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AN IMPROVED AXISYMMETRIC WILSON NONCONFORMING FINITE ELEMENT METHOD FOR STRESS ANALYSIS*

ZHANG Wu(张武)[1†], DESAI, C.S.[2], YONG, Y.K.[3]

([1]*Dept. of Civil Engineering, Zhejiang University, Hangzhou, 310027, China*)
([2]*Dept. of Civil Engineering and Engineering Mechanics, University of Arizona, Arizona, 85721, USA*)
([3]*Dept. of Civil and Environmental Engineering, Rutgers University, New Jersey, 08855, USA*)

[†]Email: zhangwu@civil.zju.edu.cn

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Abstract: The Wilson and Taylor elements Q6 and QM6, the representative nonconforming finite element method(FEM), have been successfully developed and implemented in the finite element code SAP for improved displacement and stress analysis. This paper formulates an improved convergent nonconforming axisymmetric element AQM6 over the corresponding axisymmetric Q6 and QM6 elements. The proposed modified nonconforming axisymmetric element AQM6 satisfies the engineering patch test condition for convergence, and also meets the condition for suppression of spurious shear stress by using a special remedying procedure. The numerical test results are in agreement with the element performance.

Key words: nonconforming finite element, axisymmetric Wilson element, patch test condition, convergence, spurious shear stress.
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INTRODUCTION

The nonconforming finite element Q6(Wilson et al., 1973) improves the accuracy of conforming finite element Q4 by incorporating higher order nonconforming displacements which reduce the shear locking, soften the stiffness of compatible elements and increase the interpolation accuracy. QM6(Taylor et al., 1976), an improvement of Q6 by constant Jacobian (adjoint) 'crime' scheme, further satisfies the patch test which improves the convergence and/or robustness of a nonconforming element. Consequently, QM6 has been considered to be the representative nonconforming finite element method(FEM) widely used and implemented in FE software with remarkable performance.

It seems to be more challenging to formulate a robust convergent nonconforming element for axisymmetric analysis. The successful constant Jacobian (adjoint) 'crime' scheme for QM6 cannot apply to the corresponding axisymmetric element for QM6 to pass the patch test (Taylor et al., 1976; Cook, 1981).

Furthermore, without additional special treatment, nonconforming displacements in nonconforming elements usually introduce spurious shear stresses, so the condition for suppression of spurious shear stresses should also be satisfied for an ideal nonconforming axisymmetric element (Cook, 1981).

In this paper the classic Wilson and Taylor nonconforming elements Q6 and QM6, are further modified to formulate a nonconforming axisymmetric element which satisfies both patch test condition for convergence and the condition for suppression of spurious shear stresses.

ELEMENT FORMULATION

1. Nonconforming axisymmetric finite element formulation

The following principle of minimum potential energy was applied to formulate displacement finite elements,

$$\prod_p = \int_v \frac{1}{2}\boldsymbol{\varepsilon}^T C\boldsymbol{\varepsilon} dv = \min. \qquad (1)$$

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where v is the element volume, C is material matrix and the strain vector. In isoparametric element formulation, we assume

$$U = U_q + U = N_q + M \quad (2a)$$

$$= [N, M]\{q, \lambda\}^T = (\text{compatible + incompatible) element displacement vector} \quad (2b)$$

where N *and* M are compatible and nonconforming displacement shape function matrices, q and λ are vectors of nodal displacements and internal displacement parameters respectively. Here N is also the interpolation function matrix for coordinates (r, z). The strain vector $\varepsilon = DU$ in terms of displacement vector $U = \{u, w\}^T$ is given as

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \gamma_{rz} \\ \varepsilon_\theta \end{Bmatrix} = DU = \begin{bmatrix} \partial/\partial r & 0 \\ 0 & \partial/\partial z \\ \partial/\partial z & \partial/\partial r \\ 1/r & 0 \end{bmatrix} \begin{Bmatrix} U \\ W \end{Bmatrix} \quad (3)$$

Then the resultant nonconforming element formulation is given as follows:

$$\begin{aligned} \varepsilon &= \varepsilon_q + \varepsilon_\lambda = [B_q, B_\lambda]\{q\ \lambda\}^T \\ &= (DN)q + (DM)\lambda \\ &= [B_q, B_\lambda]\{q\ \lambda\}^T \\ &= (1/J)F_q W_q^o q + (1/J)F_\lambda W_\lambda^o \lambda \end{aligned} \quad (4)$$

$$\begin{aligned} K_e &= \int_v [B_q, B_\lambda]^T C [B_q, B_\lambda] dv \\ &= \begin{bmatrix} \int_v B_q^T C B_q dv & \int_v B_q^T C B_\lambda dv \\ \int_v B_\lambda^T C B_q dv & \int_v B_\lambda^T C B_\lambda dv \end{bmatrix} = \begin{bmatrix} K_{qq} & K_{q\lambda} \\ K_{q\lambda}^T & K_{\lambda\lambda} \end{bmatrix} \end{aligned}$$

$$= \text{element stiffness matrix} \quad (5)$$

$$\sigma = C\varepsilon_q = CB_q q = \text{element stress vector} \quad (6)$$

where $J = J(\xi, \eta) = \det[J(\xi, \eta)] =$ Jacobi determinant in terms of isoparametric coordinates (ξ, η), and $(1/J)F_q$ and $(1/J)F_\lambda$ are resulting strain shape function matrices in terms of natural coordinates (ξ, η), W_q^o and W_λ^o are constant matrices in terms of nodal coordinates (r_i, z_i). $\gamma_q (= W_q^0 q)$ and $\gamma_\lambda (= W_\lambda^o \lambda)$ can be viewed as generalized compatible and incompatible strain vector parameters as in the assumed stress/strain element methods. The corresponding explicit complete expressions can be found in the Appendix of Reference(Zhang et al., 1997) for several 4-node isoparametric elements.

2. Explicit formulation of wilson nonconforming element functions

For an isoparametric nonconforming element (Fig. 1), we have the following element displacements and coordinates,

$$U_\lambda = \begin{Bmatrix} u_\lambda \\ w_\lambda \end{Bmatrix} = \begin{bmatrix} m_2 & 0 \\ 0 & m_2 \end{bmatrix} \{\lambda_1 \lambda_2 \lambda_3 \lambda_4\}^T$$

$$= \text{nonconform. displace. vector} \quad (7)$$

$$\begin{aligned} U_q &= Nq = \{u_q\ w_q\}^T = \sum_{i=1}^{4} N_i(\xi, \eta)\{u_i\ w_i\}^T \\ &= \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \end{bmatrix} \{1 \xi \eta \xi \eta\}^T \end{aligned} \quad (8)$$

$$\begin{aligned} \begin{Bmatrix} r \\ z \end{Bmatrix} &= \sum N_i(\xi, \eta) \begin{Bmatrix} r_i \\ z_i \end{Bmatrix} \\ &= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} \{1 \xi \eta \xi \eta\}^T \end{aligned} \quad (9)$$

where

$$N_i = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta),$$

$$\xi_i = \pm 1, \eta_i = \pm 1, -1 \leq \xi, \eta \leq +1, i = 1, 2, 3, 4 \quad (10)$$

$$\alpha = \{\alpha_1, \cdots, \alpha_8\}^T = \text{diag}[L, L]q \quad (11)$$

$$q = \{u_1, \cdots, u_4\quad w_1, \cdots, w_4\}^T \quad (12)$$

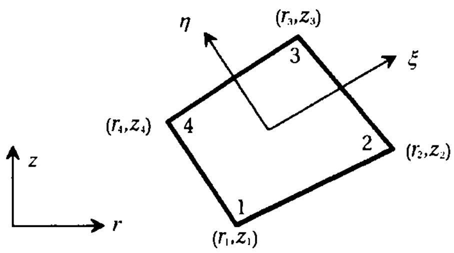

Fig. 1 Isoparametric axisymmetric quadrilateral element

$$L = \frac{1}{4} \begin{bmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & -1 \\ -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 \end{bmatrix} \quad (13)$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{bmatrix} = L \begin{bmatrix} r_1 & z_1 \\ r_2 & z_2 \\ r_3 & z_3 \\ r_4 & z_4 \end{bmatrix} \quad (14)$$

For a constant Jacobian element, $a_4 = b_4 = 0$; for the rectangular element with sides parallel to physical coordinates r and z, $a_3 = b_2 = a_4 = b_4 = 0$; and an element with diagonals parallel to physical coordinates (r and z) yields $a_2^2 - a_3^2 = 0$ and $b_2^2 - b_3^2 = 0$.

The resulting element strain vector can be obtained by differentiating the displacement vector using the following differential operator

$$\begin{Bmatrix} \partial/\partial r \\ \partial/\partial z \end{Bmatrix} = [\boldsymbol{J}]^{-1} \begin{Bmatrix} \partial/\partial \xi \\ \partial/\partial \eta \end{Bmatrix} \quad (15a)$$

$$[\boldsymbol{J}]^{-1} = \frac{[\boldsymbol{J}]^*}{J} = \frac{1}{J}\begin{bmatrix} \partial z/\partial \eta & \partial z/\partial \xi \\ -\partial r/\partial \eta & \partial r/\partial \xi \end{bmatrix}$$
$$= \frac{1}{J}\begin{bmatrix} b_3 + b_4\xi & -(b_2 + b_4\eta) \\ -(a_3 + a_4\xi) & a_2 + a_4\eta \end{bmatrix} \quad (15b)$$

$$J = \det[\boldsymbol{J}] = a + b\xi + c\eta = (a_2b_3 - a_3b_2) + (a_2b_4 - a_4b_2)\xi + (a_4b_3 - a_3b_4)\eta \quad (15c)$$

According to Reference(Zhang, 1991), the element (strain) functions can be expressed in the following form by splitting the strains into function and constant parts for convenient analysis:

$$\varepsilon = \boldsymbol{DU} = (\boldsymbol{DN})\boldsymbol{q} + (\boldsymbol{DM})\boldsymbol{\lambda}$$
$$= (1/J)\boldsymbol{F}_q\boldsymbol{W}_q^0\boldsymbol{q} + (1/J)\boldsymbol{F}_\lambda\boldsymbol{W}_\lambda^0\boldsymbol{\lambda} \quad (16a)$$

where $(1/J)\boldsymbol{F}_q$ and $(1/J)\boldsymbol{F}_\lambda$ are resulting strain shape function matrices in terms of natural coordinates (ξ, η), $\boldsymbol{W}_q^0$ and $\boldsymbol{W}_\lambda^0$ are constant matrices in terms of nodal coordinates (r_i, z_i). Together with the compatible nodal displacements $\boldsymbol{q}$ and incompatible disparameters λ, respectively, γ_q ($= \boldsymbol{W}_q^0\boldsymbol{q}$) and γ_λ ($= \boldsymbol{W}_\lambda^0\lambda$) can be viewed as generalized compatible and incompatible strain vector parameters as in the assumed stress/strain element methods. The explicit expressions for these matrices of isoparametric 4-node elements can be found in Reference(Zhang et al., 1997) . Here the concerned matrices for internal nonconforming displacements are given as follows:

$$\boldsymbol{F}_\lambda = \text{diag}[\boldsymbol{F}_\lambda^0, \boldsymbol{F}_\lambda^0, \boldsymbol{F}_\lambda^0, \boldsymbol{F}_\lambda^\theta] \quad (16b)$$

$$\boldsymbol{F}_\lambda^\theta = \frac{J}{r}\text{diag}[\boldsymbol{m}_1 \quad \boldsymbol{m}_2] \quad (16c)$$

$$\boldsymbol{W}_\lambda^0 = 2\begin{bmatrix} \boldsymbol{B}_0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_0 \\ \boldsymbol{A}_0 & \boldsymbol{B}_0 \\ \boldsymbol{C}_0 & \boldsymbol{0} \end{bmatrix} \quad (16d)$$

Axi-Q4:

$$\boldsymbol{m}_1 = \boldsymbol{m}_2 = \boldsymbol{0}, \quad \boldsymbol{F}_\lambda^0 = \boldsymbol{0} \quad (17)$$

Axi-Q6(Wilson et al., 1973):

$$\boldsymbol{m}_1 = \boldsymbol{m}_2 = [1 - \xi^2 \quad 1 - \eta^2],$$
$$\boldsymbol{F}_\lambda^0 = [\xi \quad \eta \quad \xi^2 \quad \eta^2] \quad (18a)$$

$$\boldsymbol{F}_\lambda^\theta = \left[\frac{1}{r}(1 - \xi^2) \quad \frac{1}{r}(1 - \eta^2)\right] \quad (18b)$$

$$\boldsymbol{A}_0 = \begin{bmatrix} a_4 & 0 & a_4 & 0 \\ 0 & -a_2 & 0 & -a_4 \end{bmatrix}^T,$$
$$\boldsymbol{B}_0 = \begin{bmatrix} -b_3 & 0 & -b_4 & 0 \\ 0 & b_2 & 0 & b_4 \end{bmatrix}^T \quad (18c)$$

$$\boldsymbol{C}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (18d)$$

Axi-QM6(Taylor et al., 1986):

$$\boldsymbol{m}_1 = \boldsymbol{m}_2 = [1 - \xi^2 \; 1 - \eta^2], \boldsymbol{F}_\lambda^0 = [\xi \quad \eta] \quad (19a)$$

$$\boldsymbol{F}_\lambda^\theta = \left[\frac{J}{r}(1 - \xi^2) \quad \frac{1}{r}(1 - \eta^2)\right] \quad (19b)$$

$$\boldsymbol{A}_0 = \begin{bmatrix} a_3 & 0 \\ 0 & -a_2 \end{bmatrix}^T, \boldsymbol{B}_0 = \begin{bmatrix} -b_3 & 0 \\ 0 & b_2 \end{bmatrix}^T \quad (19c)$$

$$\boldsymbol{C}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (19d)$$

Unlike QM6, the corresponding axisymmetric nonconforming elements Axi-Q6 and Axi-QM6 cannot pass the patch test(Taylor et al., 1976; Wilson et al., 1973).

IMPLEMENTATION OF THE PATCH TEST CONDITION

The constant strain/stress patch test condition(Taylor et al., 1986; Wilson et al., 1973) is

$$\int_v \boldsymbol{B}_\lambda dv = 0 \quad (20)$$

By use of the previously given explicit expressions in Eq. (16a) for resulting element strains as in Reference (Zhang et al., 1997), we can rewrite the PTC into another form:

$$\int_v \boldsymbol{B}_\lambda dv = \int_v \frac{1}{J}\boldsymbol{F}_\lambda \boldsymbol{W}_\lambda^0 dv = 2\pi\int_{-1}^{+1}\int_{-1}^{+1}\boldsymbol{F}_\lambda \boldsymbol{W}_\lambda^0 r d\xi d\eta = \boldsymbol{0} \quad (21a)$$

or

$$\int_{-1}^{+1}\int_{-1}^{+1}\boldsymbol{F}_\lambda r d\xi d\eta \boldsymbol{W}_\lambda^0 = \boldsymbol{0} \quad (21b)$$

where $\boldsymbol{W}_\lambda^0$ is constant matrix given by eq. (16d).

To enforce the above patch test condition (PTC) for plane elements, one alternative is to make some terms zero in $\boldsymbol{W}_\lambda^0$ to eliminate the violating function terms in $\boldsymbol{F}_\lambda$. For example, by adopting constant Jacobian (adjoint) to set $a_4 = b_4 = 0$ in $\boldsymbol{W}_\lambda^0$ of Wilson element Q6, the violating terms ξ^2 and η^2 in corresponding $\boldsymbol{F}_\lambda$ are consequently removed and an improved element has been successfully formulated, which is well known as QM6 satisfying the patch test.

For axisymmetric element or other arbitrary nonconforming displacements, this constant Jacobian scheme becomes invalid (Taylor et al., 1976); i.e., Axi-QM6 does not pass PTC with the resulting incompatible strains.

If we keep $\boldsymbol{W}_\lambda^0$ intact and adopt the following procedure to enforce the PTC,

$$\int_{-1}^{+1}\int_{-1}^{+1}\boldsymbol{F}_\lambda r d\xi d\eta = \boldsymbol{0} \quad (22)$$

then this alternative approach will be general and valid for all the cases. Here r and $\boldsymbol{F}_\lambda$ are given by Eqs. (11) and (16a), respectively.

We have two alternative schemes to implement this PTC.

(1) Simply omit those terms such as and (violating the PTC) in $\boldsymbol{F}_\lambda$.

(2) Add some modifying constants to the violating terms in $\boldsymbol{F}_\lambda$, so that Eq. (22) satisfies.

By using scheme (1), deleting or setting the violating terms ξ^2 and η^2 to be zero in $\boldsymbol{F}_\lambda$ of Q6, QM6 passing PTC is immediately obtained. This scheme is equivalent to the constant Jacobian 'crime' scheme (Taylor et al., 1976), but cannot be applied to the axisymmetric case; otherwise, the Wilson axisymmetric element AQ6 will turn out to be axisymmetric conforming element AQ4.

Here we try to apply the scheme (2) to Axi-Q6 and Axi-QM6 violating PTC to formulate another axisymmetric element AQM6 satisfying PTC in Eqs. (22) and (20), by adopting the following new internal strain shape functions:

$$\boldsymbol{F}_\lambda = \mathrm{diag}[\boldsymbol{F}_\lambda^0, \boldsymbol{F}_\lambda^0, \boldsymbol{F}_\lambda^0, \boldsymbol{F}_\lambda^\theta] \quad (23a)$$

where

$$\boldsymbol{F}_\lambda = \left[\left(\xi - \frac{a_2}{3a_1}\right)\left(\eta - \frac{a_3}{3a_1}\right)\left(\xi^2 - \frac{1}{3}\right)\left(\eta^2 - \frac{1}{3}\right)\right],$$

corresponding to Axi-Q6

or $\left[\left(\xi - \dfrac{a_2}{3a_1}\right)\left(\eta - \dfrac{a_3}{3a_1}\right)\right]$, corresponding to Axi-QM6 (23b)

$$\boldsymbol{F}_\lambda^\theta = \left[\frac{1}{r}\left(\frac{1}{3} - \eta^2\right)\right] \quad (23c)$$

It will be shown that when the optimal 2 × 2 Gauss integration is adopted, the derived axisymmetric element AQM6 gives the same numerical results, whether it is derived from Q6 or QM6.

ANALYSIS OF AQM6

The analysis will be made with respect to satisfaction of the patch test condition, the condition for suppressing spurious shear stress and the element rank condition for stability.

It can be readily verified that AQM6 satisfies the patch test condition in Eqs. (20) – (22).

However, it was pointed out that the coupling between $\varepsilon_\lambda^\theta$ and γ_λ^{rz} in nonconforming axisymmetric element generation may trigger spurious shear stress τ_{rz} when there should be no such shear stress. Additional special treatment for traditional Axi-Q6 and Axi-QM6 is needed to eliminate the spurious shear stress by removing all those terms related to τ_{rz} and obtaining the shear stress at the center of the elements only (Cook, 1981).

Numerical test shows AQM6 in this paper has no coupling between $\varepsilon_\lambda^\theta / u_\lambda$ and γ_{rz} ($= \gamma_q^{rz} + \gamma_\lambda^{rz}$), and $\varepsilon_\lambda^\theta / u_\lambda$ will not contribute to γ_{rz}. Therefore AQM6 based on Eq(23c) satisfies numerically the decoupling condition for suppression of spurious shear stress. We find this is because the best 2 × 2 Gauss integration for these axisymmetric elements is carried out at Gauss points $(\xi, \eta)_{\text{Gauss}} = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ in element stiffness generation and all the $\left(\frac{1}{3} - \xi^2\right)$ and $\left(\frac{1}{3} - \eta^2\right)$ in $\boldsymbol{\varepsilon}_\lambda^\theta$ and other related terms ($\boldsymbol{K}_{q\lambda}$ and $\boldsymbol{K}_{\lambda\lambda}$) of AQM6, will eventually vanish in element stiffness computation.

Consequently, we can omit the term $\boldsymbol{F}_\lambda^\theta$ in Eq. (23c) and have the following reasonably simplified formulation for AQM6:

$$\varepsilon_\lambda = \frac{1}{J}\boldsymbol{F}_\lambda \boldsymbol{W}_\lambda^0 \boldsymbol{\lambda} \tag{24a}$$

where

$$\boldsymbol{F}_\lambda = \text{diag}[\boldsymbol{F}_\lambda^0, \boldsymbol{F}_\lambda^0, \boldsymbol{F}_\lambda^0] \tag{24b}$$

$$\boldsymbol{F}_\lambda^0 = \left[\left(\xi - \frac{a_2}{3a_1}\right)\left(\eta - \frac{a_3}{3a_1}\right)\right] \tag{24c}$$

and $\boldsymbol{W}_\lambda^0$ is given by Eqs. (19c) and (16d) without $\boldsymbol{C}_0$,

$$\boldsymbol{W}_\lambda^0 = 2\begin{bmatrix} \boldsymbol{B}_0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_0 \\ \boldsymbol{A}_0 & \boldsymbol{B}_0 \end{bmatrix} \tag{24d}$$

which is actually the same as that of QM6.

Like Axi-Q6 and Axi-QM6, it can be analytically verified that AQM6 satisfies the element rank conditions (Zhang, 1993) for stability in the case of both regular and distorted element shapes.

The easiest procedure to verify element rank is to check some crucial element configurations, such as the rectangular element ($a_4 = b_4 = a_3 = b_2 = 0$) with sides parallel to physical coordinates r and z, and the element ($a_2^2 - a_3^2 = 0$ and $b_2^2 - b_3^2 = 0$) with both element diagonals parallel to physical coordinates r and z, where the geometric parameters a_i and b_j are given by Eq. (14). These two element configurations most likely cause element rank deficiency (Zhang, 1993).

NUMERICAL VERIFICATION AND TEST

The numerical test problems (Figs. 2 and 3) are included to investigate the proposed nonconforming axisymmetric element AQM6, the axisymmetric elements Axi-Q4, Axi-Q6 and Axi-QM6 corresponding to conforming element Q4, nonconforming Wilson element Q6 and its modified QM6 respectively, in respect to patch test (for both regular and irregular mesh), locking test (for almost incompressible materials) and computation of elastic limit loads.

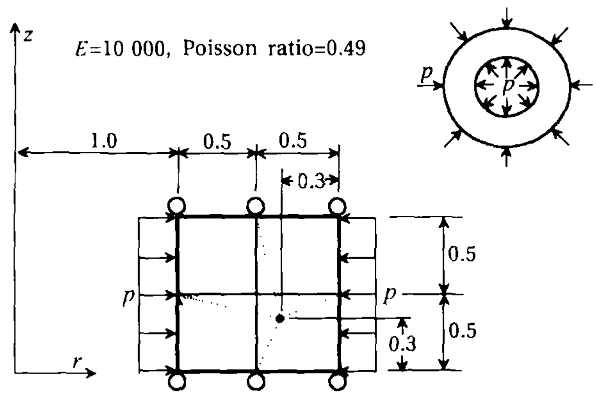

Fig. 2 Axial patch test problems and meshes
Case 1-Regular mesh Case 2-Irregular mesh (dashed line)

Table 1 Axial patch test (PT) and spurious shear stress test (SSST) results (Fig. 2)

Elements		Axi-Q4	Axi-Q6	Axi-QM6	AQM6	Exact
u_r	case 1	-3.31	-3.28	-3.28	-3.31	-3.31
	case 2	-3.31	-3.43	-3.28	-3.31	
σ_r	case 1	-636.6	No Const.	No Const.	-636.6	-636.60
	case 2	-636.6			-636.6	
τ_{rz}	case 1	0.0	No Const.	No Constnt.	0.0	0.0
	case 2	0.0			0.0	
PT		Pass	Fail	Fail	Pass	-
SSST		Pass	Fail	Fail	Pass	-

Table 1 shows that unlike the compatible element Q4 and the successful plane nonconforming element QM6, the corresponding axisymmetric elements Axi-Q6 and Axi-QM6 cannot pass the patch test (Wilson et al., 1973; Taylor et al., 1976) or satisfy the spurious shear stress test (Cook, 1981; Wu et al., 1987), even for a constant Jacobian rectangular mesh. Like compatible element Axi-Q4, the AQM6 obtained in this paper by the new PTC implemental method passes the patch test and the spurious shear stress test.

The second test problem is to investigate the element performance in relieving shear locking. For an axisymmetric problem with a 5-element mesh in Fig.3, Axi-Q4 experiences severe locking in displacement computation for an almost incompressible material when Poisson ratio tends to 0.5 (Fig.4).

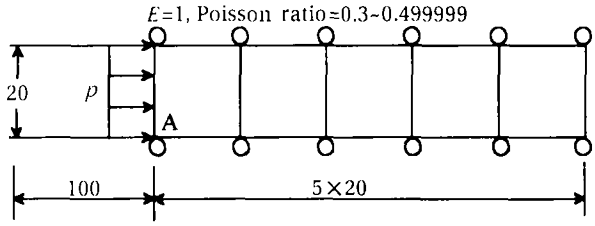

Fig.3 Axisymmetric test problem for locking test

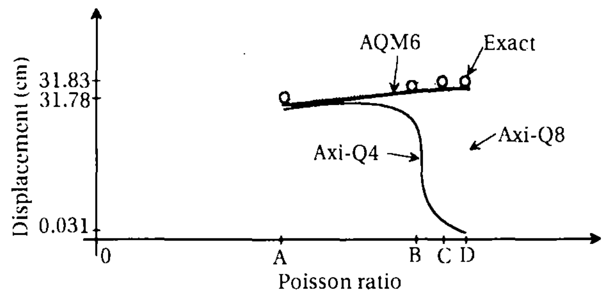

Fig.4 Radius displacement vs. Poisson ratio for an almost imcompressible material

A = 0.3; B = 0.49; C = 0.4999; D = 0.499999

In Fig. 4, the result for AQM6 is obtained without static condensation for incompatible parameters in element stiffness computation; otherwise the result will be slightly worse.

We may further adopt the test problem in Fig.3 for ideal elasto-plastic analysis with Von-Mises yield criterion, yield stress σ_y = 6000 kg/cm^2, $E = 10^2$ kg/cm^2 and $\mu = 0.49$. The results in Fig.5 show that AQM6 can provide better representation for ideal elasto-plasticity than Axi-Q4, especially in computation of plastic limit loads. According to References (Desai et al., 1997; Desai et al., 1998; Zhang et al., 2000), AQM6 can be applied to adaptive finite element analysis of highly nonlinear softening deformations and high-frequency wave propagation in linear piezoelectric materials.

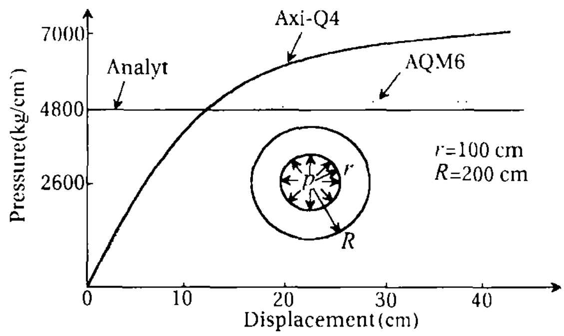

Fig.5 Ideal elasto-plastic pressure-radius displacement curve for a cylinder with uniform internal pressure

CONCLUDING REMARKS

From the theoretical and numerical analyses, we can see that, for axisymmetric elements, when Wilson incompatible displacements are introduced with the satisfaction of the patch test condition and decoupling condition for suppression of spurious shear stress, element performance can be improved by retaining the compatible axisymmetric element's merits of representing constant stress states with no false shear stress and removing the demerits of compatible axisymmetric element's overstiffness and locking.

By means of the PTC implemental procedure, the obtained axisymmetric Wilson element AQM6 in this paper passed the patch test and satisfied the decoupling condition for suppression of false shear stress. However, more comprehensive numerical comparisons between AQM6 and other existing axisymmetric elements are needed with respect to patch test, spurious shear stress, accuracy and efficiency.

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