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MIXED H_2/l_1 OPTIMIZATION PROBLEMS FOR SISO DISCRETE TIME CONTROL SYSTEMS*

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Abstract: One purpose of this work is to establish the nominal description of mixed H_2/l_1 optimization problems evolving from mixed H_2/l_1 control problems for SISO discrete time systems. Some assumptions on mixed H_2/l_1 optimization problems are made. Another purpose of this work is to study the structure of the closure of feasible region for mixed H_2/l_1 optimization problems. The feasible region is the set of a map of a free parameter which is rational stable and satisfies some constraints. It is shown that the closure is exactly the set of the same map, where the free parameter is stable and satisfies the same constraints. It is convenient to describe mixed H_2/l_1 optimization problems with a stable free parameter. For mixed H_2/l_1 optimization problems with stable free parameter, the existence and uniqueness of the solution can be easily obtained.

Key words: l_1 control, H_2 control, discrete time systems, closure
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INTRODUCTION

The design of controllers satisfying mixed performance criteria, such as mixed H_2/H_∞ problem (Kaminer et al., 1993), mixed l_1/H_∞ problem (Sznaier et al., 1996), mixed l_1/H_2 problem (Salapaka et al., 1995), have recently been the focus of researchers. Voulgaris (1995) introduced mixed H_2/l_1 optimization problems of minimizing the H_2-norm of the closed loop map while its l_1-norm is maintained at a prescribed level. Such a design would provide good rms-behavior to stochastic disturbances and also would exhibit acceptable behavior to deterministic and persistently exciting inputs. Voulgaris (1995) showed that the optimal solution of the mixed H_2/l_1 optimization problem is finite impulse response and uniqueness in the nontrivial case. A finite step procedure is given for the construction of the exact solution. Recently, much effort has been focused to developing a mixed H_2/l_1 optimization method. Wu et al. (1996) introduced a scalar discrete time general mixed H_2/l_1 optimization problem of minimizing the H_2-norm of a closed loop map while maintaining the l_1-norm of another closed loop map at a prescribed level. The motivation of general mixed H_2/l_1 optimization problems is to design controllers that provide good nominal rms-behavior as well as guaranteed stability robustness to the unmodeled dynamics of the bounded l_∞-induced norm.

The problem studied by Voulgaris (1995) is a special case of the problem studied by Wu et al. (1996). For the general mixed H_2/l_1 optimization problem developed by Wu et al. (1996), there are no research results on the existence and uniqueness of its solution. This paper deals with this point. One purpose of this work is to establish the nominal description of general mixed H_2/l_1 optimization problems evolving from mixed H_2/l_1 control problems for SISO discrete time systems. Some assumptions on general mixed H_2/l_1 optimization problems are made. Another purpose of this work is to study the structure of the closure of the feasible region for general mixed H_2/l_1 optimization problems. The

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feasible region is the set of a map of a free parameter which is rational stable and satisfies some constraints. It is shown that the closure is exactly the set of the same map, where the free parameter is stable and satisfies the same constraints. It is convenient to use a stable free parameter to describe general mixed H_2/l_1 optimization problems. For general mixed H_2/l_1 optimization problems with stable free parameter, the existence and uniqueness of the solution can be easily obtained.

NOTATION AND MATHEMATICAL PRELIMINARIES

Let R denote the field of real numbers, R^m denote the m-dimensional real vectors, C denote the field of complex numbers, $\bar{D}$ denote the closed unit disk in C, ∂D denote the unit circle in C, N denote the nonnegative integers. A causal SISO (Single Input Single Output) LSI (Linear Shift Invariant) transfer function $\hat{G}$ can be described as

$$\hat{G} = G(0) + G(1)\lambda + G(2)\lambda^2 + \cdots, \quad G(k) \in \mathrm{R}, \ \forall k \in \mathrm{N}.$$

As $\hat{G}$ can be represented uniquely by its impulse response sequence $[G(0), G(1), G(2), \cdots]^{\mathrm{T}}$, its impulse response sequence is not differentiated in notation throughout this paper. Define

$$l_e = \{\hat{G} \mid \hat{G} = G(0) + G(1)\lambda + G(2)\lambda^2 + \cdots, \ G(k) \in \mathrm{R}, \ \forall k \in \mathrm{N}\},$$

$$l_1 = \left\{\hat{G} \in l_e \mid \sum_{k=0}^{\infty} |G(k)| < \infty\right\},$$

$$l_2 = \left\{\hat{G} \in l_e \mid \sum_{k=0}^{\infty} (G(k))^2 < \infty\right\}.$$

For any $\hat{G} \in l_1$, the l_1-norm of $\hat{G}$ is given by

$$\|\hat{G}\|_1 = \sum_{k=0}^{\infty} | G(k) | \qquad (1)$$

It is easy to verify that $\forall \hat{G}_1, \hat{G}_2 \in l_1, \hat{G}_1\hat{G}_2 \in l_1$ and $\|\hat{G}_1\hat{G}_2\|_1 \le \|\hat{G}_1\|_1 \|\hat{G}_2\|_1$. For any $\hat{G} \in l_2$, the l_2-norm of $\hat{G}$ is

$$\|\hat{G}\|_2 = \sqrt{\sum_{k=0}^{\infty} (G(k))^2} \qquad (2)$$

$\|\hat{G}\|_2$ is also the H_2-norm of $\hat{G}$. Notice that $l_1 \subset l_2$ and $\|\hat{G}\|_2 \le \|\hat{G}\|_1$. Define

$$Rl_e = \{\hat{G} \in l_e \mid \hat{G} \text{ is a rational function of } \lambda\},$$
$$Rl_1 = Rl_e \cap l_1$$

The following lemma is well known.

Lemma 2.1 (Vidyasagar, 1985): For $\hat{G} \in Rl_e$, $\hat{G} \in Rl_1$ if and only if the poles of $\hat{G}$ are outside $\bar{D}$ or $\hat{G}$ has no poles (i.e. $\hat{G}$ is a polynomial).

Given an admissible plant $\hat{P}$ as

$$\begin{cases} y = \hat{P}_{11} u + \hat{P}_{12} w_1 + \hat{P}_{13} w_2 \\ z_1 = \hat{P}_{21} u + \hat{P}_{22} w_1 + \hat{P}_{23} w_2 \\ z_2 = \hat{P}_{31} u + \hat{P}_{32} w_1 + \hat{P}_{33} w_2 \end{cases} \qquad (3)$$

where $\hat{P}_{ij} \in Rl_e$, $\forall i \in \{1, 2, 3\}$, $j \in \{1, 2, 3\}$; w_1 and w_2 are single exogenous inputs, z_1 and z_2 are single regulated outputs, u is the single control input, and y is the single measured output. $\hat{P}$ is assumed to be strictly causal and stabilizable. Connect a feedback controller $\hat{C}$ from y to u. Let $\hat{\Psi}$ denote the closed loop transfer function between w_1 and z_1, $\hat{\Phi}$ denote the closed loop transfer function between w_2 and z_2. Given a constant γ, the objective is to find $\hat{C} \in Rl_e$ which stabilizes $\hat{P}$ and minimizes $\|\hat{\Phi}\|_2$ subject to $\|\hat{\Psi}\|_1 \le \gamma$. A compensator that solves this mixed H_2/l_1 problem will ensure that the quadratic cost criterion from w_2 to z_2 is optimal, and that the closed loop system is robustly stable to any finite gain stable perturbation $\hat{\Delta}$, interconnected to the system by $w_1 = \hat{\Delta} z_1$, such that $\|\hat{\Delta}\|_1 < \frac{1}{\gamma}$.

Youla parametrization of all stabilizing compensators (Francis, 1987) yields

$$\hat{\Psi} = \hat{P}_{22} + \hat{P}_{21}\hat{C}(1 - \hat{P}_{11}\hat{C})^{-1}\hat{P}_{12} = \hat{T}_1 - \hat{Q}\hat{V}_1 \qquad (4)$$

$$\hat{\Phi} = \hat{P}_{33} + \hat{P}_{31}\hat{C}(1 - \hat{P}_{11}\hat{C})^{-1}\hat{P}_{13} = \hat{T}_2 - \hat{Q}\hat{V}_2 \qquad (5)$$

where $\hat{T}_1, \hat{T}_2, \hat{V}_1, \hat{V}_2 \in Rl_1$ are fixed SISO maps that depend on plant $\hat{P}$, and $\hat{Q} \in Rl_1$ is a free SISO parameter. Define

$$\gamma_0 = \inf_{\hat{Q} \in Rl_1} \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \qquad (6)$$

Obviously, there exist $\hat{C}$ such that $\|\hat{\Psi}\|_1 \le \gamma$

when $\gamma > \gamma_0$. This paper is based throughout on the following assumption.

Assumption 2.1: $\gamma \in (\gamma_0, \infty)$.

Taking into account the nontrivial case, assume $\hat{V}_1 \neq 0$ and $\hat{V}_2 \neq 0$. Thus $\hat{V}_1, \hat{V}_2 \in Rl_1$ can be described uniquely as

$$\hat{V}_1 = \frac{b_1(0) + b_1(1)\lambda + \cdots + b_1(n_1)\lambda^{n_1}}{1 + a_1(1)\lambda + \cdots + a_1(m_1)\lambda^{m_1}} = \frac{\hat{N}_1}{\hat{M}_1} \tag{7}$$

$$\hat{V}_2 = \frac{b_2(0) + b_2(1)\lambda + \cdots + b_2(n_2)\lambda^{n_2}}{1 + a_2(1)\lambda + \cdots + a_2(m_2)\lambda^{m_2}} = \frac{\hat{N}_2}{\hat{M}_2} \tag{8}$$

where $a_1(m_1) \neq 0$, $b_1(n_1) \neq 0$, $a_2(m_2) \neq 0$, $b_2(n_2) \neq 0$. Then we can rewrite

$$\inf_{\hat{Q} \in Rl_1} \| \hat{T}_2 - \hat{Q}\hat{V}_2 \|_2$$

$$\text{s.t.} \quad \| \hat{T}_1 - \hat{Q}\hat{V}_1 \|_1 \leq \gamma \tag{9}$$

as

$$\inf_{\hat{Q} \in Rl_1} \left\| \hat{T}_2 - \left(b_2(n_2)a_1(m_1) \frac{\hat{Q}}{\hat{M}_1\hat{M}_2} \right) \times \left(\frac{1}{b_2(n_2)a_1(m_1)} \hat{N}_2\hat{M}_1 \right) \right\|_2$$

$$\text{s.t.} \quad \left\| \frac{b_2(n_2)a_1(m_1)}{b_1(n_1)a_2(m_2)} \hat{T}_1 - \left(b_2(n_2)a_1(m_1) \frac{\hat{Q}}{\hat{M}_1\hat{M}_2} \right) \times \left(\frac{1}{b_1(n_1)a_2(m_2)} \hat{N}_1\hat{M}_2 \right) \right\|_1 \leq \gamma \left| \frac{b_2(n_2)a_1(m_1)}{b_1(n_1)a_2(m_2)} \right| \tag{10}$$

From Lemma 2.1, $\hat{M}_1\hat{M}_2 \in Rl_1$ and $\frac{1}{\hat{M}_1\hat{M}_2} \in Rl_1$. Clearly, there is a one-to-one correspondence between $\hat{Q}$ and $b_2(n_2)a_1(m_1)\frac{\hat{Q}}{\hat{M}_1\hat{M}_2}$ in Rl_1, and (10) is equal to

$$\inf_{\hat{Q}' \in Rl_1} \| \hat{T}_2' - \hat{Q}'\hat{V}_2' \|_2$$

$$\text{s.t.} \quad \| \hat{T}_1' - \hat{Q}'\hat{V}_1' \|_1 \leq \gamma' \tag{11}$$

where

$$\hat{T}_1' = \frac{b_2(n_2)a_1(m_1)}{b_1(n_1)a_2(m_2)} \hat{T}_1, \quad \hat{T}_2' = \hat{T}_2 \tag{12}$$

$$\hat{V}_1' = \frac{1}{b_1(n_1)a_2(m_2)} \hat{N}_1\hat{M}_2 = V_1'(0) + V_1'(1)\lambda + \cdots + \lambda^{n_1+m_2} \tag{13}$$

$$\hat{V}_2' = \frac{1}{b_2(n_2)a_1(m_1)} \hat{N}_2\hat{M}_1 = V_2'(0) + V_2'(1)\lambda + \cdots + \lambda^{n_2+m_1} \tag{14}$$

$$\gamma' = \gamma \left| \frac{b_2(n_2)a_1(m_1)}{b_1(n_1)a_2(m_2)} \right| \tag{15}$$

Since the common stable zeros (the zeros which are outside $\bar{D}$) of $\hat{V}_1'$ and $\hat{V}_2'$ can be absorbed in $\hat{Q}'$, in (11), without loss of generality, we can assume that $\hat{V}_1'$ and $\hat{V}_2'$ have no common stable zeros.

For $\hat{V}_1'$ and $\hat{V}_2'$, suppose:

$V_1'(0) = V_1'(1) = \cdots = V_1'(\delta_1 - 1) = 0$, $V_1'(\delta_1) \neq 0$ and

$V_2'(0) = V_2'(1) = \cdots = V_2'(\delta_2 - 1) = 0$, $V_2'(\delta_2) \neq 0$, i.e.

$$\hat{V}_1' = V_1'(\delta_1)\lambda^{\delta_1} + V_1'(\delta_1 + 1)\lambda^{\delta_1+1} + \cdots + \lambda^{n_1+m_2} \tag{16}$$

$$\hat{V}_2' = V_2'(\delta_2)\lambda^{\delta_2} + V_2'(\delta_2 + 1)\lambda^{\delta_2+1} + \cdots + \lambda^{n_2+m_1} \tag{17}$$

For $\forall N \in \mathrm{N}$, define the Nth truncation operator $\Gamma_N : l_e \to \mathrm{R}^{N+1}$ as

$$\Gamma_N\hat{G} = G(0) + G(1)\lambda + \cdots + G(N)\lambda^N \tag{18}$$

Then

$$\inf_{\hat{Q}' \in Rl_1} \| \hat{T}_2' - \hat{Q}'\hat{V}_2' \|_2 =$$

$$\inf_{\hat{Q}' \in Rl_1} \| \Gamma_{\delta_2-1}\hat{T}_2' + (\hat{T}_2' - \Gamma_{\delta_2-1}\hat{T}_2') - \hat{Q}'\hat{V}_2' \|_2 =$$

$$\inf_{\hat{Q}' \in Rl_1} \sqrt{\| \Gamma_{\delta_2-1}\hat{T}_2' \|_2^2 + \| (\hat{T}_2' - \Gamma_{\delta_2-1}\hat{T}_2') - \hat{Q}'\hat{V}_2' \|_2^2} =$$

$$\sqrt{\| \Gamma_{\delta_2-1}\hat{T}_2' \|_2^2 + \left(\inf_{\hat{Q}' \in Rl_1} \| (\hat{T}_2' - \Gamma_{\delta_2-1}\hat{T}_2') - \hat{Q}'\hat{V}_2' \|_2 \right)^2} =$$

$$\sqrt{\| \Gamma_{\delta_2-1}\hat{T}_2' \|_2^2 + \left(\inf_{\hat{Q}' \in Rl_1} \| \lambda^{-\delta_2}(\hat{T}_2' - \Gamma_{\delta_2-1}\hat{T}_2') - \hat{Q}'(\lambda^{-\delta_2}\hat{V}_2') \|_2 \right)^2},$$

$$\| \hat{T}_1' - \hat{Q}'\hat{V}_1' \|_1 =$$

$$\|\Gamma_{\delta_1-1}\hat{T}_1' + (\hat{T}_1' - \Gamma_{\delta_1-1}\hat{T}_1') - \hat{Q}'\hat{V}_1'\|_1 =$$
$$\|\Gamma_{\delta_1-1}\hat{T}_1'\|_1 + \|(\hat{T}_1' - \Gamma_{\delta_1-1}\hat{T}_1') - \hat{Q}'\hat{V}_1'\|_1 =$$
$$\|\Gamma_{\delta_1-1}\hat{T}_1'\|_1 + \|\lambda^{-\delta_1}(\hat{T}_1' - \Gamma_{\delta_1-1}\hat{T}_1') - \hat{Q}'(\lambda^{-\delta_1}\hat{V}_1')\|_1 .$$

Hence (9) can be solved by solving the following problem:

$$\inf_{\hat{Q}\in Rl_1}\|\lambda^{-\delta_2}(\hat{T}_2' - \Gamma_{\delta_2-1}\hat{T}_2') - \hat{Q}'(\lambda^{-\delta_2}\hat{V}_2')\|_2$$
$$s.t.\|\lambda^{-\delta_1}(\hat{T}_1' - \Gamma_{\delta_1-1}\hat{T}_1') - \hat{Q}'(\lambda^{-\delta_1}\hat{V}_1')\|_1 \le \gamma' - \|\Gamma_{\delta_1-1}\hat{T}_1'\|_1 \quad (19)$$

This proves that we may, without loss of generality, assume

Assumption 2.2:

$$\hat{V}_1 = V_1(0) + V_1(1)\lambda + \cdots + \lambda^m, V_1(0)\neq 0$$

and

$$\hat{V}_2 = V_2(0) + V_2(1)\lambda + \cdots + \lambda^n,\ V_2(0)\neq 0.$$

Assumption 2.3: $\hat{V}_1$ and $\hat{V}_2$ have no common stable zeros.

For $\hat{V}_1 = V_1(0) + V_1(1)\lambda + \cdots + \lambda^m = \prod_{i=1}^{m}(\lambda - \lambda_i)$, let

$$\Lambda = \{\lambda_1,\cdots,\lambda_m\} \quad (20)$$

denote the roots of $\hat{V}_1$. Similar to the usual rank assumptions made in H_2 optimization and l_1 optimization, we assume that every root of $\hat{N}_1$ in (7) is not on ∂D. Notice that every root of $\hat{M}_2$ in (8) is not on ∂D yet. So this assumption can be described as

Assumption 2.4: $\Lambda \cap \partial D = \Phi$.

Finally, mixed H_2/l_1 optimization problems can be stated as: Given $\hat{T}_1, \hat{T}_2, \hat{V}_1, \hat{V}_2 \in Rl_1$, $\gamma \in R$ satisfying Assumptions 2.1, 2.2, 2.3 and 2.4, find $\hat{Q}\in Rl_1$ such that $\|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2$ is minimized and $\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \le \gamma$.

CLOSURE OF FEASIBLE REGION

Define the feasible region of the mixed H_2/l_1 optimization problem as

$$\xi(\gamma) = \{\hat{\Phi} | \hat{\Phi} = \hat{T}_2 - \hat{Q}\hat{V}_2, \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \le \gamma, \hat{Q} \in Rl_1\}.$$

It is easily seen that $\xi(\gamma) \subset Rl_1 \subset l_1 \subset l_2$ and that mixed H_2/l_1 optimization problems can described as

$$\mu(\gamma) = \inf_{\hat{\Phi}\in\xi(\gamma)}\|\hat{\Phi}\|_2 \quad (21)$$

For mixed H_2/l_1 optimization problems, define

$$\zeta(\gamma) = \{\hat{Q}\in Rl_1 | \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \le \gamma\},$$
$$L(\gamma) = \frac{\|\hat{T}_1\|_1 + \gamma}{\prod_{i=1}^{m}|1 - |\lambda_i||} \quad (22)$$

Proposition 3.1: $\forall \hat{Q}\in\zeta(\gamma), \|\hat{Q}\|_1 \le L(\gamma)$.

Proof: Space C_1 is defined as

$$\{\hat{G} | \hat{G} = G(0) + G(1)\lambda + \cdots\},$$
$$\sum_{k=0}^{\infty}|G(k)| < \infty, G(k) \in \mathrm{C}, \forall k \in \mathrm{N}.$$

For any $\hat{G}\in C_1$, the C_1-norm of $\hat{G}$ is given by $\|\hat{G}\|_1 = \sum_{k=0}^{\infty}|G(k)|$. Obviously, l_1 is a subset of C_1 and C_1-norm in l_1 space is exactly l_1-norm. With C_1-norm and $\|\hat{Q}\|_1 < \infty$, it follows that

$$\|\hat{T}_1\|_1 + \gamma \ge \|\hat{T}_1\|_1 + \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \ge$$
$$\|\hat{Q}\hat{V}_1\|_1 \ge \|\prod_{i=1}^{m}(\lambda - \lambda_i)\hat{Q}\|_1 \ge$$
$$\|-\lambda_1\prod_{i=2}^{m}(\lambda - \lambda_i)\hat{Q} + \lambda\prod_{i=2}^{m}(\lambda - \lambda_i)\hat{Q}\|_1 \ge$$
$$\left|\|\lambda\prod_{i=2}^{m}(\lambda - \lambda_i)\hat{Q}\|_1 - \|-\lambda_1\prod_{i=2}^{m}(\lambda - \lambda_i)\hat{Q}\|_1\right| \ge$$
$$\left|\|\prod_{i=2}^{m}(\lambda - \lambda_i)\hat{Q}\|_1 - |\lambda_1|\|\prod_{i=2}^{m}(\lambda - \lambda_i)\hat{Q}\|_1\right| \ge$$
$$|1 - |\lambda_1||\|\prod_{i=2}^{m}(\lambda - \lambda_i)\hat{Q}\|_1 \ge$$
$$\cdots \ge |1 - |\lambda_1||\,|1 - |\lambda_2||\,\|\prod_{i=3}^{m}(\lambda - \lambda_i)\hat{Q}\|_1 \ge$$
$$\cdots \ge \prod_{i=1}^{m}|1 - |\lambda_i||\,\|\hat{Q}\|_1$$

The above means that $\|\hat{Q}\|_1 \le \frac{\|\hat{T}_1\|_1 + \gamma}{\prod_{i=1}^{m}|1 - |\lambda_i||}$.

The following lemma is the result of l_1 optimization.

Lemma 3.1 (McDonald et al., 1991): Under Assumption 2.4, there exists a $\hat{Q}_0 \in Rl_1$ such that

$$\|\hat{T}_1 - \hat{Q}_0\hat{V}_1\|_1 = \gamma_0 = \inf_{\hat{Q}\in Rl_1}\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \quad (23)$$

Let $\bar{\xi}(\gamma)$ denote the closure of $\xi(\gamma)$ in l_2.

Proposition 3.2: For $\hat{X} \in l_2, \hat{X} \in \bar{\xi}(\gamma)$ if $\hat{X} = \hat{T}_2 - \hat{Q}\hat{V}_2, \hat{Q} \in l_1$ and $\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leq \gamma$.

Proof: $\forall \varepsilon \in (0, 2\|\hat{V}_2\|_1\|\hat{Q} - \hat{Q}_0\|_1)$, let $\alpha = \dfrac{\varepsilon}{2\|\hat{V}_2\|_1\|\hat{Q} - \hat{Q}_0\|_1}$, where $\hat{Q}_0$ is the solution of the l_1 optimization problem shown in Lemma 3.1. Let

$$\hat{Q}' = (1-\alpha)\hat{Q} + \alpha\hat{Q}_0 \in l_1 \tag{24}$$

Choose $N \in \mathrm{N}$ such that

$$\|\hat{Q}' - \Gamma_N\hat{Q}'\|_1 < \min\left(\frac{\varepsilon}{2\|\hat{V}_2\|_1}, \frac{\varepsilon(\gamma - \gamma_0)}{2\|\hat{V}_1\|_1\|\hat{V}_2\|_1\|\hat{Q} - \hat{Q}_0\|_1}\right) \tag{25}$$

Then

$$\begin{aligned}
&\|\hat{T}_1 - (\Gamma_N\hat{Q}')\hat{V}_1\|_1 \leq \\
&\|\hat{T}_1 - \hat{Q}'\hat{V}_1\|_1 + \|(\hat{Q}' - \Gamma_N\hat{Q}')\hat{V}_1\|_1 \leq \\
&(1-\alpha)\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 + \alpha\|\hat{T}_1 - \hat{Q}_0\hat{V}_1\|_1 + \\
&\|\hat{Q}' - \Gamma_N\hat{Q}'\|_1\|\hat{V}_1\|_1 \leq \\
&(1-\alpha)\gamma + \alpha\gamma_0 + \\
&\frac{\varepsilon(\gamma - \gamma_0)}{2\|\hat{V}_1\|_1\|\hat{V}_2\|_1\|\hat{Q} - \hat{Q}_0\|_1}\|\hat{V}_1\|_1 \leq \\
&\gamma - \alpha(\gamma - \gamma_0) + \frac{\varepsilon(\gamma - \gamma_0)}{2\|\hat{V}_2\|_1\|\hat{Q} - \hat{Q}_0\|_1} \leq \\
&\gamma - \frac{\varepsilon(\gamma - \gamma_0)}{2\|\hat{V}_2\|_1\|\hat{Q} - \hat{Q}_0\|_1} + \frac{\varepsilon(\gamma - \gamma_0)}{2\|\hat{V}_2\|_1\|\hat{Q} - \hat{Q}_0\|_1} \leq \gamma,
\end{aligned}$$

and $\Gamma_N\hat{Q}' \in Rl_1$. Thus $\hat{T}_2 - (\Gamma_N\hat{Q}')\hat{V}_2 \in \xi(\gamma)$. However,

$$\begin{aligned}
&\|\hat{X} - (\hat{T}_2 - (\Gamma_N\hat{Q}')\hat{V}_2)\|_2 \leq \\
&\|(\hat{Q} - \Gamma_N\hat{Q}')\hat{V}_2\|_2 \leq \\
&\|(\hat{Q} - \hat{Q}')\hat{V}_2 + (\hat{Q}' - \Gamma_N\hat{Q}')\hat{V}_2\|_2 \leq \\
&\|(\hat{Q} - \hat{Q}')\hat{V}_2\|_2 + \|(\hat{Q}' - \Gamma_N\hat{Q}')\hat{V}_2\|_2 \leq \\
&\|(\hat{Q} - \hat{Q}')\hat{V}_2\|_1 + \|(\hat{Q}' - \Gamma_N\hat{Q}')\hat{V}_2\|_1 \leq \\
&\|\hat{Q} - \hat{Q}'\|_1\|\hat{V}_2\|_1 + \|\hat{Q}' - \Gamma_N\hat{Q}'\|_1\|\hat{V}_2\|_1 < \\
&\|\hat{Q} - ((1-\alpha)\hat{Q} + \alpha\hat{Q}_0)\|_1\|\hat{V}_2\|_1 + \\
&\frac{\varepsilon}{2\|\hat{V}_2\|_1}\|\hat{V}_2\|_1 \leq \\
&\alpha\|\hat{Q} - \hat{Q}_0\|_1\|\hat{V}_2\|_1 + \frac{\varepsilon}{2} \leq \\
&\frac{\varepsilon}{2\|\hat{Q} - \hat{Q}_0\|_1\|\hat{V}_2\|_1}\|\hat{Q} - \hat{Q}_0\|_1\|\hat{V}_2\|_1 + \frac{\varepsilon}{2} \leq \\
&\varepsilon.
\end{aligned}$$

Hence $\hat{X} \in \bar{\xi}(\gamma)$.

Proposition 3.3: For $\hat{X} \in l_2, \hat{X} \notin \bar{\xi}(\gamma)$ if $\hat{Q} = \dfrac{\hat{T}_2 - \hat{X}}{\hat{V}_2} \notin l_1$.

Proof: As $V_2(0) \neq 0$, $\hat{Q} = \dfrac{\hat{T}_2 - \hat{X}}{\hat{V}_2} \in l_e$ which means we can write

$$\hat{X} = \hat{T}_2 - (Q(0) + Q(1)\lambda + \cdots)\hat{V}_2 \tag{26}$$

Choose $N \in \mathrm{N}$ such that $\|\Gamma_N\hat{Q}\|_1 > L(\gamma)$. Denote

$$\mathscr{V}_{2N} = \begin{bmatrix} V_2(0) & & & 0 \\ V_2(1) & V_2(0) & & \\ \cdots & \cdots & \ddots & \\ V_2(N) & V_2(N-1) & \cdots & V_2(0) \end{bmatrix} \tag{27}$$

Note that $\mathscr{V}_{2N}$ is invertible. For

$$\mathscr{V}_{2N}^{-1} = \begin{bmatrix} v_{00} & \cdots & v_{0N} \\ \cdots & \cdots & \cdots \\ v_{N0} & \cdots & v_{NN} \end{bmatrix} \tag{28}$$

define

$$\|\mathscr{V}_{2N}^{-1}\|_1 = \max_{j \in \{0, \cdots, N\}} \sum_{i=0}^{N} |v_{ij}| \tag{29}$$

Let $\varepsilon = \dfrac{\|\Gamma_N\hat{Q}\|_1 - L(\gamma)}{\sqrt{N+1}\,\|\mathscr{V}_{2N}^{-1}\|_1}$ and $B(\hat{X}, \varepsilon) = \{\hat{\Phi} \in l_2 \mid \|\hat{X} - \hat{\Phi}\|_2 < \varepsilon\}$. The Proposition will follow if we can prove that $B(\hat{X}, \varepsilon) \cap \xi(\gamma) = \Phi$. We will show that $B(\hat{X}, \varepsilon) \cap \xi(\gamma) = \Phi$ by contradiction. Suppose that there exists $\hat{\Phi} \in B(\hat{X}, \varepsilon)$ such that $\hat{\Phi} = \hat{T}_2 - \hat{Q}'\hat{V}_2$, $\hat{Q}' \in Rl_1$ and $\|\hat{T}_1 - \hat{Q}'\hat{V}_1\|_1 \leq \gamma$. Then

$$\Gamma_N(\hat{\Phi} - \hat{X}) = \Gamma_N\hat{\Phi} - \Gamma_N\hat{X} = \Gamma_N(\hat{Q}\hat{V}_2) - \Gamma_N(\hat{Q}'\hat{V}_2)$$

$$= \mathscr{V}_{2N}(\Gamma_N\hat{Q}) - \mathscr{V}_{2N}(\Gamma_N\hat{Q}') = \mathscr{V}_{2N}(\Gamma_N\hat{Q} -$$

$\Gamma_N\hat{Q}^{'})$.

It follows that

$$\Gamma_N C\hat{Q}^{'} = \Gamma_N\hat{Q} - \mathscr{T}_{2N}^{-1}(\Gamma_N(\hat{\Phi} - \hat{X})) \quad (30)$$

Thus

$$\|\hat{Q}^{'}\|_1 \geq \|\Gamma_N\hat{Q}^{'}\|_1 \geq$$
$$\|\Gamma_N\hat{Q} - \mathscr{T}_{2N}^{-1}(\Gamma_N(\hat{\Phi} - \hat{X}))\|_1 \geq$$
$$\|\Gamma_N\hat{Q}\|_1 - \|\mathscr{T}_{2N}^{-1}(\Gamma_N(\hat{\Phi} - \hat{X}))\|_1 \geq$$
$$\|\Gamma_N\hat{Q}\|_1 - \|\mathscr{T}_{2N}^{-1}\|_1\|\Gamma_N(\hat{\Phi} - \hat{X})\|_1 \geq$$
$$\|\Gamma_N\hat{Q}\|_1 - \sqrt{N+1}\|\mathscr{T}_{2N}^{-1}\|_1\|\Gamma_N(\hat{\Phi} - \hat{X})\|_2 \geq$$
$$\|\Gamma_N\hat{Q}\|_1 - \sqrt{N+1}\|\mathscr{T}_{2N}^{-1}\|_1\|\hat{\Phi} - \hat{X}\|_2 >$$
$$\|\Gamma_N\hat{Q}\|_1 - \sqrt{N+1}\|\mathscr{T}_{2N}^{-1}\|_1\varepsilon \geq$$
$$\|\Gamma_N\hat{Q}\|_1 - \sqrt{N+1}\|\mathscr{T}_{2N}^{-1}\|_1\frac{\|\Gamma_N\hat{Q}\|_1 - L(\gamma)}{\sqrt{N+1}\|\mathscr{T}_{2N}^{-1}\|_1} \geq$$

$L(\gamma)$.

From Proposition 3.1, it follows that $\|\hat{T}_1 - \hat{Q}^{'}\hat{V}_1\|_1 > \gamma$ which is a contradiction.

Proposition 3.4: For $\hat{X} \in l_2, \hat{X} \notin \xi(\gamma)$ if $\hat{X} = \hat{T}_2 - \hat{Q}\hat{V}_2, \hat{Q} \in l_1$ and $\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 > \gamma$.

Proof: Choose $N \in \mathrm{N}$ such that $\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 > \gamma$. It is easily seen that $\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1) = \Gamma_N\hat{T}_1 - \Gamma_N(\hat{Q}\hat{V}_1) = \Gamma_N\hat{T}_1 - \mathscr{T}_{1N}(\Gamma_N\hat{Q})$, where

$$\mathscr{T}_{1N} = \begin{bmatrix} V_1(0) & & 0 \\ \cdots & \ddots & \\ V_1(N) & \cdots & V_1(0) \end{bmatrix} \quad (31)$$

Similar to (28) and (29), we can get $\|\mathscr{T}_{1N}\|_1$.

Let $\varepsilon = \dfrac{\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 - \gamma}{\sqrt{N+1}\|\mathscr{T}_{1N}\|_1\|\mathscr{T}_{2N}^{-1}\|_1}$. The Proposition will follow if we can prove that $B(\hat{X}, \varepsilon) \cap \xi(\gamma) = \Phi$. We will show that $B(\hat{X}, \varepsilon) \cap \xi(\gamma) = \Phi$ by contradiction. Suppose that there exists $\hat{\Phi} \in B(\hat{X}, \varepsilon)$ such that $\hat{\Phi} = \hat{T}_2 - \hat{Q}^{'}\hat{V}_2, \hat{Q}^{'} \in Rl_1$ and $\|\hat{T}_1 - \hat{Q}^{'}\hat{V}_1\|_1 \leq \gamma$. Then

$$\|\hat{T}_1 - \hat{Q}^{'}\hat{V}_1\|_1 \geq \|\Gamma_N(\hat{T}_1 - \hat{Q}^{'}\hat{V}_1)\|_1 \geq$$
$$\|\Gamma_N\hat{T}_1 - \Gamma_N(\hat{Q}\hat{V}_1) + \Gamma_N(\hat{Q}\hat{V}_1) - \Gamma_N(\hat{Q}^{'}\hat{V}_1)\|_1 \geq$$
$$\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 - \|\mathscr{T}_{1N}(\Gamma_N\hat{Q}) - \mathscr{T}_{1N}(\Gamma_N\hat{Q}^{'})\|_1 \geq$$
$$\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 - \|\mathscr{T}_{1N}(\Gamma_N\hat{Q} - \Gamma_N\hat{Q}^{'})\|_1 \geq$$
$$\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 - \|\mathscr{T}_{1N}\mathscr{T}_{2N}^{-1}(\Gamma_N(\hat{\Phi} - \hat{X}))\|_1 \geq$$
$$\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 - \|\mathscr{T}_{1N}\|_1\|\mathscr{T}_{2N}^{-1}\|_1\|\Gamma_N(\hat{\Phi} - \hat{X})\|_1 \geq$$
$$\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 - \sqrt{N+1}\|\mathscr{T}_{1N}\|_1\|\mathscr{T}_{2N}^{-1}\|_1\|\Gamma_N(\hat{\Phi} - \hat{X})\|_2 \geq$$
$$\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 - \sqrt{N+1}\|\mathscr{T}_{1N}\|_1\|\mathscr{T}_{2N}^{-1}\|_1\|\hat{\Phi} - \hat{X}\|_2 >$$
$$\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 - \sqrt{N+1}\|\mathscr{T}_{1N}\|_1\|\mathscr{T}_{2N}^{-1}\|_1\varepsilon \geq$$
$$\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 - \sqrt{N+1}\|\mathscr{T}_{1N}\|_1\|\mathscr{T}_{2N}^{-1}\|_1 .$$
$$\frac{\|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 - \gamma}{\sqrt{N+1}\|\mathscr{T}_{1N}\|_1\|\mathscr{T}_{2N}^{-1}\|_1} \geq \gamma,$$

which is a contradiction.

From Propositions 3.2, 3.3 and 3.4, we have the following

Proposition 3.5: $\bar{\xi}(\gamma) = \{\hat{\Phi} \in l_2 | \hat{\Phi} = \hat{T}_2 - \hat{Q}\hat{V}_2, \hat{Q} \in l_1, \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leq \gamma\}$.

With regard to the more general case for mixed H_2/l_1 problems, $\hat{T}_1 \in l_1$ does not violate $\|\hat{\Psi}\|_1 < \infty$, and $\hat{T}_2 \in l_2$ does not violate $\|\hat{\Phi}\|_2 < \infty$. Notice that $\inf_{\hat{\Phi} \in \xi(\gamma)}\|\hat{\Phi}\|_2 = \inf_{\hat{\Phi} \in \bar{\xi}(\gamma)}\|\hat{\Phi}\|_2$. From Proposition 3.5, we may extend mixed H_2/l_1 optimization problems in Rl_1 to ones in l_1 as follows: Given $\hat{T}_1 \in l_1$, $\hat{T}_2 \in l_2$, $\hat{V}_1, \hat{V}_2 \in Rl_1, \gamma \in R$ satisfying Assumptions 2.1, 2.2, 2.3 and 2.4, find $\hat{Q} \in l_1$ such that $\|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2$ is minimized and $\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leq \gamma$, i.e.

$$\mu(\gamma) = \inf_{\hat{Q} \in l_1}\|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2$$
$$\text{s.t.} \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leq \gamma \quad (32)$$

Next we will show the existence and uniqueness of the solution of (32).

Lemma 3.2 (Conway, 1990): If E is a Hilbert space with norm $\|\cdot\|$, A is a closed convex nonempty subset of E, then there is a unique point $x^* \in A$ such that $\|x^*\| = \inf_{x \in A}\|x\|$.

It can be verified easily that $\bar{\xi}(\gamma)$ is a closed convex nonempty subset of l_2. Applying Lemma 3.2, it is plain that

Proposition 3.6: For problem (32), There is a unique element $\hat{Q}^*$ in l_1 such that $\|\hat{T}_2 - \hat{Q}^*\hat{V}_2\|_2 = \mu(\gamma)$ and $\|\hat{T}_1 - \hat{Q}^*\hat{V}_1\|_1 \leq \gamma$.

CONCLUSIONS

This paper established the nominal description of mixed H_2/l_1 optimization problems from mixed H_2/l_1 control problems for SISO discrete time systems, and extended mixed H_2/l_1 optimization problems in Rl_1 to ones in l_1. A mixed

H_2/l_1 optimization problem in Rl_1 and the corresponding one in l_1 have the same optimal value. However, there may not be solution for mixed H_2/l_1 optimization problem in Rl_1, while there is an unique solution for mixed H_2/l_1 optimization problem in l_1. Hence mixed H_2/l_1 optimization problems in l_1 are more convenient for study.

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