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MIXED l_2/l_1 CONTROL FOR DISCRETE-TIME SYSTEMS VIA LAGRANGE MULTIPLIER THEORY*

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Abstract: The dual formulation of the discrete-time mixed l_2/l_1 control design problem was achieved by using the duality theory of Lagrange multipliers. For some special dual mixed l_2/l_1 problems, an approximation method for the optimal value is introduced. A suboptimal value of the infinite-dimensional dual problem can be obtained by solving a sequence of truncated problems. The convergence property of the solution scheme is investigated. This paper gives a low approximation method for the primal problem.

Key words: mixed l_2/l_1 control, approximate analysis, lagrange multiplier theory
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INTRODUCTION

The l_1 optimal control theory for linear discrete-time control systems where the maximum amplitude of system signals is constrainted to be bounded, was studied in recent years(McDonald et al., 1991; Dullerud et al., 1992; Staffans, 1993; Dahleh et al., 1993). In H_∞ optimal control theory, the system signals are constrained to have finite energy. However, inputs in practice such as steps, sinusoids of known frequency, do not satisfy this condition. l_1 optimal control problems are of considerable practical value for treating the above signals. The mixed l_2/l_1 control problem is formulated on the minimization of the l_2-norm of one part of the impulse response of a system, under a constraint on the l_1-norm of another part of the impulse response(Wu et al., 1996). This problem was shown to have a unique solution when the feasible region is close. An upper approximation method of this solution is devised. The difficulty with this procedure is that it is not clear how far the suboptimal solution is from the optimal solution. This paper considers the dual formulation of the mixed l_2/l_1 optimization problem, and explores another solution method for some special problems that approximates the optimal value from below. Combined with the upper approximation method, this lower approximation method can give both an upper bound and a lower bound on the optima.

PRELIMINARIES

R denotes the field of real numbers, R^N the set of $N \times 1$ vectors with elements in R. A causal LTI (Linear Time Invariant) transfer function $\hat{G} = G(0) + G(1)\lambda + G(2)\lambda^2 + \cdots$ is BIBO (Bounded Input Bounded Output) stable if and only if $\sum_{k=0}^{\infty} | G(k) | < \infty$. l_1 denotes the real normed linear space of all BIBO stable causal LTI transfer functions. For any $\hat{G} \in l_1$, the l_1 norm of $\hat{G}$ is given by $\| \hat{G} \|_1 = \sum_{k=0}^{\infty} | G(k) |$. $Rl_1 = \{ \hat{G} | \hat{G} \in l_1, \hat{G} \text{ is a rational function of } \lambda \}$. Real linear normed space

$$l_2 = \{ \hat{G} \mid \| \hat{G} \|_2 = (\sum_{k=0}^{\infty} (G(k))^2)^{1/2} < \infty \}.$$

$\hat{G}$ can be represented uniquely by its impulse response sequence $(G(0), G(1), G(2), \cdots)^T$. So $\hat{G}$ and its impulse response sequence are not differentiated in notation.

l_∞ denotes the set of all real sequences $d = (d(0), d(1), d(2), \cdots)^T$ such that $\sup_k | d(k) | < \infty$. For any $d \in l_\infty$, the l_∞ norm of d is given by $\| d \|_\infty = \sup_k | d(k) |$. c_0 denotes the subspace of l_∞ of sequences conve-

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rging to zero.

Let X be a normed linear space. The space of all bounded linear functionals on X is denoted X^*, equiped with the natural induced norm. For any $x \in X$ and $y \in X^*$, $\langle x, y \rangle$ denotes the value of the bounded linear functional y at the point x. It is convenient to put on X^* a weaker topology which makes $X^{**} = X$. This is the weak*-topology. $l_1^* = l_\infty$, $c_0^* = l_1$, $l_2^* = l_2$ and $Rl_1 \subset l_1 \subset l_2 \subset c_0 \subset l_\infty$. For any $x \in l_1$ and $y \in l_\infty$ (or for any $x \in l_2$ and $y \in l_2$),

$$\langle x, y \rangle = \sum_{k=0}^{\infty} (x(k)y(k)).$$

Let X be a vector space. Given a convex cone S_1, it is possible to define an ordering relation on X as follows: $x_1 \geq x_2$ if and only if $x_1 - x_2 \in S_1$. Then it is natural to define a convex cone S_2 inside X^* in the following way:

$$S_2 = \{y \in X^* \mid \langle x, y \rangle \geq 0, \ \forall x \in S_1\}.$$

This in turn defines an ordering relation on X^*. For any vector space in this paper, the positive cone which defines an ordering relation is the set consisting of elements with nonnegative pointwise components.

Let f be a convex function from X to R and G a convex map from X to another normed space Z. Also, let Ω be a convex subset of X. Assume that there exists $x_1 \in X$ such that $G(x_1) < 0$ (the inequality with respect to some cone in Z). This is generally known as the regularity assumption. Define the minimization problem:

$$\nu = \inf f(x) \text{ subject to } x \in \Omega, \ G(x) \leq 0. \tag{1}$$

Then the dual problem is

$$\nu = \sup_{z^* \geq 0} \inf_{x \in \Omega} \{f(x) + \langle G(x), z^* \rangle\} \tag{2}$$

$$\text{s.t. } z^* \in Z^*$$

In the case where the infinization problem contains equality constraints, we will replace them by two inequality constraints. Care should be taken in this case since the assumption that the constraint set has an interior point will be violated; however under mild assumptions, if the equality constraints are given in terms of linear operators, the result will still hold without the regularity conditions (Dahleh, 1993).

PROBLEM DEFINITION

Given an admissible plant $\hat{P}$ as

$$\begin{cases} y = -\hat{P}_{11} u + \hat{P}_{12} w_1 + \hat{P}_{13} w_2 \\ z_1 = \hat{P}_{21} u + \hat{P}_{22} w_1 + \hat{P}_{23} w_2 \\ z_2 = \hat{P}_{31} u + \hat{P}_{32} w_1 + \hat{P}_{33} w_2 \end{cases} \tag{3}$$

where $\hat{P}_{ij}(i = 1, 2, 3; j = 1, 2, 3)$ are rational causal LTI transfer functions, w_1 and w_2 are single exogenous inputs, z_1 and z_2 are single regulated outputs, u is the single control input, and y is the single measured output. $\hat{P}$ is assumed to be stabilizable, and $\hat{P}_{11}$ is assumed to be strictly causal. The compensator is $u = \hat{C}y$.

Let $\hat{\Phi}$ denote the closed-loop transfer function between w_1 and z_1, $\hat{\Psi}$ denote the closed-loop transfer function between w_2 and z_2. Given a constant γ, the objective is to find a rational causal LTI compensator $\hat{C}$ which stabilizes $\hat{P}$ and minimizes $\|\hat{\Phi}\|_2$ subject to $\|\hat{\Psi}\|_1 \leq \gamma$.

Incorporating the Youla parametrization of all stabilizing compensators (Francis, 1987) yields

$$\begin{aligned} \hat{\Phi} &= \hat{P}_{22} + \hat{P}_{21}\hat{C}(1 + \hat{P}_{11}\hat{C})^{-1}\hat{P}_{12} \\ &= \hat{T}_2 - \hat{Q}\hat{V} \end{aligned} \tag{4}$$

$$\begin{aligned} \hat{\Psi} &= \hat{P}_{33} + \hat{P}_{31}\hat{C}(1 + \hat{P}_{11}\hat{C})^{-1}\hat{P}_{13} \\ &= \hat{T}_1 - \hat{Q}\hat{U} \end{aligned} \tag{5}$$

$$\hat{Q} \in Rl_1.$$

Hence the mixed l_2/l_1 control problem can be stated as: Given $\hat{T}_1$, $\hat{U}$, $\hat{T}_2$, $\hat{V} \in Rl_1$ and a constant γ, where $\hat{U}$, $\hat{V} \neq 0$, find $\hat{Q} \in Rl_1$ such that $\|\hat{T}_2 - \hat{Q}\hat{V}\|_2$ is minimized and $\|\hat{T}_1 - \hat{Q}\hat{U}\|_1 \leq \gamma$. Without loss of generality (McDonald et al. 1991), it is assumed that

$$\hat{V} = (V(0), \cdots, V(m))^T \in R^{m+1},$$

$$\hat{U} = (U(0), \cdots, U(n))^T \in R^{n+1}.$$

For $\hat{Q} \in Rl_1$, define

$$\xi = \{\hat{\Phi} \mid \hat{\Phi} = \hat{T}_2 - \hat{Q}\hat{V}, \ \|\hat{T}_1 - \hat{Q}\hat{U}\|_1 \leq \gamma\}.$$

The mixed l_2/l_1 problem is described as

$$\sqrt{\mu} = \inf_{\Phi \in \xi} \|\hat{\Phi}\|_2 \tag{6}$$

Throughout this paper, it is assumed that ξ is nonempty. Obviously, ξ is nonempty when

$$\gamma > \inf_{\hat{Q}\in Rl_1} \| \hat{T}_1 - \hat{Q}\hat{U} \|_1 .$$

Notice that $\mu \in [0, \infty)$ when ξ is nonempty.

Let $W = (1, 1, \cdots)$. Define two operators $\mathscr{U}$: $l_1 \to l_1$ and $\mathscr{V}$: $l_1 \to l_1$ as follows:

$$\mathscr{U} = \begin{bmatrix} U(0) & & & 0 \\ \vdots & \ddots & & \\ U(n) & \ddots & U(0) & \\ & \ddots & \vdots & \ddots \\ & & U(n) & \ddots \\ 0 & & & \ddots \end{bmatrix}$$

$$\mathscr{V} = \begin{bmatrix} V(0) & & & 0 \\ \vdots & \ddots & & \\ V(m) & \ddots & V(0) & \\ & \ddots & \vdots & \ddots \\ & & V(m) & \ddots \\ 0 & & & \ddots \end{bmatrix}$$

Then (6) can be posed as

$$\mu = \inf\langle \hat{\Phi}, \hat{\Phi} \rangle$$

$$\text{s.t.} \begin{cases} \hat{\Phi} + \mathscr{V}\hat{Q} = \hat{T}_2 \\ \hat{\Psi}_+ - \hat{\Psi}_- + \mathscr{U}\hat{Q} = \hat{T}_1 \\ W(\hat{\Psi}_+ + \hat{\Psi}_-) \le \gamma \\ \hat{\Phi} \in Rl_1, \ \hat{\Psi}_+ \in Rl_1, \ \hat{\Psi}_- \in Rl_1, \\ \hat{Q} \in Rl_1, \ \hat{\Psi}_+, \ \hat{\Psi}_- \ge 0. \end{cases} \tag{7}$$

DUAL PROBLEM

Let $X = l_1 \times l_1 \times l_1 \times l_1$, $Z = l_1 \times l_1 \times l_1 \times l_1 \times R$,

$$x = \begin{pmatrix} \hat{\Phi} \\ \hat{\Psi}_+ \\ \hat{\Psi}_- \\ \hat{Q} \end{pmatrix} = \begin{pmatrix} \hat{\Phi} \\ \hat{\Psi}_0 \\ \hat{Q} \end{pmatrix}, \quad b = \begin{pmatrix} \hat{T}_2 \\ -\hat{T}_2 \\ \hat{T}_1 \\ -\hat{T}_1 \\ \gamma \end{pmatrix},$$

$\Omega = \{ x \mid x = (\hat{\Phi}^T, \hat{\Psi}_0^T, \hat{Q}^T)^T, \hat{\Phi} \in Rl_1, \hat{\Psi}_0 \in Rl_1 \times Rl_1, \hat{Q} \in Rl_1, \hat{\Psi}_0 \ge 0 \}$,

$F = (E \ \ 0 \ \ 0 \ \ 0)$: $X \to l_1$,

where $E = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{bmatrix}$.

Define the operator

$$A = \begin{bmatrix} E & 0 & 0 & \mathscr{V} \\ -E & 0 & 0 & -\mathscr{V} \\ 0 & E & -E & \mathscr{U} \\ 0 & -E & E & -\mathscr{U} \\ 0 & W & W & 0 \end{bmatrix} = [A_1 \ \ A_2 \ \ A_3]: X \to Z.$$

With these definitions, (7) becomes

$$\mu = \inf\langle Fx, Fx \rangle \tag{8}$$

s.t. $Ax - b \le 0$, $x \in \Omega$.

It is easy to see that: $\langle Fx, Fx \rangle$ is a convex function from X to R, $Ax - b$ is a convex map from X to Z, Ω is a convex subset. Let $Z^* = c_0 \times c_0 \times c_0 \times c_0 \times R$. Here Z is equipped with the weak*-topology, not the norm topology (Dahleh et al., 1993). From the theory of Lagrange multipliers, the minimum solution of (8) can be obtained by performing an unconstrained minimization of its Lagrangian, i.e.

$$\begin{aligned} \mu &= \sup_{z^* \ge 0} \inf_{x \in \Omega} \{ \langle Fx, Fx \rangle + \langle Ax - b, z^* \rangle \} \\ &= \sup_{z^* \ge 0} \inf_{x \in \Omega} \{ \langle \hat{\Phi}, \hat{\Phi} \rangle + \langle A_1 \hat{\Phi} + A_2 \hat{\Psi}_0 + A_3 \hat{Q} - b, z^* \rangle \} \\ &= \sup_{z^* \ge 0} \inf_{x \in \Omega} \{ \langle \hat{\Phi}, \hat{\Phi} + A_1^* z^* \rangle + \langle \hat{\Psi}_0, A_2^* z^* \rangle + \langle \hat{Q}, A_3^* z^* \rangle - \langle b, z^* \rangle \} \end{aligned} \tag{9}$$

s.t. $z^* \in Z^*$,

where $A_1^* = [E \ \ -E \ \ 0 \ \ 0 \ \ 0]$: $Z^* \to l_\infty$

$$A_2^* = \begin{bmatrix} 0 & 0 & E & -E & W^* \\ 0 & 0 & -E & E & W^* \end{bmatrix}: Z^* \to l_\infty \times l_\infty$$

$A_3^* = [\mathscr{V}^* \ \ -\mathscr{V}^* \ \ \mathscr{U}^* \ \ -\mathscr{U}^* \ \ 0]$: $Z^* \to l_\infty$

$W^* = (1, 1, \cdots)^T$

$$\mathscr{U}^* = \begin{bmatrix} U(0) & \cdots & U(n) & & 0 \\ & \ddots & \ddots & \ddots & \\ & & U(0) & \cdots & U(n) \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\mathscr{V}^* = \begin{bmatrix} V(0) & \cdots & V(m) & & 0 \\ & \ddots & \ddots & \ddots & \\ & & V(0) & \cdots & V(m) \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix}$$

are the adjoint operators of A_1, A_2, A_3, W, $\mathscr{U}$, $\mathscr{V}$ respectively. This result is true despite the fact that the constraints do not satisfy the regularity conditions (Luenberger, 1969). In the case where $A_3^* z^* \neq 0$, $\hat{Q}$ can be chosen such that $\mu < 0$. However $\mu \in [0, \infty]$, it is evident that $A_3^* z^* = 0$. Similarly, it can be obtained that $A_2^* z^* \geq 0$ and the above infinization is achieved for $\hat{\Psi}_0 = 0$. Let $A_1^* z^* = (h(0), h(1), \cdots)^T$. If $A_1^* z^* \notin l_2$, i.e. $\sum_{k=0}^{\infty}(h(k))^2 = \infty$. For any positive integer M, construct

$$\hat{\Phi}_M = \frac{-1}{\sqrt{\sum_{k=0}^{M-1}(h(k))^2}}(h(0), \cdots, h(M-1), 0, \cdots)^T \in l_1.$$

Then

$$\langle \hat{\Phi}_M, \hat{\Phi}_M + A_1^* z^* \rangle = 1 - \sqrt{\sum_{k=0}^{M-1}(h(k))^2}$$

and $\lim_{M\to\infty}\langle \hat{\Phi}_M, \hat{\Phi}_M + A_1^* z^* \rangle = -\infty$. It follows that $A_1^* z^* \in l_2$, moreover

$$\langle \hat{\Phi}, \hat{\Phi} + A_1^* z^* \rangle = \langle \hat{\Phi} + (A_1^* z^*/2), \hat{\Phi} + (A_1^* z^*/2) \rangle - \langle A_1^* z^*/2, A_1^* z^*/2 \rangle.$$

For any $\varepsilon > 0$ and $A_1^* z^* = (h(0), h(1), \cdots)^T \in l_2$, δ can be found such that

$$\hat{\Phi} = -\frac{1}{2}(h(0), \cdots, h(\delta), 0, \cdots)^T \in Rl_1$$

and

$$0 \leq \langle \hat{\Phi} + (A_1^* z^*/2), \hat{\Phi} + (A_1^* z^*/2) \rangle < \varepsilon.$$

This means

$$\inf_{\Phi \in Rl_1} \langle \hat{\Phi}, \hat{\Phi} + A_1^* z^* \rangle = -\frac{1}{4}\langle A_1^* z^*, A_1^* z^* \rangle.$$

Consequently,

$$\mu = \sup_{z^* \geq 0}\{-\frac{1}{4}\langle A_1^* z^*, A_1^* z^* \rangle - \langle b, z^* \rangle\}$$

$$\text{s.t.} \begin{cases} A_1^* z^* \in l_2 \\ A_2^* z^* \geq 0 \\ A_3^* z^* = 0 \\ z^* \in Z^* \end{cases} \qquad (10)$$

Let z^* be given by $z^* = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \eta \end{pmatrix}$. By direct substitution, (10) is converted to

$$\mu = \sup\{-\frac{1}{4}\langle \alpha_1 - \alpha_2, \alpha_1 - \alpha_2 \rangle - \langle \hat{T}_2, \alpha_1 - \alpha_2 \rangle - \langle \hat{T}_1, \beta_1 - \beta_2 \rangle - \langle \gamma, \eta \rangle\}$$

$$\text{s.t.} \begin{cases} \beta_1 - \beta_2 + W^* \eta \geq 0 \\ -\beta_1 + \beta_2 + W^* \eta \geq 0 \\ \mathscr{V}^* \alpha_1 - \mathscr{V}^* \alpha_2 + \mathscr{U}^* \beta_1 - \mathscr{U}^* \beta_2 = 0 \\ \alpha_1, \alpha_2, \beta_1, \beta_2 \in c_0, \eta \in R \\ \alpha_1 - \alpha_2 \in l_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \eta \geq 0 \end{cases} \qquad (11)$$

Finally, substituting $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$, the dual formulation of (7) is

$$\mu = \sup\{-\frac{1}{4}\langle \alpha, \alpha \rangle - \langle \hat{T}_2, \alpha \rangle - \langle \hat{T}_1, \beta \rangle - \gamma\eta\}$$

$$\text{s.t.} \begin{cases} -W^* \eta \leq \beta \leq W^* \eta \\ \mathscr{V}^* \alpha + \mathscr{U}^* \beta = 0 \\ \alpha \in l_2, \beta \in c_0, \eta \in R, \eta \geq 0 \end{cases} \qquad (12)$$

LOWER APPROXIMATION

Assumption # : For $\hat{U} = (U(0), \cdots, U(n))^T$, there is $\rho \in \{0, \cdots, n\}$ such that $U(\rho) \neq 0$ and $U(k) = 0$ when $k \neq \rho$.

Under this assumption,

$$\mathscr{U}^* = \left[\begin{array}{ccc|ccc} 0 & \cdots & & U(\rho) & & 0 \\ \vdots & \cdots & & & U(\rho) & \\ \vdots & \cdots & & 0 & & \ddots \end{array}\right] = [\mathscr{U}_1^* \quad \mathscr{U}_2^*]$$

and $(\mathscr{U}_2^*)^{-1} = \begin{bmatrix} \frac{1}{U(\rho)} & & 0 \\ & \frac{1}{U(\rho)} & \\ 0 & & \ddots \end{bmatrix}$. Let $\hat{T}_{1\alpha} = (T_1(\rho), T_1(\rho+1), \cdots)^T$. (12) can be writen

$$\mu = \sup\{-\frac{1}{4}\langle \alpha, \alpha \rangle - \langle \hat{T}_2, \alpha \rangle + \eta \sum_{k=0}^{\rho-1} |T_1(k)| - \langle \hat{T}_{1\alpha}, -(\mathscr{U}_2^*)^{-1}\mathscr{V}\hat{\alpha} \rangle - \gamma\eta\}$$

$$\text{s.t.} \begin{cases} -W^* \eta \leq -(\mathscr{U}_2^*)^{-1}\mathscr{V}\hat{\alpha} \leq W^* \eta \\ \alpha \in l_2, \eta \in R, \eta \geq 0 \end{cases} \qquad (13)$$

The N-th truncated problem of (13) can be constructed as follows.

$$\mu_N = \max\{-\frac{1}{4}\langle\alpha, \alpha\rangle - \langle\hat{T}_2, \alpha\rangle - \langle\hat{T}_{1\alpha}, -(\mathscr{U}_2^*)^{-1}\mathscr{V}_\alpha^*\rangle + (\sum_{k=0}^{\rho-1}|T_1(k)| - \gamma)\eta\}$$

$$\text{s.t.} \begin{cases} -W^*\eta \le -(\mathscr{U}_2^*)^{-1}\mathscr{V}_\alpha^* \le W^*\eta \\ \alpha \in R^{N+1}, \ \eta \in R, \ \eta \ge 0 \end{cases} \quad (14)$$

Obviously, the N-th truncated problem is a finite-dimensional optimization problem which can be successfully solved with many numerical optimization techniques. The following results are due to the fact that $R^{N+1} \times 0 \subset R^{N+2}$ and finite sequences are dense in l_2.

Proposition: Given assumption #, then $\mu_0 \le \mu_1 \le \cdots$ and $\lim_{N\to\infty}\mu_N = \mu$.

CONCLUSION

The dual formulation sheds a new light on the mixed l_2/l_1 control problem. It may provide an important approximation to the optima of a general mixed l_2/l_1 problem from below. Using both upper approximation and lower approximation is a convenient and valid way for computing the minimum performance to any prescribed accuracy.

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