mathrm{R3}_{N,M}(x,y)\\ =&\oint_{C_\mathrm{j}}\frac{f(x,z_2)}{(z_2-y)\sin^2(\frac{z_2\pi}{2h_2})}\frac{\sin^2(\frac{y\pi}{2h_2})}{2\pi\mathrm{j}}\mathrm{d}z_2+\frac{\sin^2(\frac{x\pi}{2h_1})}{2\pi\mathrm{i}}\\ &\cdot\oint_{C_\mathrm{i}}\frac{f(z_1,y)}{(z_1-x)\sin^2(\frac{z_1\pi}{2h_1})}\mathrm{d}z_1-\frac{\sin^2(\frac{x\pi}{2h_1})}{2\pi\mathrm{i}}\\ &\cdot\oint_{C_\mathrm{i}}\oint_{C_\mathrm{j}}\frac{f(z_1,z_2)}{(z_1-x)(z_2-y)\sin^2(\frac{z_1\pi}{2h_1})\sin^2(\frac{z_2\pi}{2h_2})}\mathrm{d}z_1\mathrm{d}z_2\\ &\cdot\frac{\sin^2(\frac{y\pi}{2h_2})}{2\pi\mathrm{j}}. \end{aligned} \tag{25}$$

Here C_i, shown in Fig. 1, is a simple closed contour enclosing both the point $z_1=x$ and the zero point $z_1=nh_1$ for all integers $K_1(x)-N\le n\le K_1(x)+N$. C_j shown in Fig. 1 is also a simple closed contour enclosing both the point $z_2=y$ and the zero point $z_2=mh_2$ for all integers $K_2(y)-M\le m\le K_2(y)+M$.

The proof of Theorem 14 is available in the supplementary materials.

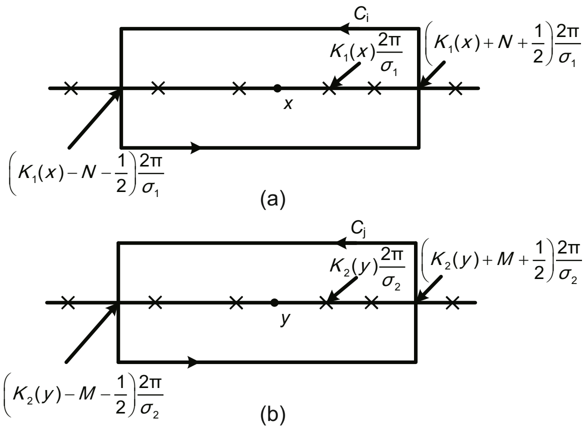

Fig. 1 The simple closed contours C_i (a) and C_j (b)
C_i and C_j belong to $\mathbb{H}_\mathrm{i}$ and $\mathbb{H}_\mathrm{j}$ planes respectively, and are used to calculate the truncation error bound $\mathrm{R3}_{N,M}(x,y)$

Theorem 15 If quaternionic function $f(x,y)$ is BL to $(r_1\sigma_1, r_2\sigma_2)$ in the QFT sense, where $0\le r_i<1$ $(i=1,2)$ and $|f(x,y)|\le C$, for $(x,y)\in\mathbb{R}^2$ and $C>0$, then an upper bound for the truncation error $\mathrm{R3}_{N,M}$ at point (x,y) is given by

$$\begin{aligned} |\mathrm{R3}_{N,M}(x,y)|\le&\frac{2C\left|\sin\left(\frac{y\pi}{2h_2}\right)\right|^2}{\frac{\pi M\sin(\pi r_2)}{r_2}}+\frac{2C\left|\sin\left(\frac{x\pi}{2h_1}\right)\right|^2}{\frac{\pi N\sin(\pi r_1)}{r_1}}\\ &+\frac{4C\left|\sin\left(\frac{x\pi}{2h_1}\right)\sin\left(\frac{y\pi}{2h_2}\right)\right|^2}{\frac{\pi N\sin(\pi r_1)}{r_1}\frac{\pi M\sin(\pi r_2)}{r_2}}. \end{aligned}$$

Before giving the proof of Theorem 15, we need the following lemma:

Lemma 12 If $f\in L^2(\mathbb{R}^2,\mathbb{H})$ is BL to (σ_1,σ_2), then

$$(1)\ |f(z_1,z_2)|\le C_1\mathrm{e}^{\sigma_1|z_1|}\mathrm{e}^{\sigma_2|z_2|},$$

(2) $|f(z_1, y_1)| \le C_1 \mathrm{e}^{\sigma_1|z_1|}$,

(3) $|f(x_1, z_2)| \le C_1 \mathrm{e}^{\sigma_2|z_2|}$,

where $C_1 = \frac{1}{4\pi^2}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}|\mathcal{F}_{\mathrm{R}}(v,u)|\mathrm{d}v\mathrm{d}u$.

The proof of Lemma 12 is available in the supplementary materials.

From this lemma, using the technique in Yao and Thomas (1966), we obtain Theorem 15.

4.2 Truncation errors associated with QLCT

If $f(x,y)$ is BL to (σ_1,σ_2) in the two-sided QLCT sense, Lemma 8 implies that $\mathrm{e}^{\mathrm{i}a_1x^2/(2b_1)}f(x,y)\mathrm{e}^{\mathrm{j}a_2y^2/(2b_2)}$ is BL to $(-\frac{\sigma_1}{b_1},\frac{\sigma_2}{b_2})$ in the two-sided QFT sense. So, we have the following truncation errors in the QLCT sense from Theorems 13–15. Let $\breve{\mathrm{R}}1_{N,M}, \breve{\mathrm{R}}2_{N,M}$, and $\breve{\mathrm{R}}3_{N,M}$ denote the truncation errors of $f(x,y)$ as follows (Assume $\breve{h}_1 := b_1\pi/\sigma_1$ and $\breve{h}_2 = b_2\pi/\sigma_2$) :

$$\begin{aligned}&\breve{\mathrm{R}}1_{N,M}(x,y)\\ =&f(x,y)-\sum_{n=-N}^{N}\sum_{m=-M}^{M}\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}f(s_n,t_m)\\ &\cdot\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\mathrm{sinc}\frac{\sigma_1(x-s_n)}{b_1\pi}\mathrm{sinc}\frac{\sigma_2(y-t_m)}{b_2\pi},\end{aligned}\tag{26}$$

$$\begin{aligned}&\breve{\mathrm{R}}2_{N,M}(x,y)\\ =&f(x,y)-\sum_{n=-N}^{N}\sum_{m=-M}^{M}\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\\ &\cdot\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\cos\frac{\sigma_2\widetilde{T}_m}{2b_2}\cos\frac{\sigma_1\widetilde{s}_n}{2b_1}-\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}\\ &\cdot\mathcal{H}^x_{\boldsymbol{A}_1}[f](\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\cos\frac{\sigma_2\widetilde{T}_m}{2b_2}\sin\frac{\sigma_1\widetilde{S}_n}{2b_1}\\ &-\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}\mathcal{H}^y_{\boldsymbol{A}_2}[f](\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\\ &\cdot\sin\frac{\sigma_2\widetilde{T}_m}{2b_2}\cos\frac{\sigma_1\widetilde{S}_n}{2b_1}+\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}\mathcal{H}^{xy}_{\boldsymbol{A}_1\boldsymbol{A}_2}[f](\widetilde{s}_n,\widetilde{t}_m)\\ &\cdot\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\sin\frac{\sigma_2\widetilde{T}_m}{2b_2}\sin\frac{\sigma_1\widetilde{S}_n}{2b_1}\Bigg]\\ &\cdot\mathrm{sinc}\frac{\sigma_1\widetilde{S}_n}{2b_1\pi}\mathrm{sinc}\frac{\sigma_2\widetilde{T}_m}{2b_2\pi},\end{aligned}\tag{27}$$

$$\begin{aligned}&\breve{\mathrm{R}}3_{N,M}\\ =&f(x,y)-\sum_{|n|\le N}\sum_{|m|\le M}\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\\ &+(x-\widetilde{s}_n)\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial x}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\\ &+\mathrm{i}\frac{a_1\widetilde{s}_n}{b_1}\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\Bigg]\\ &+(y-\widetilde{t}_m)\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial y}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\\ &+\mathrm{e}^{i\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\mathrm{j}\frac{a_2\widetilde{t}_m}{b_2}\Bigg]\\ &+(x-\widetilde{s}_n)(y-\widetilde{t}_m)\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial^2 f}{\partial x\partial y}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\\ &+\mathrm{i}\frac{a_1\widetilde{s}_n}{b_1}\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial y}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\\ &+\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial x}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\mathrm{j}\frac{a_2\widetilde{t}_m}{b_2}\\ &+\mathrm{i}\frac{a_1\widetilde{s}_n}{b_1}\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\mathrm{j}\frac{a_2\widetilde{t}_m}{b_2}\Bigg]\Bigg]\\ &\cdot\left(\mathrm{sinc}\left(\frac{\sigma_1x}{2b_1\pi}-n\right)\right)^2\left(\mathrm{sinc}\left(\frac{\sigma_2y}{b_22\pi}-m\right)\right)^2.\end{aligned}$$

Corollary 2 Suppose that $f\in L^2(\mathbb{R}^2,\mathbb{H})$ is BL to (σ_1,σ_2) in the two-sided QLCT sense. Let $\breve{\mathrm{R}}1_{N,M}$ be defined by Eq. (26), where $|x|<N\breve{h}_1, |y|<M\breve{h}_2, N\ge1, M\ge1$,

$$\breve{K}_N=\left(\breve{h}_1^2\sum_{|n|>N}\sum_{m=-\infty}^{+\infty}|f(n\breve{h}_1,m\breve{h}_2)|^2\right)^{\frac12},$$

$$\breve{L}_M=\left(\breve{h}_2^2\sum_{|m|>M}\sum_{n=-\infty}^{+\infty}|f(n\breve{h}_1,m\breve{h}_2)|^2\right)^{\frac12},$$

$$\breve{J}_{N,M}=\left(\breve{h}_1^2\breve{h}_2^2\sum_{|m|>M}\sum_{|n|>N}|f(n\breve{h}_1,m\breve{h}_2)|^2\right)^{\frac12}.$$

Then

$$|\breve{\mathrm{R}}1_{N,M}(x,y)|\le\breve{I}_1(x,y,N)+\breve{I}_2(x,y,M)+\frac{2\breve{J}_{N,M}\sqrt{MN}}{\pi^2\sqrt{((N\breve{h}_1)^2-x^2)((M\breve{h}_2)^2-y^2)}},$$

where

$$\breve{I}_1(x,y,N):=\frac{2\sqrt{2N}}{\pi}\left|\sin\left(\frac{\sigma_1x}{b_1}\right)\right|\frac{\breve{K}_N}{\sqrt{(N\breve{h}_1)^2-x^2}},$$

$$\breve{I}_2(x,y,M):=\frac{2\sqrt{2M}}{\pi}\left|\sin\left(\frac{\sigma_2y}{b_2}\right)\right|\frac{\breve{L}_M}{\sqrt{(M\breve{h}_2)^2-y^2}}.$$

Corollary 3 Let $\breve{\mathrm{R}}2_{N,M}$ be defined by Eq. (27),

$|x| < N\check{h}_1, |y| < M\check{h}_2, N \geq 1, M \geq 1,$

$$\check{\mathrm{K}}i_N = \frac{1}{2}\left(\check{h}_1^2 \sum_{|n|>N} \sum_{m=-\infty}^{+\infty} |f_i(2n\check{h}_1, 2m\check{h}_2)|^2\right)^{\frac{1}{2}},$$

$$\check{\mathrm{L}}i_M = \frac{1}{2}\left(\check{h}_2^2 \sum_{|m|>M} \sum_{n=-\infty}^{+\infty} |f_i(2n\check{h}_1, 2m\check{h}_2)|^2\right)^{\frac{1}{2}},$$

$$\check{\mathrm{J}}i_{N,M} = \frac{1}{4}\left(\check{h}_1^2\check{h}_2^2 \sum_{|m|>M} \sum_{|n|>N} |f_i(2n\check{h}_1, 2m\check{h}_2)|^2\right)^{\frac{1}{2}},$$

where $i = 2, 3, 4, 5, f_2 = f, f_3 = \mathcal{H}^x_{\boldsymbol{A}_1}[f], f_4 = \mathcal{H}^y_{\boldsymbol{A}_2}[f]$, and $f_5 = \mathcal{H}^{xy}_{\boldsymbol{A}_1\boldsymbol{A}_2}[f]$. Then we have

$$\begin{aligned}&|\check{\mathrm{R}}2_{N,M}(x,y)|\\ &\leq \check{S}_2(x,y) + \check{S}_3(x,y) + \check{S}_4(x,y) + \check{S}_5(x,y),\end{aligned}$$

where

$$\begin{aligned}\check{S}_j(x,y,N) = &\frac{2\sqrt{2N}}{\pi}\left|\sin\left(\frac{\sigma_1 x}{2b_1}\right)\right|\frac{\check{\mathrm{K}}j_N}{\sqrt{(\frac{N\check{h}_1}{2})^2 - x^2}}\\ &+ \frac{2\sqrt{2M}}{\pi}\left|\sin\left(\frac{\sigma_2 y}{2b_2}\right)\right|\frac{\check{\mathrm{L}}j_M}{\sqrt{(Mh_2)^2 - y^2}}\\ &+ \frac{2\sqrt{MN}}{\pi^2}\frac{\check{\mathrm{J}}j_{N,M}}{\sqrt{((\frac{N\check{h}_1}{2})^2 - x^2)((\frac{M\check{h}_2}{2})^2 - y^2)}}.\end{aligned}$$

Herein, $j = 2, 3, 4, 5$.

Corollary 4 If quaternionic function $f(x,y)$ is BL to $(r_1\sigma_1/b_1, r_2\sigma_2/b_2)$ in the QFT sense, where $0 \leq r_i < 1$ $(i = 1, 2)$ and $|f(x,y)| \leq C$, for $(x,y) \in \mathbb{R}^2$ and $C > 0$, then an upper bound for the truncation error $\check{\mathrm{R}}3_{N,M}$ at point (x,y) is given by

$$\begin{aligned}|\check{\mathrm{R}}3_{N,M}(x,y)| \leq &\frac{2C\left|\sin\left(\frac{y\pi}{2\check{h}_2}\right)\right|^2}{\frac{\pi M\sin(\pi r_2)}{r_2}} + \frac{2C\left|\sin\left(\frac{x\pi}{2\check{h}_1}\right)\right|^2}{\frac{\pi N\sin(\pi r_1)}{r_1}}\\ &+ \frac{4C\left|\sin\left(\frac{x\pi}{2\check{h}_1}\right)\sin\left(\frac{y\pi}{2\check{h}_2}\right)\right|^2}{\frac{\pi N\sin(\pi r_1)}{r_1}\frac{\pi M\sin(\pi r_2)}{r_2}}.\end{aligned}$$

5 Examples

In this section, we use mainly sampling formula (8) in Theorem 3 to produce a high-resolution image from its corresponding low-resolution version. The quality of the high-resolution image is measured by the structural similarity index measure (SSIM) and feature similarity index measure (FSIM) in Algorithm 1.

By Figs. 2 and 3, our sampling formula can recover the color image from low to high resolution. The quantitative measurements in Table 1 show the effectiveness of the proposed sampling formula.

Algorithm 1 Image reconstruction

1: Input the test color image $f(t_1, t_2)$ and convert the color image into the quaternion form.
2: The test image is downsampled by factor 2.
3: Generate a high-resolution (HR) image from the downsampled image by Eq. (8).
4: Compute the SSIM and FSIM to evaluate the quality of the generated HR image.

Fig. 2 Reconstructed images for Lena, flower, and bird by Algorithm 1
The first row shows the original images. The second row shows the degraded images with the resolution of 128×128. The third row shows the reconstructed images

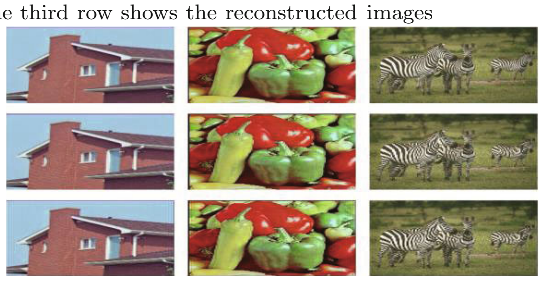

Fig. 3 Reconstructed images for house, pepper, and horse by Algorithm 1
The first row shows the original images. The second row shows the degraded images with the resolution of 128×128. The third row shows the reconstructed images

Table 1 SSIM and FSIM values of the reconstructed images

Image	SSIM	FSIM
Lena	0.9440	0.8912
Flower	0.9554	0.9495
Bird	0.9462	0.9353
House	0.9357	0.8822
Pepper	0.9559	0.8953
Horse	0.8531	0.8751

6 Conclusions

First, by Lemma 1, if the quaternionic function is bandlimited to a rectangle that is symmetric about the origin in the right-sided QFTs, then it is also bandlimited to this rectangle in the left- and two-sided QFTs, and vice versa. Therefore, if the quaternionic function is bandlimited to this rectangle, then the sampling formula associated with various QFTs is identical. However, if the quaternionic function is bandlimited to a rectangle that is not symmetric about the origin, then the sampling formulas associated with various QFTs are different. Second, we obtained not only the sampling formulas using the samples, but also the sampling series using samples of the partial derivatives and quaternion partial and total Hilbert transforms. Third, the sampling formulas associated with various QLCTs were obtained by the relationships of QFTs and QLCTs. Fourth, the truncation errors of those sampling formulas were derived. At last, by Algorithm 1, the sampling formula was applied to color image reconstruction.

In the future, we will apply the sampling series to color images, and multi-dimensional signals will be explored.

Contributors

Xiaoxiao HU designed the research. Xiaoxiao HU and Dong CHENG processed the data. Xiaoxiao HU drafted the paper. Kit Ian KOU helped organize the paper. Xiaoxiao HU and Dong CHENG revised and finalized the paper.

Compliance with ethics guidelines

Xiaoxiao HU, Dong CHENG, and Kit Ian KOU declare that they have no conflict of interest.

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List of electronic supplementary materials

Proof S1 Proof of Lemma 4
Proof S2 Proof of Theorem 13
Proof S3 Proof of Theorem 14
Proof S4 Proof of Lemma 12