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Pseudo-evolute curves and caustic surfaces*

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Abstract: In this study, osculating caustic developable and rectifying caustic developable surfaces were obtained by considering space curves and curves on the surface as base curves and changing the direction of the light source reflected by the mirror surface. It was proved that the pseudo-evolute curves represent the striction curves (regression edges) of these surfaces. In the case of developable surfaces based on curves on the surface, it was observed that osculating caustic developable surfaces are equivalent to rectifying caustic developable surfaces if the curve is geodesic. In addition, when the base curve was taken over any surface, the caustic surfaces were characterized as flat or normal approximation surfaces, depending on the direction of the light source.

Key words: Pseudo-evolute curves; Caustic surfaces; Developable surfaces; Rectifying caustic developable surfaces; Osculating caustic developable surfaces; Normal caustic developable surfaces
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1 Introduction

Traditionally, conical, cylindrical, and tangent surfaces of space curves are considered to be the three principal categories of developable surfaces. In this study, we examined the tangent surfaces of space curves, which are the most common and well-studied type of these surfaces. In the study by (Hoffmann et al., 2022), the caustic surfaces of all three types of developable surfaces were successfully generated, and all the required parameters have been defined and characterized. In the study by (Hoffmann et al., 2022), the equation of the caustic surfaces is presented as follows, assuming that the developable surfaces obtained from the tangent surfaces of space curves are mirror surfaces.

$$\boldsymbol{\varepsilon}(s,u)=\boldsymbol{\gamma}(s)+u\boldsymbol{f}(s); \quad t\in[a,b], u\in\mathbb{R} \tag{1}$$

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where $\boldsymbol{\gamma}$ is an arbitrary space curve and

$$\begin{aligned}\boldsymbol{f}(s)=&<\boldsymbol{\gamma}'(s)\times\boldsymbol{d}(s),\boldsymbol{d}'(s)>\boldsymbol{\gamma}'(s)\\&-<\boldsymbol{\gamma}'(s)\times\boldsymbol{d}(s),\boldsymbol{\gamma}''(s)>\boldsymbol{d}(s)\end{aligned} \tag{2}$$

where $\boldsymbol{d}$ is the reflected vector.

What is the precise meaning of the term "reflected vector" in this context? The direction of reflection of light rays transmitted from a light source to a mirror surface, which is a developable surface, is represented by the *reflected vector*. In this study, we have followed the methodology proposed by Hoffmann et al. in (2022) and fixed the directions of the light sources and the reflected vectors at different angles.

The question is how the reflected vector can be obtained. If the mirror surface $\boldsymbol{\Phi}$ of the surface $\boldsymbol{M}(s,u)$ based on the curve $\boldsymbol{\gamma}$ and the normal vector field of the mirror surface is $\boldsymbol{S}=(s_1,s_2,s_3)$, we can find the reflected vector $\boldsymbol{d}$ of the transmitted light ray in the $\boldsymbol{X}$ direction as follows:

$$\boldsymbol{d}=-(\boldsymbol{I}_3+2S^2)\boldsymbol{X},$$

where $\boldsymbol{I}$ is a unit matrix and

$$(s_1, s_2, s_3) \cong \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}.$$

The curve formed by the motion of the rectifying planes of a space curve $\boldsymbol{\gamma}$ is defined as a *pseudo-evolute curve*. In the event that the base curve $\boldsymbol{\gamma}$ is situated on a surface designated as $\boldsymbol{M}(s,u)$, the pseudo-evolute curve is defined as the motion of the tangent planes of the given surface $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}$. For more detailed information about the pseudo-evolute curves of an arbitrary space curve, see (Fuchs, 2013) and (Fuchs et al., 2024).

In analyzing the caustic surfaces, we were interested in observing the generation of the rectifying caustic developable, the osculating caustic developable, and the normal caustic developable surfaces.

In the process of our investigation into rectifying caustic surfaces, two key questions emerged. The first is determining the optimal direction for the light source to create these caustic surfaces. The second is exploring the potential tangent surfaces of curves that could be utilized to generate these surfaces. As a consequence of our research, we discovered that if we put the light source in the direction of the unit binormal of the base curve and use the tangent surfaces of the pseudo-evolute curve of the base curve, we can obtain these surfaces (Section 3).

The Darboux frame of a surface $\boldsymbol{M}(s,u)$ along a curve $\boldsymbol{\gamma}$ is given by the set $\{\boldsymbol{e_1}, \boldsymbol{y}, \boldsymbol{u}\}$, where the unit normal of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}$ is $\boldsymbol{u}$. In this case, a similar approach to that previously described for caustic surfaces yields the same results when the light source is positioned in the $\boldsymbol{y}$ direction for osculating caustic developable surfaces and in the $\boldsymbol{u}$ direction for normal caustic developable surfaces. Furthermore, the tangent surfaces of the pseudo-evolute curves on the surface are also applicable to both caustics (Section 4).

2 Basic Concepts

Let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^3$ be a unit speed space curve. In this context, the Frenet frame of the curve $\boldsymbol{\gamma}(s)$ will be denoted by the set $\{\boldsymbol{e_1}(s), \boldsymbol{e_2}(s), \boldsymbol{e_3}(s)\}$, where $\boldsymbol{e_1} = \boldsymbol{\gamma}'(s)$ is the unit tangent vector, $\boldsymbol{e_2}(s) = \frac{\boldsymbol{\gamma}''(s)}{||\boldsymbol{\gamma}''(s)||}$ is the unit normal vector, $\boldsymbol{e_3}(s) = \boldsymbol{e_1}(s) \times \boldsymbol{e_2}(s)$ is the unit binormal vector of $\boldsymbol{\gamma}(s)$, and $\times$ is the cross product. In this case, Frenet—Serret formulas:

$$\begin{aligned} \boldsymbol{e_1'}(s) &= \boldsymbol{\kappa}(s)\boldsymbol{e_2}(s) \\ \boldsymbol{e_2'}(s) &= -\boldsymbol{\kappa}(s)\boldsymbol{e_1}(s) + \boldsymbol{\tau}(s)\boldsymbol{e_3}(s) \\ \boldsymbol{e_3'}(s) &= -\boldsymbol{\tau}(s)\boldsymbol{e_2}(s) \end{aligned}$$

where $\boldsymbol{\kappa}(s)$ is the curvature and $\boldsymbol{\tau}(s)$ is the torsion of the curve $\boldsymbol{\gamma}(s)$. The vector field $\boldsymbol{D}(s)$ for any unit speed space curve $\boldsymbol{\gamma}(s)$, which is written in the form:

$$\boldsymbol{D}(s) = \big(\boldsymbol{\tau}\boldsymbol{e_1} + \boldsymbol{\kappa}\boldsymbol{e_3}\big)(s) \tag{3}$$

is called *the modified Darboux vector*. *The unit Darboux vector* can also be formulated as:

$$\overline{\boldsymbol{D}}(s) = \frac{\big(\boldsymbol{\tau}\boldsymbol{e_1} + \boldsymbol{\kappa}\boldsymbol{e_3}\big)(s)}{\sqrt{\boldsymbol{\tau}^2(s) + \boldsymbol{\kappa}^2(s)}}. \tag{4}$$

It is known that a constant ratio $\frac{\boldsymbol{\tau}(s)}{\boldsymbol{\kappa}(s)}$, where $\boldsymbol{\kappa}(s) \neq 0$, means that the curve is a *generalized helix*. Furthermore, if the value of

$$q(s) = \left(\frac{\boldsymbol{\kappa}^2}{\big(\boldsymbol{\kappa}^2 + \boldsymbol{\tau}^2\big)^{3/2}} \left(\frac{\boldsymbol{\tau}}{\boldsymbol{\kappa}}\right)' \right)(s)$$

is a constant, the curve will assume the form of a *slant helix*, where $\boldsymbol{\kappa}(s) \neq 0$.

Let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^3$ be *the base curve*, $\boldsymbol{v}\colon I \to \mathbb{R}^3 \setminus \{0\}$ be *the director curve*, and $\boldsymbol{\Phi}(s,u)$ be defined as $\boldsymbol{\Phi}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{v}(s)$, then the surfaces expressed as $\boldsymbol{\Phi}(s,u)$ are referred to as *ruled surfaces*. If these surfaces are represented in the form $\boldsymbol{\Phi}(s,u) = \boldsymbol{\gamma}(s) + \boldsymbol{u}\boldsymbol{D}(s)$, they are known as *rectifying developable surfaces*. For further details, please refer to (Izumiya and Takeuchi, 2004).

One of the fundamental formulas utilized in this study is the formula of *the pseudo-evolute curve of a space curve*, which is expressed as follows:

$$\boldsymbol{R}(s) = \boldsymbol{\gamma}(s) - \frac{1}{\left(\frac{\boldsymbol{\tau}}{\boldsymbol{\kappa}}\right)'(s)} \left(\left(\frac{\boldsymbol{\tau}}{\boldsymbol{\kappa}}\right)\boldsymbol{e_1} + \boldsymbol{e_3} \right)(s) \tag{5}$$

$$= \boldsymbol{\gamma}(s) - \frac{1}{\left(\frac{\boldsymbol{\tau}}{\boldsymbol{\kappa}}\right)'(s)} \boldsymbol{D}(s). \tag{6}$$

Let $\boldsymbol{M}(s,u)$ be any surface and let $\boldsymbol{\gamma}\colon I \to \boldsymbol{M}$ be a unit speed curve on this surface. In this context, the Darboux frame of the surface $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$ is denoted by the set $\{\boldsymbol{e_1}(s), \boldsymbol{y}(s), \boldsymbol{u}(s)\}$, where $\boldsymbol{u}(s)$ is the unit normal of the surface $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$ and $\boldsymbol{y}(s) = \boldsymbol{u}(s) \times \boldsymbol{e_1}(s)$. The following formulas are provided for *the normal curvature* $\boldsymbol{\kappa_n}(s)$, *the geodesic curvature* $\boldsymbol{\kappa_g}(s)$, and *the geodesic torsion* $\boldsymbol{\tau_g}(s)$ of the $\boldsymbol{\gamma}(s)$:

$$\begin{aligned}\boldsymbol{\kappa_n}(s) &= \frac{<\boldsymbol{\gamma}''(s), \boldsymbol{u}(s)>}{||\boldsymbol{\gamma}'(s)||^2}\\ \boldsymbol{\kappa_g}(s) &= \frac{det(\boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{u}(s))}{||\boldsymbol{\gamma}'(s)||^3}\\ \boldsymbol{\tau_g}(s) &= \frac{det(\boldsymbol{\gamma}'(s), \boldsymbol{u}(s), \boldsymbol{u}'(s))}{||\boldsymbol{\gamma}'(s)||^2}.\end{aligned}$$

Then, the Darboux formulas are presented as follows:

$$\begin{aligned}\boldsymbol{e_1'}(s) &= \boldsymbol{\kappa_g}(s)\boldsymbol{y}(s) + \boldsymbol{\kappa_n}(s)\boldsymbol{u}(s)\\ \boldsymbol{y}'(s) &= -\boldsymbol{\kappa_g}(s)\boldsymbol{e_1}(s) + \boldsymbol{\tau_g}(s)\boldsymbol{u}(s)\\ \boldsymbol{u}'(s) &= -\boldsymbol{\kappa_n}(s)\boldsymbol{e_1}(s) - \boldsymbol{\tau_g}(s)\boldsymbol{y}(s).\end{aligned}$$

In the study by (Izumiya and Otani, 2015), the authors defined two vector fields, *osculating Darboux vector* $\boldsymbol{D_O}(s)$ along $\boldsymbol{\gamma}$ and *unit osculating Darboux vector* $\overline{\boldsymbol{D_O}}(s)$, which we will utilize in this work. These are defined as follows:

$$\boldsymbol{D_O}(s) = \Big(\boldsymbol{\tau_g e_1} - \boldsymbol{\kappa_n y}\Big)(s) \tag{7}$$

$$\overline{\boldsymbol{D_O}}(s) = \left(\frac{\boldsymbol{\tau_g e_1} - \boldsymbol{\kappa_n y}}{\sqrt{\boldsymbol{\tau_g^2} + \boldsymbol{\kappa_n^2}}}\right)(s). \tag{8}$$

They also defined the *osculating developable surface* of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}$ in the study by (Izumiya and Otani, 2015) and provided the following equation for the surface:

$$\begin{aligned}\boldsymbol{OD_\gamma}(s,u) &= \boldsymbol{\gamma}(s) + u\left(\frac{\boldsymbol{\tau_g e_1} - \boldsymbol{\kappa_n y}}{\sqrt{\boldsymbol{\tau_g^2} + \boldsymbol{\kappa_n^2}}}\right)(s) &(9)\\ &= \boldsymbol{\gamma}(s) + u\overline{\boldsymbol{D_O}}(s) &(10)\end{aligned}$$

In the study by (Hananoi and Izumiya, 2017), similar methodologies are employed to derive the $\boldsymbol{D_R}$ *rectifying Darboux vector* along $\boldsymbol{\gamma}$, $\overline{\boldsymbol{D_R}}$ *unit rectifying Darboux vector*

$$\boldsymbol{D_R}(s) = \Big(\boldsymbol{\tau_g e_1} + \boldsymbol{\kappa_g u}\Big)(s) \tag{11}$$

$$\overline{\boldsymbol{D_R}}(s) = \left(\frac{\boldsymbol{\tau_g e_1} + \boldsymbol{\kappa_g u}}{\sqrt{\boldsymbol{\tau_g^2} + \boldsymbol{\kappa_g^2}}}\right)(s) \tag{12}$$

and $\boldsymbol{ND_\gamma}$ *normal developable surface*,

$$\begin{aligned}\boldsymbol{ND_\gamma}(s,u) &= \boldsymbol{\gamma}(s) + u\left(\frac{\boldsymbol{\tau_g e_1} + \boldsymbol{\kappa_g u}}{\sqrt{\boldsymbol{\tau_g^2} + \boldsymbol{\kappa_g^2}}}\right)(s) &(13)\\ &= \boldsymbol{\gamma}(s) + u\overline{\boldsymbol{D_R}}(s) &(14)\end{aligned}$$

for which the formulas are provided. For a more comprehensive understanding please refer to (Hananoi and Izumiya, 2017) and, (Köse and Yayli, 2023).

Another term that will be repeatedly referenced in this study is *the striction curve (the regression edge)* of the surface $\boldsymbol{M}(s,u)$, for which the following formula will be used.

$$S(s) = \boldsymbol{\gamma}(s) - \frac{<\boldsymbol{\gamma}'(s), \boldsymbol{v}'(s)>}{<\boldsymbol{v}'(s), \boldsymbol{v}'(s)>}\boldsymbol{v}(s)$$

where $\boldsymbol{\gamma}$ is *the base curve* and $\boldsymbol{v}\colon I \to \mathbb{R}^3 \setminus \{0\}$ is *the director curve.*

In conclusion, we will present a fundamental overview of approximation surfaces. If the normal vector field of any ruled surface accepting $\boldsymbol{\gamma}$ as the base curve coincides with the normal vector field of any surface $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}$, this ruled surface is called the *flat approximation surface* of the surface $\boldsymbol{M}(s,u)$ and is described as below:

$$\begin{aligned}F_b(s,u) &= b(s) + u\left(\frac{\boldsymbol{\tau_g e_1} - \boldsymbol{\kappa_n y}}{\sqrt{\boldsymbol{\tau_g^2} + \boldsymbol{\kappa_n^2}}}\right)(s) &(15)\\ &= b(s) + u\overline{\boldsymbol{D_O}}(s) &(16)\end{aligned}$$

where

$$b(s) = \boldsymbol{\gamma}(s) + r\boldsymbol{u}(s), \quad r \in \mathbb{R}$$

Similarly, if the normal vector field of the surface $\boldsymbol{M}(s,u)$ and the normal vector field of the ruled surface that takes $\boldsymbol{\gamma}$ as the base curve are orthogonal to each other along the curve $\boldsymbol{\gamma}$, then this ruled surface is called the normal approximation surface of $\boldsymbol{M}(s,u)$ and can be expressed as follows:

$$\begin{aligned}N_b(s,u) &= b(s) + u\left(\frac{\boldsymbol{\tau_g e_1} + \boldsymbol{\kappa_g u}}{\sqrt{\boldsymbol{\tau_g^2} + \boldsymbol{\kappa_g}^2}}\right)(s) &(17)\\ &= b(s) + u\overline{\boldsymbol{D_R}} &(18)\end{aligned}$$

where

$$b(s) = \boldsymbol{\gamma}(s) + r\boldsymbol{y}(s), \quad r \in \mathbb{R}$$

The second fundamental formula that will be utilized is the pseudo-evolute curve $\overline{\boldsymbol{R}}(s)$ of the $\boldsymbol{\gamma}(s)$

on a surface $\boldsymbol{M}(s,u)$.

$$\begin{aligned}\overline{\boldsymbol{R}}(s) &= \boldsymbol{\gamma}(s) \\ &\quad - \left(\frac{\boldsymbol{\kappa_n}}{\left(\boldsymbol{\kappa}_g + \frac{\boldsymbol{\kappa}_n\boldsymbol{\kappa}_g' - \boldsymbol{\kappa}_n'\boldsymbol{\kappa}_g}{\boldsymbol{\kappa}_n^2+\boldsymbol{\kappa}_g^2}\right)\sqrt{\boldsymbol{\kappa}_n^2+\boldsymbol{\kappa}_g^2}}\right)(s)\overline{\boldsymbol{D_O}}(s) \\ &= \boldsymbol{\gamma}(s) - \frac{\boldsymbol{\kappa}_n(s)\Big(\boldsymbol{\kappa}_g\boldsymbol{e_1} - \boldsymbol{\kappa_n y}\Big)(s)}{\Big(\boldsymbol{\kappa}_g^3 + \boldsymbol{\kappa}_g\boldsymbol{\kappa_n}^2 + \boldsymbol{\kappa_n}\boldsymbol{\kappa}_g' - \boldsymbol{\kappa_n}'\boldsymbol{\kappa}_g\Big)(s)}.\end{aligned} \tag{19}$$

3 Rectifying Caustic Developable Surfaces

In this section, we considered a unit speed space curve $\boldsymbol{\gamma}\colon I \to \mathbb{R}^3$ and analyzed: when a rectifying developable surface becomes a caustic surface. In which direction of the light source does the reflected surface become a rectifying caustic developable surface? Furthermore, it will be demonstrated that the surface derived from the tangents of the pseudo-evolute curve of $\boldsymbol{\gamma}$ is a rectifying caustic developable surface, i.e., the base curve of this surface is the pseudo-evolute curve of $\boldsymbol{\gamma}$.

Theorem 1 Let $\gamma\colon I \to \mathbb{E}^3$ be a unit speed space curve and the light source be in the negative direction of the unit binormal $-\boldsymbol{e_3(s)}$ of the base curve $\boldsymbol{\gamma}$ along the tangent plane of the mirror surface

$$\boldsymbol{\Phi}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\gamma}'(s), \quad u \in \mathbb{R} \tag{20}$$

and the reflected vector be the unit binormal $\boldsymbol{e_3(s)}$ of $\boldsymbol{\gamma}$ along the tangent plane. In this case, the caustic of the mirror surface is

$$\begin{aligned}\boldsymbol{\varepsilon}_R(s,u) &= \boldsymbol{\gamma}(s) + u\big(\boldsymbol{\tau e_1} + \boldsymbol{\kappa e_3}\big)(s) && (21)\\ &= \boldsymbol{\gamma}(s) + u\boldsymbol{D}(s) && (22)\end{aligned}$$

which is a rectifying developable surface. Furthermore, the caustic surface can be presented as:

$$\boldsymbol{\varepsilon}_R(s,u) = \boldsymbol{R}(s) + u\boldsymbol{R}'(s) \tag{23}$$

where $\boldsymbol{R}(s)$ is the pseudo-evolute curve of $\boldsymbol{\gamma}$.

Proof. From Eqs (1) and (2), we know the equation of caustic of a developable surface, which is

$$\boldsymbol{\varepsilon}_R(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{f}(s), \quad u \in \mathbb{R} \tag{24}$$

where

$$\begin{aligned}\boldsymbol{f}(s) = &< \boldsymbol{\gamma}'(s) \times \boldsymbol{e_3}(s), \boldsymbol{e_3}'(s) > \boldsymbol{\gamma}'(s) \\ &- < \boldsymbol{\gamma}'(s) \times \boldsymbol{e_3}(s), \boldsymbol{\gamma}''(s) > \\ =&\big(\boldsymbol{\tau e_1} + \boldsymbol{\kappa e_3}\big)(s).\end{aligned} \tag{25}$$

Thus, the caustic equation of a developable surface can take the form of the following equation

$$\boldsymbol{\varepsilon}_R(s,u) = \boldsymbol{\gamma}(s) + u\big(\boldsymbol{\tau e_1} + \boldsymbol{\kappa e_3}\big)(s)).$$

The striction curve of the caustic of the developable surface $\boldsymbol{c_R}(s)$ is also known from Eq. (50) in the study by (Hoffmann et al., 2022).

$$\boldsymbol{c_R}(s) = \boldsymbol{\gamma}(s) + \boldsymbol{\lambda_R}(s)\boldsymbol{f}(s) \tag{26}$$

Furthermore, according to our previous assumptions, we can calculate $\boldsymbol{\lambda_R}(s)$ from Eq. (57) in the study by (Hoffmann et al., 2022) as follows:

$$\boldsymbol{\lambda_R}(s) = \frac{- < \boldsymbol{e_1}(s), \boldsymbol{e_1}'(s) \times \boldsymbol{e_3}(s) >}{< \boldsymbol{f}'(s), \boldsymbol{e_1}'(s) \times \boldsymbol{e_3}(s) + \boldsymbol{e_1}(s) \times \boldsymbol{e_3}'(s) >} \tag{27}$$

$$= \left(\frac{\boldsymbol{\kappa}}{\boldsymbol{\kappa}'\boldsymbol{\tau} - \boldsymbol{\kappa}\boldsymbol{\tau}'}\right)(s) \tag{28}$$

Then, from Eqs (26) and (5)

$$\begin{aligned}\boldsymbol{c_R}(s) &= \boldsymbol{\gamma}(s) + \left(\frac{\boldsymbol{\kappa}}{\boldsymbol{\kappa}'\boldsymbol{\tau} - \boldsymbol{\kappa}\boldsymbol{\tau}'}\right)(s)\big(\boldsymbol{\tau e_1} + \boldsymbol{\kappa e_3}\big)(s) \\ &= \boldsymbol{R}(s).\end{aligned}$$

Thus, from the study by (Hoffmann et al., 2022), we have

$$\begin{aligned}\boldsymbol{\varepsilon_R}(s,u) &= \boldsymbol{c_R}(s) + u\boldsymbol{c_R'}(s), \quad u \in \mathbb{R} && (29)\\ &= \boldsymbol{R}(s) + u\boldsymbol{R}'(s). && (30)\end{aligned}$$

□

4 Osculating and Normal Caustic Developable Surfaces

In Section 3, the curve $\boldsymbol{\gamma}$ was taken to be a space curve. We will now modify our question by considering $\boldsymbol{\gamma}$ on any surface $\boldsymbol{M}(s,u)$ as follows: in which the direction of the light source is the reflected surface an oscillating caustic developable or a normal caustic developable surface? Also in this section, it will be shown that for osculating caustic developable and normal caustic developable surfaces, even if the direction of the light source is changed, the basis curves of these surfaces will still be the pseudo-evolute curve of $\boldsymbol{\gamma}$.

4.1 Osculating Caustic Developable Surfaces

Theorem 2 Let $\boldsymbol{\gamma}\colon I \to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s,u)$ and the light source be in the direction of $-\boldsymbol{y(s)}$ of the base curve $\boldsymbol{\gamma}$ along the tangent plane of the mirror surface

$$\boldsymbol{\Phi}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\gamma}'(s), \quad u \in \mathbb{R} \tag{31}$$

and the reflected vector be in the direction of $\boldsymbol{y(s)}$ of $\boldsymbol{\gamma}$ along the tangent plane. In this case, the caustic of the mirror surface is

$$\boldsymbol{\varepsilon}_O(s,u) = \boldsymbol{\gamma}(s) + u\Big(\boldsymbol{\tau_g e_1} - \boldsymbol{\kappa_n y}\Big)(s) \tag{32}$$

$$= \boldsymbol{\gamma}(s) + u\boldsymbol{D_O}(s) \tag{33}$$

which is an osculating developable surface. Furthermore, the caustic surface can be presented as:

$$\boldsymbol{\varepsilon_O}(s,u) = \boldsymbol{R}(s) + u\boldsymbol{R}'(s), \tag{34}$$

where $\boldsymbol{R}(s)$ is the pseudo-evolute curve of $\boldsymbol{\gamma}$ on the surface $\boldsymbol{M}(s,u)$.

Proof. From Eq. (1), caustic of $\boldsymbol{\Phi}(s,u)$ is

$$\boldsymbol{\varepsilon_O}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{f}(s). \tag{35}$$

According to our assumptions, from Eq. (2), the function $\boldsymbol{f}$ here is

$$\begin{aligned}\boldsymbol{f}(s) &= <\boldsymbol{e_1}(s)\times\boldsymbol{y}(s), \boldsymbol{y}'(s)>\boldsymbol{e_1}(s)\\ &\quad - <\boldsymbol{e_1}(s)\times\boldsymbol{y}(s), \boldsymbol{e_1}'(s)>\boldsymbol{y}(s)\\ &=\Big(\boldsymbol{\tau_g e_1} - \boldsymbol{\kappa_n y}\Big)(s),\end{aligned} \tag{36}$$

then we have

$$\boldsymbol{\varepsilon_O}(s,u) = \boldsymbol{\gamma}(s) + u\Big(\boldsymbol{\tau_g e_1} - \boldsymbol{\kappa_n y}\Big)(s).$$

In consideration of Eq. (16), it becomes clear that $\boldsymbol{\varepsilon}(s,u)$ can also be identified as the flat approximation surface of the surface $\boldsymbol{M}(s,u)$. The striction curve of the caustic is

$$\begin{aligned}\boldsymbol{c_O}(s) &= \boldsymbol{\gamma}(s) + \boldsymbol{\lambda_O}(s)\boldsymbol{f}(s)\\ &= \boldsymbol{\gamma}(s) + \boldsymbol{\lambda_O}(s)\Big(\boldsymbol{\tau_g e_1} - \boldsymbol{\kappa_n y}\Big)(s).\end{aligned}$$

From Eq. (57) in the study by (Hoffmann et al., 2022), we also have

$$\begin{aligned}\boldsymbol{\lambda_O}(s) &= \frac{-<\boldsymbol{e_1}(s), \boldsymbol{e_1}'(s)\times\boldsymbol{y}(s)>}{<\boldsymbol{f}'(s), \boldsymbol{e_1}'(s)\times\boldsymbol{y}(s) + \boldsymbol{e_1}(s)\times\boldsymbol{y}'(s)>}\\ &= \left(\frac{\boldsymbol{\kappa_n}}{-\boldsymbol{\kappa_g}\big(\boldsymbol{\kappa_n}^2 + \boldsymbol{\tau_g}^2\big) - \boldsymbol{\kappa_n}\boldsymbol{\tau_g}' + \boldsymbol{\kappa_n}'\boldsymbol{\tau_g}}\right)(s).\end{aligned} \tag{37}$$

Therefore, the equation of the striction curve of the caustic surface takes the form

$$\begin{aligned}\boldsymbol{c_O}(s) =&\boldsymbol{\gamma}(s)\\ &-\left(\frac{\boldsymbol{\kappa_n}}{-\boldsymbol{\kappa_g}\big(\boldsymbol{\kappa_n}^2 + \boldsymbol{\tau_g}^2\big) - \boldsymbol{\kappa_n}\boldsymbol{\tau_g}' + \boldsymbol{\kappa_n}'\boldsymbol{\tau_g}}\right)(s)\\ &\Big(\boldsymbol{\tau_g e_1} - \boldsymbol{\kappa_n y}\Big)(s)\\ =&\boldsymbol{\gamma}(s)\\ &-\left(\frac{\boldsymbol{\kappa_n}}{\Big(\boldsymbol{\kappa}_g + \frac{\boldsymbol{\kappa}_n\boldsymbol{\kappa}_g' - \boldsymbol{\kappa}_n'\boldsymbol{\kappa}_g}{\boldsymbol{\kappa}_n^2 + \boldsymbol{\kappa}_g^2}\Big)\sqrt{\boldsymbol{\kappa}_n^2 + \boldsymbol{\kappa}_g^2}}\right)(s)\overline{\boldsymbol{D_0}}(s)\end{aligned}$$

which is the pseudo-evolute curve of the developable surface from Eq. (19).

The caustic surface formed by reflected rays in the direction of $\boldsymbol{y}(s)$ on the mirror surface $\boldsymbol{\Phi}(s,u)$ is

$$\boldsymbol{\varepsilon_O}(s,u) = \overline{\boldsymbol{R}}(s) + u\overline{\boldsymbol{R}}'(s).$$

□

For instance, in the event of the selection of $\boldsymbol{M}(s,u)$ as:

$$\boldsymbol{M}(s,u) = \left\{\frac{2}{3}\sqrt{1+\frac{3u^2}{4}}cos(s), \frac{2}{3}\sqrt{1+\frac{3u^2}{4}}sin(s), u\right\} \tag{38}$$

which is a circular hyperboloid of one sheet, the osculating developable surface $\boldsymbol{\varepsilon_O}(s,u)$ of the surface $\boldsymbol{M}(s,u)$ and the mirror surface $\boldsymbol{\Phi}(s,u)$ and the base curve $\boldsymbol{\gamma}$ as mentioned in Theorem 2, are illustrated in Figs. 1, 2, 3 and 5.

To create a more understandable visual, Figs. 3, 4 and 5 are zoomed in, and the value ranges are narrowed to the region where the curve is on the surface.

Corollary 1 If we take $\boldsymbol{\kappa_g}(s) = 0$ in (4.1), we have the striction curve of the curve $\boldsymbol{\gamma}(s)$, which is a space curve. In this particular instance, it can be observed that the Frenet frame and the Darboux frame coincide.

Proposition 1 Let $\boldsymbol{\gamma}\colon I \to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s,u)$, then the following statements are satisfied for curve $\boldsymbol{\gamma}$.

(i) The developable surface $\boldsymbol{\Phi}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\gamma}'(s)$ is the mirror surface.

(ii) Surface $\boldsymbol{\varepsilon_O}(s,u) = \boldsymbol{\gamma}(s) + u\Big(\boldsymbol{\tau_g e_1} - \boldsymbol{\kappa_n y}\Big)(s)$ is the flat approximation surface of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$.

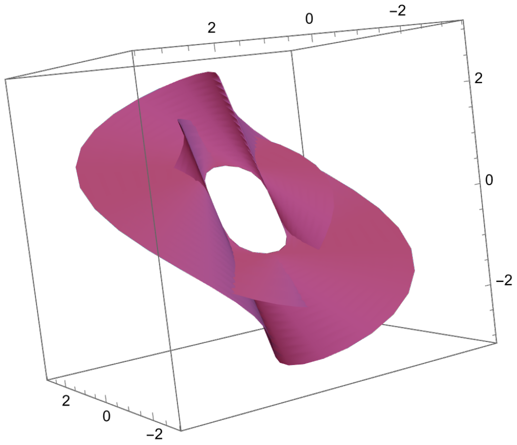

Fig. 1 **The mirror surface $\boldsymbol{\Phi}(s,u)$**

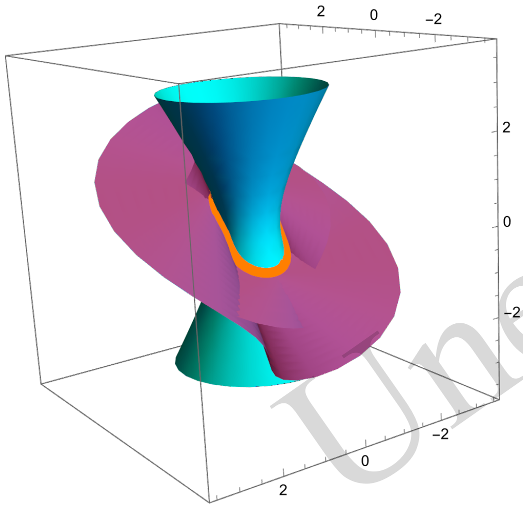

Fig. 2 **The base curve γ (orange), the mirror surface $\boldsymbol{\Phi}(s,u)$ (purple), and the surface $\boldsymbol{M}(s,u)$ (cyan).**

(iii) $\boldsymbol{\varepsilon_O}(s,u)$ is the osculating caustic surface of the mirror surface $\boldsymbol{\Phi}(s,u)$.

(iv) $\boldsymbol{c_O}(s)$ is the pseudo-evolute curve of the curve $\boldsymbol{\gamma}$ on the surface $\boldsymbol{M}(s,u)$.

4.2 Normal Caustic Developable Surfaces

Theorem 3 Let $\boldsymbol{\gamma}\colon I \to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s,u)$ and the reflected vector be in the direction $\boldsymbol{u}(s)$ of the base curve $\boldsymbol{\gamma}$ along the tangent plane of the mirror surface,

$$\boldsymbol{\Phi}(s,u) = \gamma(s) + u\gamma'(s), \quad u \in \mathbb{R} \tag{39}$$

and the reflected vector be in the direction $\boldsymbol{u}(s)$ of $\boldsymbol{\gamma}$ along the tangent plane. In this case, the caustic

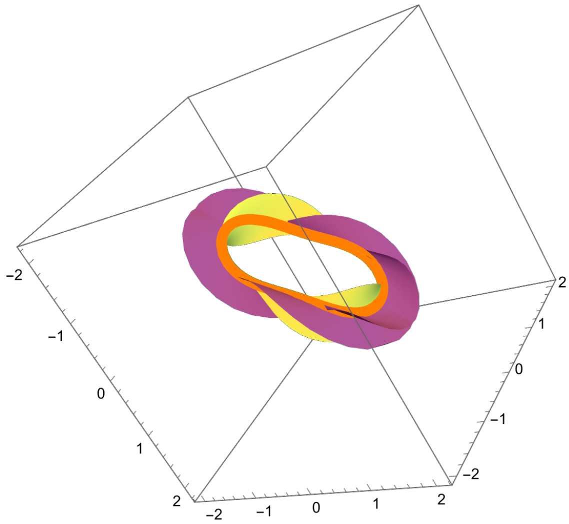

Fig. 3 **The osculating caustic developable $\boldsymbol{\varepsilon_O}(s,u)$ (yellow), the mirror surface $\boldsymbol{\Phi}(s,u)$ (purple), and the base curve $\gamma(s)$ (orange).**

surface of the mirror surface is

$$\boldsymbol{\varepsilon_N}(s,u) = \gamma(s) + u\Big(\boldsymbol{\tau_g e_1} + \boldsymbol{\kappa_g u}\Big)(s) \tag{40}$$

$$= \gamma(s) + u\boldsymbol{D_R}(s) \tag{41}$$

which is a normal developable surface.

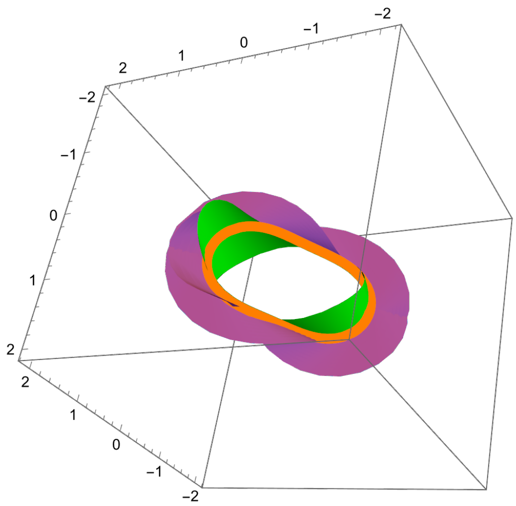

Fig. 4 **The normal caustic developable $\boldsymbol{\varepsilon_O}(s,u)$ (green), the mirror surface $\boldsymbol{\Phi}(s,u)$ (purple), and the base curve $\gamma(s)$ (orange).**

Proposition 2 Let $\boldsymbol{\gamma}\colon I \to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s,u)$ and the reflected vector be in the direction $\boldsymbol{u(s)}$ of the base curve $\boldsymbol{\gamma}$ along the tangent plane of the developable surface $\boldsymbol{\Phi}(s,u)$.

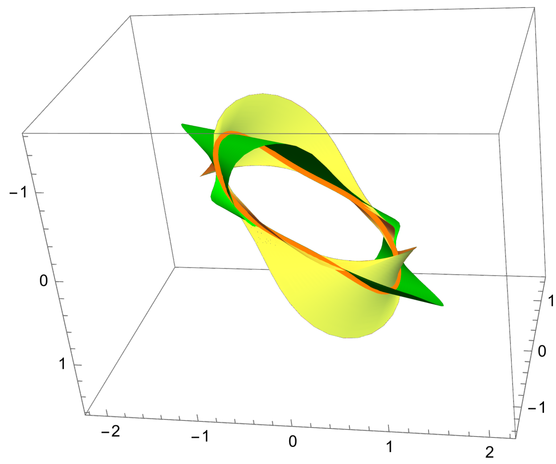

Fig. 5 The osculating caustic developable surface $\varepsilon_O(s,u)$ (yellow), the normal caustic developable surface $\varepsilon_N(s,u)$ (green), and the curve $\gamma(s)$ (orange) are displayed together.

Then, the caustic surface $\boldsymbol{\varepsilon_N}(s,u)$ is the normal approximation surface of $\boldsymbol{M}(s,u)$.

The surface $\boldsymbol{M}(s,u)$ in Eq. (38) and the normal caustic developable surface $\boldsymbol{\varepsilon_N}(s,u)$ of the surface $\boldsymbol{M}(s,u)$ as mentioned in Theorem 3 and Proposition 2 are illustrated in Figs. 4 and 5.

Theorem 4 Let $\gamma\colon I\to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s,u)$ and the reflected vector be in the direction $\boldsymbol{u}(s)$ of the base curve γ along the tangent plane of the developable surface $\boldsymbol{\Phi}(s,u)$. The striction curve of the caustic surface $\boldsymbol{\varepsilon_N}(s,u)$ is

$$\begin{aligned}\overline{\boldsymbol{R}}(s) =&\gamma(s)\\ &+\left(\frac{-\boldsymbol{\kappa_g}}{\boldsymbol{\kappa_g}\left(\boldsymbol{\tau_g'}-\boldsymbol{\kappa_n}\boldsymbol{\kappa_g}\right)-\boldsymbol{\tau_g}\left(\boldsymbol{\kappa_g'}+\boldsymbol{\tau_g}\boldsymbol{\kappa_n}\right)}\right)(s)\\ &\left(\boldsymbol{\tau_g}\boldsymbol{e_1}+\boldsymbol{\kappa_g}\boldsymbol{u}\right)(s). \qquad (42)\end{aligned}$$

Proof. The striction curve of $\boldsymbol{\varepsilon_N}(s,u)$ is

$$\boldsymbol{c_N}(s)=\gamma(s)+\boldsymbol{\lambda_N}(s)\boldsymbol{f}(s).$$

From Eqs (2) and (27), we have

$$\boldsymbol{\lambda_N}(s)=\frac{-\boldsymbol{\kappa_g}}{\boldsymbol{\kappa_g}\left(\boldsymbol{\tau_g'}-\boldsymbol{\kappa_n}\boldsymbol{\kappa_g}\right)-\boldsymbol{\tau_g}\left(\boldsymbol{\kappa_g'}+\boldsymbol{\tau_g}\boldsymbol{\kappa_n}\right)}(s), \qquad (43)$$

$$\boldsymbol{f}(s)=\left(\boldsymbol{\tau_g}\boldsymbol{e_1}+\boldsymbol{\kappa_g}\boldsymbol{u}\right)(s). \qquad (44)$$

□

Corollary 2 If we take $\boldsymbol{\kappa_n}(s)=0$ in Eq. (42), we have the striction curve of the curve $\gamma(s)$, which is a space curve. In this particular instance, it can be observed that the Frenet frame and the Darboux frame coincide.

Proposition 3 Let $\gamma\colon I\to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s,u)$ and the reflected vector be in the direction $\boldsymbol{u(s)}$ of the base curve $\boldsymbol{\gamma}$ along the tangent plane of the developable surface $\boldsymbol{\Phi}(s,u)$. The striction curve of the caustic surface, denoted as $\boldsymbol{c_N}(s)$, is considered as the pseudo-evolute curve of $\boldsymbol{\varepsilon_N}(s,u)$, denoted as $\overline{\boldsymbol{R}}(s)$, simultaneously.

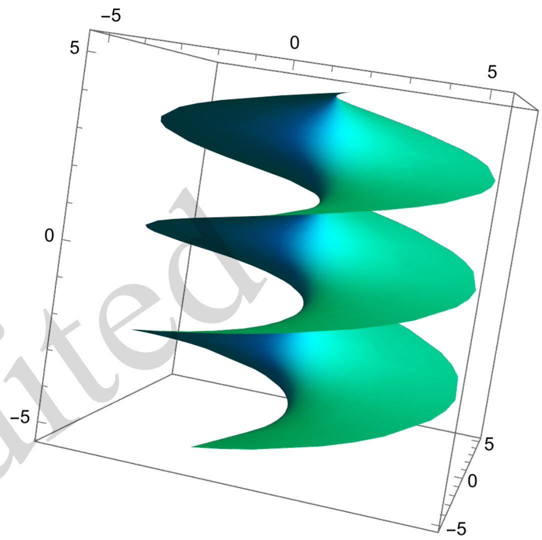

Fig. 6 The surface $M(s,u)$, which is given with Eq. (46) in Example 1.

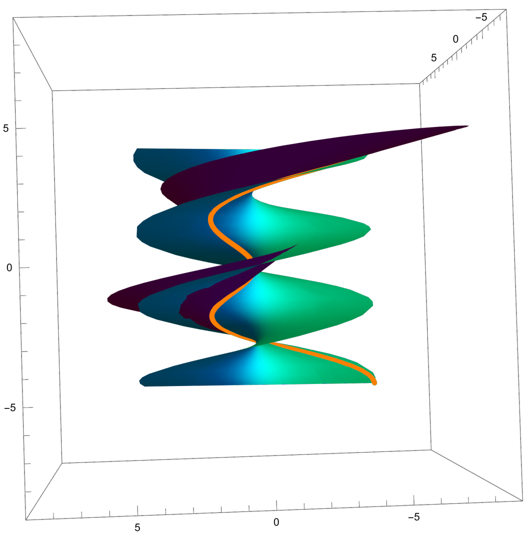

Fig. 7 The base curve $\gamma(s)$ Eq. (45) (orange) on the mirror surface $\boldsymbol{\Phi}(s,u)$ (purple) and the surface $M(s,u)$ Eq. (46) (cyan).

Example 1 Let $\boldsymbol{\gamma}: I \to \boldsymbol{M}$ be a space curve as:

$$\boldsymbol{\gamma} = \Big\{scos(s), ssin(s), s\Big\}. \tag{45}$$

The curve $\boldsymbol{\gamma}$ is located on the surface $\boldsymbol{M}(s,u)$ (Fig. 6), which is a helicoid:

$$\boldsymbol{M}(s,u) = \Big\{scos(u), ssin(u), u\Big\}. \tag{46}$$

As can be observed in Fig. 7, the curve $\boldsymbol{\gamma}$ is located above the mirror surface and the surface $\boldsymbol{M}(s,u)$. The Darboux frame of the surface $\boldsymbol{M}(s,u)$ along the $\boldsymbol{\gamma}(s)$ is

$$\boldsymbol{e_1}(s) = \frac{1}{\sqrt{1+s^2}}\Big\{cos(s) - ssin(s), scos(s) + sin(s), 1\Big\} \tag{47}$$

$$\boldsymbol{y}(s) = \Big\{ -cos(s) - \frac{ssin(s)}{1+s^2}, -sin(s) + \frac{scos(s)}{1+s^2}, \frac{1}{1+s^2}\Big\} \tag{48}$$

$$\boldsymbol{u}(s) = \frac{1}{\sqrt{1+s^2}}\Big\{sin(s), -cos(s), s\Big\} \tag{49}$$

where

$$\boldsymbol{\kappa}_n(s) = -\frac{2}{(1+s^2)^{\frac{3}{2}}} \tag{50}$$

$$\boldsymbol{\kappa}_g(s) = \frac{s(3+s^2)}{(1+s^2)^2} \tag{51}$$

$$\boldsymbol{\tau}_g(s) = \frac{s}{(1+s^2)^2}. \tag{52}$$

Osculating caustic developable surface of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}$ is

$$\begin{aligned}\boldsymbol{\varepsilon}_O(s,u) = &\frac{1}{(1+s^2)^{\frac{5}{2}}}\Big\{scos(s) + u((-2+s^2)(1+s^2)cos(s) \\ &- s(2+s^2+s^4)sin(s)), ssin(s) \\ &+ u(s(2+s^2+s^4)cos(s) + (-2+s^2)(1+s^2)sin(s), \\ &s + u(2+s^2+s^4)\Big\},\end{aligned} \tag{53}$$

and normal caustic developable surface of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}$ is

$$\begin{aligned}\boldsymbol{\varepsilon}_N(s,u) = &\Big\{s\Big(cos(s) + \frac{u((s+s^3)cos(s) - (-3+s^4)sin(s)}{(1+s^2)^{\frac{5}{2}}}\Big), \\ &s\Big(sin(s) + \frac{u((-3+s^4)cos(s) + s(1+s^2)sin(s)}{(1+s^2)^{\frac{5}{2}}}\Big), \\ &s + \frac{2s^2(2+s^2)u}{(1+s^2)^{\frac{5}{2}}}\Big\}.\end{aligned} \tag{54}$$

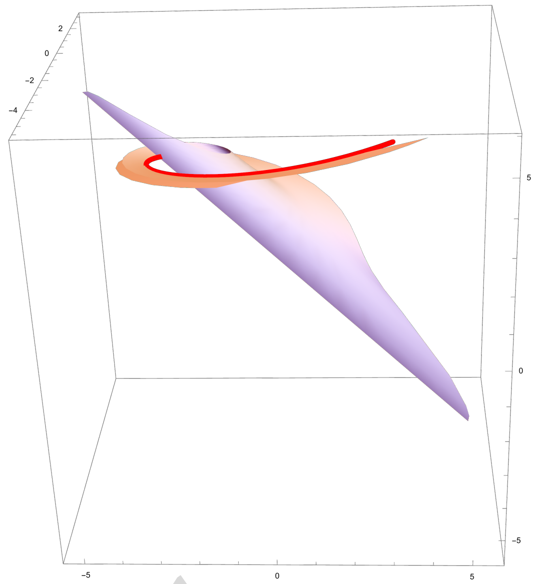

Fig. 8 The pseudo-evolute curve $c_O(s)$ (red) of the osculating caustic developable surface $\varepsilon_O(s,u)$ Eq. (53).

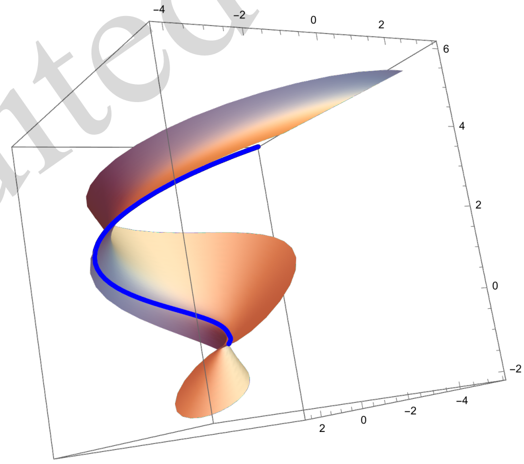

Fig. 9 The pseudo-evolute curve $c_N(s)$ (blue) of the normal caustic developable surface $\varepsilon_N(s,u)$ Eq. (54).

In Figs. 8 and 10, the pseudo-evolute curve $\boldsymbol{c_O}(s)$ is observed as the striction curve of the osculating caustic developable surface $\boldsymbol{\varepsilon}_O(s,u)$. Similarly, as can be observed in Figs. 9 and 11, the pseudo-evolute curve $\boldsymbol{c_N}(s)$ corresponds to the striction curve of the normal caustic developable surface $\boldsymbol{\varepsilon}_N(s,u)$.

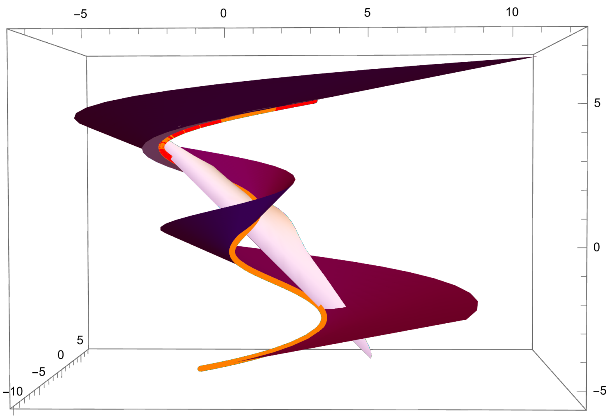

Fig. 10 The base curve γ (orange) Eq. (45), the pseudo-evolute curve $c_O(s)$ (red), the osculating caustic developable surface $\varepsilon_O(s,u)$ Eq. (53), and the mirror surface $\Phi(s,u)$ (purple).

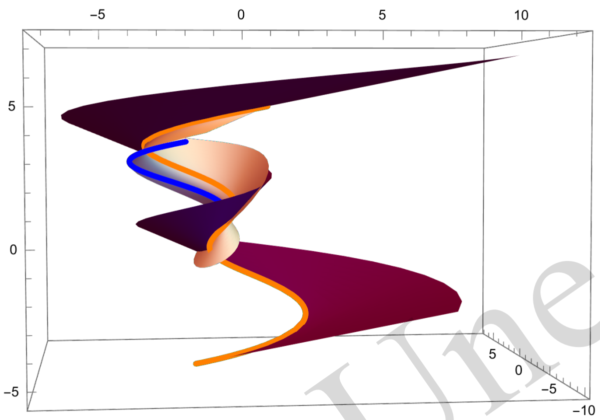

Fig. 11 The base curve γ (orange) Eq. (45), the pseudo-evolute curve $c_N(s)$ (blue), the normal caustic developable surface $\varepsilon_N(s,u)$ Eq. (54), and the mirror surface $\Phi(s,u)$ (purple).

5 Conclusions

In this study, an investigation was conducted to which base curves can be employed to create these mirror surfaces and in which direction the light sources are positioned so that the resulting surfaces can be characterized as rectifying caustic, osculating caustic or normal caustic developable surfaces. The methods used to obtain these surfaces are fully detailed in this paper. In our forthcoming research, we intend to apply the methods described here to the Lorentz-Minkowski space.

Contributors

The research was a collaborative effort involving Hande Nur DALKILIÇ and Yusuf YAYLI, who made equal contributions to the project. They were responsible for developing the mathematical computations and creating the experimental figures.

Conflict of interest

Both the authors declare that they have no conflict of interest.

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