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A LIMIT RESULT FOR SELF-NORMALIZED RANDOM SUMS*

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Abstract: Suppose $\{X, X_n; n \geq 1\}$ is a sequence i.i.d.r.v. with $EX=0$ and $EX^2<\infty$. Shao (1995) proved a conjecture of Révész (1990): if $P(X=\pm 1)=1/2$, then

$$\lim_{n\to\infty}\max_{0\leq j<n}\max_{1\leq k\leq n-j}\frac{\sum_{i=j+1}^{i=j+k}X_i}{(2k\log n)^{1/2}}=1 \quad \text{a.s.}$$

Furthermore he conjectured that

$$1\leq \lim_{n\to\infty}\max_{0\leq j<n}\max_{1\leq k\leq n-j}\frac{\sum_{i=j+1}^{i=j+k}X_i}{\left\{\sum_{i=j+1}^{i=j+k}X_i^2(2k\log n)\right\}^{1/2}}=K<\infty \quad \text{a.s.}$$

In this paper we prove that if $\sup_{b>0}P(X=b)\geq P(X=0)$ then this conjecture is ture.

Key words: self-normalized, i.i.d. random variables, Chernoff function
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INTRODUCTION

Suppose $\{X, X_n; n\geq 1\}$ is a sequence i.i.d.r.v. Let $S_n=\sum_{i=1}^n X_i$ and $V_n^2=\sum_{i=1}^n X_i^2$. Révész (1990) studied the limit behavior of the sequence

$$L_n=\max_{0\leq j<n}\max_{1\leq k<n-j}k^{-1/2}(S_{j+k}-S_j) \quad (1)$$

and proved that if $P(X=\pm 1)=\frac{1}{2}$ then

$$1\leq \liminf_{n\to\infty}\frac{L_n}{(2\log n)^{1/2}}\leq \limsup_{n\to\infty}\frac{L_n}{(2\log n)^{1/2}}=K<\infty \quad \text{a.s.}, \quad (2)$$

where the exact value of K is unknown (cf. Révész 1990, p171). Révész (1990) conjectured that $K=1$. Shao (1995) corroborated Révész conjecture and proved a general result as follows.

Theorem A. Suppose that $EX=0$, $EX^2=1$ and $Ee^{tX}<\infty$ for some $t>0$. Let $\rho(x)=\inf_{\theta\geq 0}e^{-\theta x}Ee^{\theta X}$ be the Chernoff function of X. Define

$$\alpha(C)=\sup\{x;\rho(x)\geq e^{-1/C}\}(C>0),$$

$$\lambda=\sup_{0<x<\infty}\frac{x^{1/2}\alpha(x)}{\sqrt{2}}.$$

Then $\lambda\geq 1$ and

$$\lim_{n\to\infty}\max_{0\leq j<n}\max_{1\leq k<n-j}\frac{\sum_{i=j+1}^{i=j+k}X_i}{(2k\log n)^{1/2}}=\lambda \quad \text{a.s.} \quad (3)$$

and, $\lambda<\infty$ if and only if $Ee^{tX^2}<\infty$ for some $t>0$.

Zhang (1998) proved some general results on the lag sums of i.i.d.r.v.s, his Theorem 5 extended the Theorem A above.

From Theorem A, we see that if we want to get a finite limit as in (2) for general random variables we must add a very strong moment condition $Ee^{tX^2}<\infty$, which can not be weakened. But in the past several years, many authors studied the so-called self-normalized limit theorems.

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For example, Griffin and Kuelbs (1989) obtained the self-normalized law of iterated logarithm, Csörgö and Shao (1994) studied the self-normalized Erdös-Rényi law of large numbers, Shao (1997) studied self-normalized large deviations. The previous self-normalized limit theorems show that when the normalizing constants in the classical limit theorem are replaced by an appropriate sequence of random variables, a similar result may still hold under less or even without any moment conditions. The significance of the self-normalized limit theorems is obvious. So, one may ask if the following self-normalized result related to (3) is true or not: for i.i.d. random variables $\{X, X_n; n \geq 1\}$, if $EX = 0$ and $EX^2 < \infty$ then

$$1 \leq \lim_{n\to\infty} \max_{0\leq j<n} \max_{1\leq k\leq n-j} \frac{S_{j+k} - S_j}{\{(V_{j+k}^2 - V_j^2)(2\log n)\}^{1/2}} = K < \infty \quad \text{a.s.} \tag{4}$$

This was also a conjecture of Shao (1995).

In this paper we prove that under suitable conditions this conjecture is true.

Theorem 1.

Let $f(x) = \sup_{b\geq 0} \inf_{t\geq 0} Ee^{t(bX - x(X^2+b^2)/2)}$,

$c_0 = 1/\ln(1/P(X=0))$,

$k(c) = \inf\{x \geq 0; f(x) < e^{-1/c}\}$ and

$$\lambda(c) = \begin{cases} k(c) & \text{for} \quad c > c_0 \\ 1 & \text{for} \quad c \in [0, c_0], \end{cases}$$

$$\lambda^* = \sup_{0<c<\infty} \frac{\sqrt{c}\lambda(c)}{\sqrt{2}}.$$

If $EX = 0$, $EX^2 I\{|X| \leq x\}$ slowly varies as $x \to \infty$ and $\sup_{b>0} P(X=b) \geq P(X=0)$ then $\lambda^* < \infty$ *and*

$$1 \leq \lim_{n\to\infty} \max_{0\leq j<n} \max_{1\leq k\leq n-j} \frac{S_{j+k} - S_j}{\{(V_{j+k}^2 - V_j^2)(2\log n)\}^{1/2}} = \lambda^* \quad \text{a.s.} \tag{5}$$

Remark. Obviously, if $EX^2 < \infty$, then $EX^2 I\{|X| \leq x\}$ slowly varies as $x \to \infty$.

PROOFS

We start the proof with several preliminary Lemmas.

Lemma 2.1. If $EX = 0$ and $EX^2 I\{|X| \leq x\}$ slowly varies as $x \to \infty$. Then for any $0 < \epsilon < \frac{1}{2}$, there exist $0 < \delta < 1$, $x_0 > 1$, $\theta_0 > 1$ and n_0 such that for any $n \geq n_0$, $x_0 < x < \delta\sqrt{n}$ and $1 < \theta \leq \theta_0$

$$e^{-(1+\epsilon)x^2}/2 \leq P\left(\frac{S_n}{V_n} \geq x\right) \leq e^{-(1-\epsilon)x^2}/2;$$

$$P\left(\max_{n\leq k\leq \theta n} \frac{S_k}{V_k} \geq x\right) \leq e^{-(1-\epsilon)x^2}/2. \tag{6}$$

Proof. See Remark 4.1 and Remark 4.2 of Shao (1997).

Lemma 2.2. Assume that $EX = 0$ or $EX^2 = \infty$. Then

$$\lim_{n\to\infty} P\left(\frac{S_n}{V_n n^{1/2}} \geq x\right)^{1/n} = f(x) \tag{7}$$

for $x > 0$, and $f(1) = \sup_{b\geq 0} P(X=b)$, $f(x) = P(X=0)$ for $x > 1$.

Proof. See Corollary 1.1 and Lemma 8.1 of Shao (1997).

Lemma 2.3. Assume that $EX = 0$ or $EX^2 = \infty$. Then

$$\lim_{n\to\infty} \max_{1\leq k\leq n-[c\log n]} \frac{S_{k+[c\log n]} - S_k}{\sqrt{[c\log n] \sum_{i=k+1}^{i=k+[c\log n]} X_i^2}} = k(c) \text{ a.s.} \tag{8}$$

for any $c > c_0$. Furthermore, if $\sup_{b>0} P(X=b) \geq P(X=0)$ then

$$\lim_{n\to\infty} \max_{1\leq k\leq n-[c\log n]} \frac{S_{k+[c\log n]} - S_k}{\sqrt{[c\log n] \sum_{i=k+1}^{i=k+[c\log n]} X_i^2}} = 1 \quad \text{a.s.} \tag{9}$$

for any $0 < c \leq c_0$.

Proof. (8) follows from Theorem 8.1 of Shao (1997). If $P(X=0) = 0$ then there is nothing to prove. Suppose $P(X=0) > 0$. Noting that

$$\limsup_{n\to\infty} \max_{1\leq k\leq n-[c\log n]} \frac{S_{k+[c\log n]} - S_k}{\sqrt{[c\log n] \sum_{i=k+1}^{i=k+[c\log n]} X_i^2}} \leq$$

$$\limsup_{n\to\infty} \max_{1\leq k\leq n-[c\log n]} \frac{\sqrt{[c\log n] \sum_{i=k+1}^{i=k+[c\log n]} X_i^2}}{\sqrt{[c\log n] \sum_{i=k+1}^{i=k+[c\log n]} X_i^2}} = 1 \text{ a.s.}$$

we only need to prove for any $0<\epsilon<1$

$$\liminf_{n\to\infty}\max_{1\leq k\leq n-[c\log n]}\frac{S_{k+[c\log n]}-S_k}{\sqrt{[c\log n]\sum_{i=k+1}^{i=k+[c\log n]}X_i^2}}\geqslant 1-\epsilon \quad \text{a.s.}$$

Suppose that $f(1-\epsilon)>P(X=0)$. Then there exists $\eta>0$ such that $f(1-\epsilon)-\eta>\exp\left(-\frac{1-\eta}{c_0}\right)$. Then by Lemma 2.2 for m large enough

$$P\left(\max_{1\leq j\leq e^m-[cm]}\frac{S_{j+[cm]}-S_j}{\{(V_{j+[cm]}^2-V_j^2)(cm)\}^{1/2}}\leqslant 1-\epsilon\right)\leqslant$$
$$P\left(\max_{0\leq l\leq e^m/[cm]-1}\frac{S_{(l+1)[cm]}-S_{l[cm]}}{\{(V_{(1+l)[cm]}^2-V_{l[cm]}^2)(cm)\}^{1/2}}\leqslant 1-\epsilon\right)\leqslant$$
$$P\left(\frac{S_{[cm]}}{V_{[cm]}}\leqslant(1-\epsilon)(cm)^{1/2}\right)^{[e^m/[cm]]-1}\leqslant$$
$$(1-(f(1-\epsilon)-\eta)^{[cm]})^{[e^m/[cm]]-1}\leqslant$$
$$(1-\exp(1-(1-\eta)cm/c_0))^{[e^m/[cm]]-1}\leqslant$$
$$\exp\left(-K\frac{e^m}{cm}\exp(-(1-\eta)cm/c_0)\right)\leqslant$$
$$\exp(-Ke^{\eta m}/m),$$

which implies that

$$\liminf_{n\to\infty}\max_{1\leq k\leq n-[c\log n]}\frac{S_{k+[c\log n]}-S_k}{\sqrt{[c\log n]\sum_{i=k+1}^{i=k+[c\log n]}X_i^2}}\geqslant$$
$$\liminf_{m\to\infty}\max_{1\leq j\leq e^m-[cm]}\frac{S_{j+[cm]}-S_j}{\{(V_{j+[cm]}^2-V_j^2)(cm)\}^{1/2}}\geqslant 1-\epsilon \quad \text{a.s.}$$

At last, we need to prove $f(1-\epsilon)>P(X=0)$. Since $\sup_{b>0}P(X=b)\geqslant P(X=0)$, there exists $b>0$ such that $P(X=b)\geqslant P(X=0)$. Otherwise, there would exist a sequence $\{b_i\}$ such that $P(X=b_i)\geqslant\frac{1}{2}P(X=0)$. Then $P(\bigcup\{X=b_i\})=\sum_i P(X=b_i)=\infty$ which is a contradiction. Define

$$\tau(x)=\inf_{t\geq 0}Ee^{t(bX-x(X^2+b^2)/2)},\quad x\geqslant 0,$$

let $t_x\geqslant 0$ be given by

$$\tau(x)=Ee^{t_x(bX-x(X^2+b^2)/2)},\quad x\geqslant 0.$$

It is easy to show that $P(bX=x(X^2+b^2)/2)\neq 1$. Thus according to lemmas 1 and 3 of Chernoff (1952), t_x exists and is unique.

We now show that for $0<x<1$, t_x is finite, Put $x'=\frac{1}{2}(x+1)$. Note that

$$Ee^{t(bX-x(X^2+b^2)/2)}\geqslant$$
$$\int_{by-x'(y^2+b^2)/2>0}e^{t(by-x(y^2+b^2)/2)}\,dF(y)\geqslant$$
$$e^{t(x'-x)}P(bX-x'(X^2+b^2)/2>0)=$$
$$e^{\frac{t}{2}(1-x)}P\left(\frac{b}{x'}(1-\sqrt{1-x'^2})<X<\frac{b}{x'}(1+\sqrt{1-x'^2})\right)\geqslant$$
$$e^{\frac{t}{2}(1-x)}P(X=b)\geqslant e^{\frac{t}{2}(1-x)}P(X=0),$$

which implies the finiteness of t_x. Now for $0<x<1$, if $t_x=0$ then

$$\tau(x)=Ee^{t_x(bX-x(X^2+b^2)/2)}=1>P(X=0);$$

if $t_x>0$ then

$$\tau(x)=Ee^{t_x(bX-x(X^2+b^2)/2)}\geqslant e^{\frac{t_x}{2}(1-x)}P(X=0)>P(X=0).$$

Hence for $0<x<1$,

$$f(x)=\sup_{b\geqslant 0}\inf_{t\geqslant 0}Ee^{t_x(bX-x(X^2+b^2)/2)}>P(X=0).$$

The proof of Lemma 2.3 is now complete.

Remark: If one could prove that $f(x)$ is strictly decreasing for $0<x\leqslant 1$, then the condition $\sup_{b>0}P(X=b)\geqslant P(X=0)$ is superfluous.

Proof of the Theorem 1

Step 1. For any $\epsilon>0$, then we have for c_1 large enough,

$$\limsup_{n\to\infty}\max_{0\leq j<n}\max_{c_1\log n\leq k\leq n-j}\frac{|S_{j+k}-S_j|}{\{(V_{j+k}^2-V_j^2)(2\log n)\}^{1/2}}\leqslant 1+\epsilon \quad \text{a.s.} \tag{10}$$

Proof. By Lemma 2.1, for c_1 large enough and then large enough,

$$P\left(\max_{0\leq j\leq e^{m+1}}\max_{c_1 m\leq k\leq e^{m+1}}\frac{|S_{j+k}-S_j|}{\{(V_{j+k}^2-V_j^2)(2m)\}^{1/2}}\geqslant 1+\epsilon\right)\leqslant$$
$$P\left(\max_{0\leq j\leq e^{m+1}}\max_{0\leq l\leq\frac{\log(e^{m+1}/c_1 m)}{\log\theta}+1}\max_{c_1 m\theta^l\leq k\leq c_1 m\theta^{l+1}}\frac{|S_{j+k}-S_j|}{\{(V_{j+k}^2-V_j^2)(2m)\}^{1/2}}\geqslant 1+\epsilon\right)\leqslant$$
$$Ke^m m\max_l P\left(\max_{c_1 m\theta^l\leq k\leq c_1 m\theta^{l+1}}\frac{|S_k|}{V_k}\geqslant(1+\epsilon)(2m)^{1/2}\right)\leqslant$$
$$Kme^m\exp(-(1+\epsilon)m)=Kme^{-\epsilon m}.$$

It follows that

$$\sum_{m=1}^{\infty} P\Big(\max_{0\le j\le e^{m+1}} \max_{c_1 m\le k\le e^{m+1}} \cdot \frac{|S_{j+k}-S_j|}{\{(V_{j+k}^2-V_j^2)(2m)\}^{1/2}} \ge 1+\epsilon \Big) < \infty,$$

which implies

$$\limsup_{n\to\infty} \max_{0\le j\le n} \max_{c_1 \log n\le k\le n-j} \frac{|S_{j+k}-S_j|}{\{(V_{j+k}^2-V_j^2)(2m)\}^{1/2}} \le$$

$$\limsup_{m\to\infty} \max_{0\le j\le e^{m+1}} \max_{c_1 m\le k\le e^{m+1}} \cdot \frac{|S_{j+k}-S_j|}{\{(V_{j+k}^2-V_j^2)(2m)\}^{1/2}}$$

$$\le 1+\epsilon \quad \text{a.s.}$$

Step 2. It is easy to see that for c small enough,

$$\limsup_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le (n-j)\wedge(c\log n)} \frac{|S_{j+k}-S_j|}{\{(V_{j+k}^2-V_j^2)(2\log n)\}^{1/2}} \le$$

$$\limsup_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le (n-j)\wedge(c\log n)} \frac{\{(V_{j+k}^2-V_j^2)k\}^{1/2}}{\{(V_{j+k}^2-V_j^2)(2\log n)\}^{1/2}} \le$$

$$=\sqrt{\frac{c}{2}} < \epsilon. \tag{11}$$

Step 3. For any $0<c<c_1<\infty$ we have

$$\limsup_{n\to\infty} \max_{0\le j<n} \max_{c\log n\le k\le c_1\log n} \frac{S_{j+k}-S_j}{\{(V_{j+k}^2-V_j^2)(2\log n)\}^{1/2}} \le \sup_{c\le x\le c_1+1} \frac{\sqrt{x}\lambda(x)}{\sqrt{2}} \quad \text{a.s.} \tag{12}$$

Proof. For $\eta>0$ small enough such that $\eta c<1$ by Lemma 2.3 we have

$$\limsup_{n\to\infty} \max_{0\le j<n} \max_{c\log n\le k\le c_1\log n} \frac{S_{j+k}-S_j}{\{(V_{j+k}^2-V_j^2)(2\log n)\}^{1/2}} \le$$

$$\limsup_{n\to\infty} \max_{0\le l\le \frac{c_1-c}{\eta c}} \max_{0\le j<n} \max_{(1+l\eta)c\log n\le k\le (1+(l+1)\eta)c\log n} \frac{S_{j+k}-S_j}{\{(V_{j+k}^2-V_j^2)(2\log n)\}^{1/2}} \le \max_{0\le l\le \frac{c_1-c}{\eta c}} \limsup_{n\to\infty}$$

$$\max_{0\le j<n} \frac{S_{j+(1+l\eta)c\log n}-S_j}{\{(V_{j+(1+l\eta)c\log n}^2-V_j^2)(2\log n)\}^{1/2}} +$$

$$\max_{0\le l\le \frac{c_1-c}{\eta c}} \limsup_{n\to\infty} \max_{0\le j<n} \max_{(1+l\eta)c\log n\le k\le (1+(l+1)\eta)c\log n} \frac{|S_{j+k}-S_{j+(1+l\eta)c\log n}|}{\{(V_{j+k}^2-V_j^2)(2\log n)\}^{1/2}} \le$$

$$\max_{0\le l\le \frac{c_1-c}{\eta c}} \frac{\sqrt{(1+l\eta)c}\lambda((1+l\eta)c)}{\sqrt{2}} +$$

$$\max_{0\le l\le \frac{c_1-c}{\eta c}} \limsup_{n\to\infty} \max_{0\le j<n} \max_{(1+l\eta)c\log n\le k\le (1+(l+1)\eta)c\log n} \frac{(k-(1+l\eta)c\log n)^{1/2}}{(2\log n)^{1/2}} \le$$

$$\sup_{c\le x\le c_1+1} \frac{\sqrt{x}\lambda(x)}{\sqrt{2}} + (\eta c/2)^{1/2},$$

which implies (12) immediately.

Step 4. We have

$$\liminf_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le n-j} \frac{S_{j+k}-S_j}{\{(V_{j+k}^2-V_j^2)(2\log n)\}^{1/2}} \ge \lambda^* \quad \text{a.s.} \tag{13}$$

Proof. For any $c>0$, by Lemma 2.3 we have

$$\liminf_{n\to\infty} \max_{0\le j<n} \max_{1\le k\le n-j} \frac{S_{j+k}-S_j}{\{(V_{j+k}^2-V_j^2)(2\log n)\}^{1/2}} \ge$$

$$\liminf_{n\to\infty} \max_{0\le j<n-[c\log n]} \frac{S_{j+[c\log n]}-S_j}{\{(V_{j+[c\log n]}^2-V_j^2)(2\log n)\}^{1/2}} \ge \frac{\sqrt{c}\lambda(c)}{\sqrt{2}} \quad \text{a.s.}$$

which implies (13) immediately.

Now noting that

$$\liminf_{n\to\infty} \max_{0\le j<n} \max_{c_1\log n\le k\le n-j} \frac{S_{j+k}-S_j}{\{(V_{j+k}^2-V_j^2)(2\log n)\}^{1/2}} \ge$$

$$\liminf_{n\to\infty} \max_{0\le j<n-[c_1\log n]} \frac{S_{j+[c_1\log n]}-S_j}{\{(V_{j+[c_1\log n]}^2-V_j^2)(2\log n)\}^{1/2}} = \frac{\sqrt{c_1}\lambda(c_1)}{\sqrt{2}},$$

thus by Step 1, we know that for c_1 large enough

$$\frac{\sqrt{c_1}\lambda(c_1)}{\sqrt{2}} \le 1+\epsilon. \tag{14}$$

Then, combining Step 1, Step 2, Step 3 and Step 4 implies (5).

Since $k(c)\le k_\delta(c)$ for any $c>0$ and $\delta>0$, where $k(c)=\inf\{x\ge 0; f_\delta(x)<e^{-1/c}$ and $f_\delta(x)=\sup_{b\ge 0}\inf_{t\ge 0} e^{(t\delta)^2/2}Ee^{t(bX-x(X^2+b^2)/2)}$, we have for $0<c<c_1<\infty$

$$\sup_{c\le x\le c_1} \frac{\sqrt{x}k(x)}{\sqrt{2}} \le \sup_{c\le x\le c_1} \frac{\sqrt{x}k_\delta(x)}{\sqrt{2}} < \infty \tag{15}$$

by the fact that $k_\delta(c)$ is continuous (cf. Lemma 8.1 of Shao (1997)). Hence by (11), (14) and (15) we know $\lambda^*<\infty$.

On the other hand, noting that by Lemma (2.1), for $c_1 > 0$ large enough and then m large enough

$$P\left(\max_{1\le j\le e^m-[c_1 m]} \frac{S_{j+[c_1 m]} - S_j}{\{(V_{j+[c_1 m]}^2 - V_j^2)(2m)\}^{1/2}} \le 1-\epsilon\right) \le$$

$$P\left(\max_{0\le l\le e^m/[c_1 m]-1} \frac{S_{(l+1)[c_1 m]} - S_{l[c_1 m]}}{\{(V_{(1+l)[c_1 m]}^2 - V_{l[c_1 m]}^2)(2m)\}^{1/2}} \le 1-\epsilon\right) \le$$

$$P\left(\frac{S_{[c_1 m]}}{V_{[c_1 m]}} \le (1-\epsilon)(2m)^{1/2}\right)^{[e^m/[c_1 m]]-1} \le$$

$$(1-\exp(-(1-\epsilon)m))^{[e^m/[c_1 m]]-1} \le \exp(-Ke^{\epsilon m}/m),$$

we have

$$\sum_{m=1}^{\infty} P\left(\max_{0\le j\le e^m-[c_1 m]} \frac{S_{j+[c_1 m]} - S_j}{\{(V_{j+[c_1 m]}^2 - V_j^2)(2m)\}^{1/2}} \le 1-\epsilon\right) < \infty,$$

which implies

$$\frac{\sqrt{c_1}\,k(c_1)}{\sqrt{2}} = \lim_{n\to\infty} \max_{0\le j<n-[c_1 \log n]} \frac{S_{j+[c_1\log n]} - S_j}{\{(V_{j+[c_1\log n]}^2 - V_j^2)(2\log n)\}^{1/2}} \ge$$

$$\liminf_{m\to\infty} \max_{0\le j<e^m-[c_1 m]} \frac{S_{j+[c_1 m]} - S_j}{\{(V_{j+[c_1 m]}^2 - V_j^2)(2\log n)\}^{1/2}} \ge 1-\epsilon \quad \text{a.s.}$$

So $\lambda^* \ge 1$. The proofs are now complete.

Similarly, we can prove:

Theorem 1′. Assume that there exist $0<\alpha<2$, $c_1 \ge 0$, $c_2 \ge 0$, $c_1 + c_2 > 0$ and a slowly varying function $h(x)$ such that

$$P(X \ge x) = \frac{c_1 + o(1)}{x^\alpha} h(x)$$

and

$$P(X \le -x) = \frac{c_2 + o(1)}{x^\alpha} h(x) \quad \text{as } x \to \infty.$$

Moreover, assume that $EX = 0$ if $1<\alpha<2$, X is symmetric if $\alpha = 1$ and that $c_1 > 0$ if $0<\alpha<1$. Furthermore, assume that $\sup_{b>0} P(X=b) \ge P(X=0)$. Then (5) holds and $1/\sqrt{2\beta(\alpha, c_1 c_2)} \le \lambda^* < \infty$,

where $\beta(\alpha, c_1, c_2)$ is the solution of $\Gamma(\beta,\alpha) = 0$ and

$$\Gamma(\beta,\alpha) = \begin{cases} c_1\int_0^\infty \frac{1+2x-e^{(2x-x^2/\beta)}}{x^{\alpha+1}}dx + c_2\int_0^\infty \frac{1-2x-e^{(-2x-x^2/\beta)}}{x^{\alpha+1}}dx, & \text{if } 1<\alpha<2, \\ c_1\int_0^\infty \frac{2-e^{(2x-x^2/\beta)}-e^{(-2x-x^2/\beta)}}{x^2}dx, & \text{if } \alpha = 1, \\ c_1\int_0^\infty \frac{1-e^{(2x-x^2/\beta)}}{x^{\alpha+1}}dx + c_2\int_0^\infty \frac{1-e^{(-2x-x^2/\beta)}}{x^{\alpha+1}}dx, & \text{if } 0<\alpha<1. \end{cases}$$

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