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A PENNY-SHAPED CRACK IN AN INFINITE PIEZOELECTRIC BODY UNDER ANTISYMMETRIC POINT LOADS*

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Abstract: In this study, Fabrikant(1989, 1991)'s new results in potential theory were used to obtain the exact and complete solution for the problem of a penny-shaped crack in an infinite transversely isotropic piezoelectric body subjected to antisymmetric point loads (point charges and normal point forces); then the complete solution for the problem of one-sided loading of a penny-shaped crack was obtained by the superposition of the symmetric loading solution in Chen and Shioya (1999) and the antisymmetric one presented here; and then the reciprocity theorem of piezoelectric media was used to deal with the problem of interaction between arbitrarily located point forces and a point charge with a penny-shaped crack and obtained the exact expressions of the crack faces' normal displacement in terms of elementary functions and some non-singular integrals; and finally obtained the normal displacement of the positive and negative faces of the crack under many loading cases as shown in figures for an infinite PZT-4 piezoelectric ceramic body weakened by a penny-shaped crack.

Key words: transversely isotropic piezoelectric body, penny-shaped crack, point force, point charge
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INTRODUCTION

Piezoelectric ceramic materials have many applications in modern technologies because of their excellent piezoelectric properties. But cracks arising from their intrinsic brittleness often cause piezoelectric components failure. Study of what causes this requires theoretical analysis and accurate quantitative knowledge of the elastic and electric fields in the area of the cracks in the piezoelectric ceramic, from the view point of electromechanical coupling.

The potential theory method developed by Fabrikant (1989, 1991) is efficient for analysing various mixed boundary value problems in pure elastic theory. Fabrikant (1991) used this method to obtain solution for the problem of a circular crack in an infinite transversely isotropic elastic body subjected to antisymmetric normal point forces. He derived the complete solution for the problem of one-sided loading of a circular crack by the superposing of his symmetric loading solution (Fabrikant, 1989) and the antisymmetric one, and gave the exact expressions (Fabrikant, 1991) of the normal displacement of the two faces of a crack for arbitrarily located normal point forces.

In this paper, we first further generalize Fabrikant's potential theory method to analyze corresponding mixed boundary value problems in three-dimensional piezoelectricity; then obtain the exact and complete solution for the problem of a penny-shaped crack in an infinite transversely isotropic piezoelectric body subjected to antisymmetric point charges and normal point forces. The completed solution for the problem of one-sided loading of a penny-shaped crack was obtained by the superposition of the symmetric loading solution in Chen and Shioya (1999) and the antisymmetric one presented here. And then used the reciprocity theorem of piezoelectric media to consider the problems of interaction between arbitrarily located point forces and a point charge with a penny-shaped crack and derive the exact expressions of the normal displacement of the crack in terms of elementary functions and

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some non-singular integrals, and finally get the normal displacement of the positive and negative faces of the crack under many loading cases as shown in the figures for an infinite PZT-4 piezoelectric ceramic body weakened by a penny-shaped crack.

GENERAL SOLUTION FOR TRANSVERSELY ISOTROPIC PIEZOELECTRIC MEDIA

For the characteristic roots $s_1 \neq s_2 \neq s_3 \neq s_1$, Ding et al. (1996) gave the general solutions of the displacement and electric potential of transversely isotropic piezoelectric media in terms of four displacement functions Ψ_j satisfying respectively, the following equations:

$$\left(\Delta + \frac{\partial^2}{\partial z_j^2}\right)\Psi_j = 0 \quad (j = 0,1,2,3) \qquad (1)$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $z_j = s_j z\,(j = 0,1,2,3)$ and $s_0 = \sqrt{c_{66}/c_{44}}$, $s_j\,(j = 1,2,3)$ are the three characteristic roots of a sixth degree equation defined in Ding et al. (1996) and satisfy $\mathrm{Re}(s_j) > 0$.

The constitutive relation was used to obtain the general solutions for the stress and the electric displacement expressed by four displacement functions. For the sake of convenience, the following notations were introduced:

$$U = u + iv, w_1 = w, w_2 = \Phi, \sigma_1 = \sigma_x + \sigma_y,$$
$$\sigma_2 = \sigma_x - \sigma_y + 2i\tau_{xy}, \sigma_{z1} = \sigma_z, \sigma_{z2} = D_z,$$
$$\tau_{z1} = \tau_{zx} + i\tau_{zy}, \tau_{z2} = D_x + iD_y \qquad (2)$$

where u, v and w are the displacement components in the directions of the x, y and z axis, respectively; Φ is electric potential; σ_x, σ_y, σ_z, τ_{xy}, τ_{zx} and τ_{zy} are stress components; D_x, D_y and D_z are the electric displacement components in the directions of the x, y and z axis, respectively. Then the general solution can be concisely written as follows (Ding et al., 2000):

$$U = \Lambda\left(i\Psi_0 + \sum_{j=1}^{3}\Psi_j\right), w_m = \sum_{j=1}^{3} s_j k_{mj}\frac{\partial \Psi_j}{\partial z_j}$$

$$\sigma_1 = 2\sum_{j=1}^{3}(m_j - c_{66})\frac{\partial^2 \Psi_j}{\partial z_j^2} = -2\sum_{j=1}^{3}(m_j - c_{66})\Delta\Psi_j,$$

$$\sigma_2 = 2c_{66}\Lambda^2\left(i\Psi_0 + \sum_{j=1}^{3}\Psi_j\right)$$

$$\sigma_{zm} = \sum_{j=1}^{3}\omega_{mj}\frac{\partial^2 \Psi_j}{\partial z_j^2} = -\sum_{j=1}^{3}\omega_{mj}\Delta\Psi_j$$

$$\tau_{zm} = \Lambda\left(s_0\rho_m i\frac{\partial \Psi_0}{\partial z_0} + \sum_{j=1}^{3}s_j\omega_{mj}\frac{\partial \Psi_j}{\partial z_j}\right)$$
$$(m = 1,2) \qquad (3)$$

where k_{mj} are constants dependent on material constants and characteristic roots and are defined in Ding et al. (1996),

$$\omega_{1j} = c_{44}(1 + k_{1j}) + e_{15}k_{2j}$$
$$\omega_{2j} = e_{15}(1 + k_{1j}) - \varepsilon_{11}k_{2j}$$
$$m_j = 2c_{66} - \omega_{1j}s_j^2, \rho_1 = c_{44}$$
$$\rho_2 = e_{15}, \Lambda = \partial/\partial x + i\partial/\partial y$$
$$\Lambda^2 = \partial^2/\partial x^2 - \partial^2/\partial y^2 + 2i\partial^2/\partial x\partial y$$
$$(j = 1,2,3) \qquad (4)$$

and c_{ij}, e_{ij} and ε_{ij} are elastic, piezoelectric and dielectric constants, respectively.

PENNY-SHAPED CRACK UNDER ANTISYMMETRIC POINT CHARGES AND NORMAL POINT FORCES

1. The Green's function for penny-shaped crack under antisymmetric point charges and normal point forces

Consider an infinite transversely isotropic piezoelectric body weakened by a penny-shaped crack of radius a in the plane $z = 0$ (Fig. 1). Let two normal point forces P_z be applied to the crack faces antisymmetrically in the positive z direction at the points with cylinder coordinates $(\rho_0, \phi_0, 0^{\pm})$. Meanwhile, two point charges $+Q$ and $-Q$ are also applied antisymmetrically at these two points, respectively.

The problem, due to antisymmetric loading, can be reduced to that of a half infinite body $z \geqslant 0$ with the boundary conditions at the plane $z = 0$

$$\begin{aligned} U &= 0, && \text{for } a \leqslant \rho < \infty, 0 \leqslant \phi < 2\pi \\ \sigma_{zm} &= 0, && \text{for } a \leqslant \rho < \infty, 0 \leqslant \phi < 2\pi \\ \sigma_{z1} &= P_z\delta(\rho - \rho_0, \phi - \phi_0), && \text{for } 0 \leqslant \rho \leqslant a, 0 \leqslant \phi < 2\pi \\ \sigma_{z2} &= Q\delta(\rho - \rho_0, \phi - \phi_0), && \text{for } 0 \leqslant \rho \leqslant a, 0 \leqslant \phi < 2\pi \\ \tau_{z1} &= 0 && \text{for } 0 \leqslant \rho \leqslant a, 0 \leqslant \phi < 2\pi \end{aligned} \qquad (5)$$

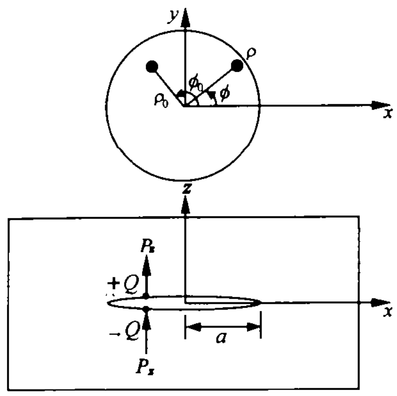

Fig. 1 Penny-shaped crack and the antisymmetric normal point forces and point charges

It is known (Ding et al., 2000) that in the case of a half infinite transversely isotropic piezoelectric body subjected to a general point force with the components P_x, P_y, P_z and a point charge Q at the point $(r, \varphi, 0)$, the complete solution can be expressed through the four displacement functions:

$$\Psi_0 = \frac{i}{4\pi s_0 c_{44}}(T\Lambda - \bar{T}\Lambda)\chi(z_0) = \frac{1}{2\pi s_0 c_{44}}\mathrm{Im}\left(\frac{dT}{D_0^*}\right)$$

$$\Psi_j = (\alpha_j P_z + \beta_j Q)\ln D_j^* - \frac{\zeta_j}{2}(T\Lambda + \bar{T}\Lambda)\chi(z_j) = (\alpha_j P_z + \beta_j Q)\ln D_j^* + \zeta_j Re\left(\frac{d\bar{T}}{D_j^*}\right), \quad (j = 1,2,3) \tag{6}$$

where overbar denotes complex conjugate, and

$$D_j^* = D_j + z_j, D_j = \sqrt{\rho^2 + r^2 - 2r\rho\cos(\phi - \varphi) + z_j^2}$$
$$T = P_x + iP_y, \chi(z_j) = z_j \ln D_j^* - D_j$$
$$d = \rho e^{i\phi} - re^{i\varphi} (j = 0,1,2,3) \tag{7}$$

and

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix} = \frac{1}{2\pi}\begin{bmatrix} s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13}\\ \omega_{11} & \omega_{12} & \omega_{13}\\ \omega_{21} & \omega_{22} & \omega_{23}\end{bmatrix}^{-1}\begin{Bmatrix}0\\1\\0\end{Bmatrix}$$

$$\begin{Bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{Bmatrix} = -\frac{1}{2\pi}\begin{bmatrix} s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13}\\ \omega_{11} & \omega_{12} & \omega_{13}\\ \omega_{21} & \omega_{22} & \omega_{23}\end{bmatrix}^{-1}\begin{Bmatrix}0\\0\\1\end{Bmatrix}$$

$$\begin{Bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{Bmatrix} = \frac{1}{2\pi}\begin{bmatrix} s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13}\\ \omega_{11} & \omega_{12} & \omega_{13}\\ \omega_{21} & \omega_{22} & \omega_{23}\end{bmatrix}^{-1}\begin{Bmatrix}1\\0\\0\end{Bmatrix} \tag{8}$$

Substituting Eq. (6) into Eq. (3), we obtain the displacements and electric potential in the half infinite piezoelectric body as follows:

$$U = \frac{1}{4\pi c_{44} s_0}\left[\frac{T}{D_0} + \frac{d^2\bar{T}}{D_0 D_0^{*2}}\right] - \frac{1}{2}\sum_{j=1}^{3}\zeta_j \cdot \left[-\frac{T}{D_j} + \frac{d^2\bar{T}}{D_j D_j^{*2}}\right] + \sum_{j=1}^{3}(\alpha_j P_z + \beta_j Q)\frac{d}{D_j D_j^*}$$

$$w_m = -\frac{1}{2}\sum_{j=1}^{3}\zeta_j s_j k_{mj}\frac{(T\bar{d} + \bar{T}d)}{D_j D_j^*} + \sum_{j=1}^{3}(\alpha_j P_z + \beta_j Q)s_j k_{mj}\frac{1}{D_j} \tag{9}$$

When $z = 0$, Eq. (9) simplifies to

$$U = \frac{G^P P_z + G^Q Q}{\bar{d}} + \frac{1}{2}G_1^T\frac{T}{D} + \frac{1}{2}G_2^T\frac{\bar{T}d^2}{D^3}$$

$$w_m = \frac{H_m^P P_z + H_m^Q Q}{D} + H_m^T \mathrm{Re}\left(\frac{T}{d}\right) \tag{10}$$

where

$$D = [\rho^2 + r^2 - 2r\rho\cos(\phi - \varphi)]^{1/2} \tag{11}$$

and

$$G^P = \sum_{j=1}^{3}\alpha_j, \quad G^Q = \sum_{j=1}^{3}\beta_j$$
$$G_1^T = \frac{1}{2\pi s_0 c_{44}} + \sum_{j=1}^{3}\zeta_j, G_2^T = \frac{1}{2\pi s_0 c_{44}} - \sum_{j=1}^{3}\zeta_j$$
$$H_m^P = \sum_{j=1}^{3}s_j k_{mj}\alpha_j, H_m^Q = \sum_{j=1}^{3}s_j k_{mj}\beta_j$$
$$H_m^T = -\sum_{j=1}^{3}s_j k_{mj}\zeta_j, \quad (m = 1,2) \tag{12}$$

The first expression of Eq. (10) can be used for the integral equation formulation of the problem. The governing integral equation will take the form

$$\frac{G_1^T}{2}\int_0^{2\pi}\int_a^{\infty}\frac{\tau(r,\varphi)}{D}r\mathrm{d}r\mathrm{d}\varphi + \frac{G_2^T}{2}\int_0^{2\pi}\int_a^{\infty}\frac{d^2\tau(r,\varphi)}{D^3}r\mathrm{d}r\mathrm{d}\varphi = -\frac{G^P P_z + G^Q Q}{\bar{g}} \tag{13}$$

$$g = \rho e^{i\phi} - \rho_0 e^{i\phi_0} \tag{14}$$

where τ stands for $-\tau_{z1}$ as it was defined in Eq. (3). And its exact solution in this case is as follows (Fabrikant, 1989):

$$\tau(\rho,\phi) = -\frac{(G^P P_z + G^Q Q)e^{i\phi}}{\pi G_1^T \rho(\rho^2 - a^2)^{1/2}} \cdot$$

$$\left[\frac{G_2^T}{G_1^T - G_2^T} + \frac{1}{(1-\zeta)^{3/2}}\right] \quad (15)$$

where $\zeta = \frac{\rho_0}{\rho} e^{i(\phi_0 - \phi)}$ (16)

Expressions (6) can be used to obtain formulae for the displacement functions in the case of distributed loading. The related integral was given by Fabrikant (1991) as follows:

$$\int_0^{2\pi}\int_a^{\infty} \frac{\bar{d}}{D_j^*}\tau(r,\varphi) r \mathrm{d}r \mathrm{d}\varphi = \frac{2(G^P P_z + G^Q Q)}{G_1^T} \cdot \left[f(z_j) + \frac{G_2^T}{G_1^T - G_2^T} f_0(z_j)\right] \quad (17)$$

where

$$f(z) = -\frac{z}{a^2 - b^2}\left[aE\left(\sin^{-1}\left(\frac{a}{l^2}\right), \frac{b}{a}\right) - \frac{b^2(l_2^2 - a^2)^{1/2}}{l^2(l_2^2 - b^2)^{1/2}}\right] + \frac{(l_2^2 - \rho^2)^{1/2}}{(l_2^2 - b^2)^{1/2}} - \ln[(l_2^2 - b^2)^{1/2} + (l_2^2 - \rho^2)^{1/2}]$$

$$f_0(z) = -\frac{z}{a}\sin^{-1}\left(\frac{a}{l_2}\right) + \frac{(a^2 - l_1^2)^{1/2}}{a} - \ln[l_2 + (l_2^2 - \rho^2)^{1/2}] \quad (18)$$

where $E(\cdot,\cdot)$ stands for the incomplete elliptic integral of the second kind, and

$b^2 = \rho\rho_0 e^{i(\phi - \phi_0)}$, $l_1 = l_1(a,\rho,z)$, $l_2 = l_2(a,\rho,z)$

$$l_1(x,\rho,z) = \{[(\rho + x)^2 + z^2]^{1/2} - [(\rho - x)^2 + z^2]^{1/2}\}/2$$

$$l_2(x,\rho,z) = \{[(\rho + x)^2 + z^2]^{1/2} + [(\rho - x)^2 + z^2]^{1/2}\}/2 \quad (19)$$

Equation (17) allows us to define the displacement functions of this problem as follows:

$$\Psi_0 = \frac{G^P P_z + G^Q Q}{\pi s_0 c_{44} G_1^T}\mathrm{Im}[f(z_0)]$$

$$\Psi_j = \frac{2\zeta_j(G^P P_z + G^Q Q)}{G_1^T - G_2^T} \cdot \left\{\left(1 - \frac{G_2^T}{G_1^T}\right)\mathrm{Re}[f(z_j)] + \frac{G_2^T}{G_1^T} f_0(z_j)\right\} + (\alpha_j P_z + \beta_j Q)\ln R_j^* \quad (20)$$

where $f(z)$ and $f_0(z)$ are defined in Eq. (18), and

$$R_j^* = R_j + z_j, R_j = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z_j^2]^{1/2} (j = 1,2,3) \quad (21)$$

Substitution of Eq. (20) into Eq. (3), we obtain the complete solution as follows:

$$U = \frac{2(G^P P_z + G^Q Q)}{G_1^T - G_2^T}\sum_{j=1}^{3}\zeta_j \cdot \left[\left(1 - \frac{G_2^T}{G_1^T}\right)\Lambda\mathrm{Re}[f(z_j)] + \frac{G_2^T}{G_1^T}\Lambda f_0(z_j)\right] + \sum_{j=1}^{3}(\alpha_j P_z + \beta_j Q)\frac{g}{R_j R_j^*} + \frac{i(G^P P_z + G^Q Q)}{\pi s_0 c_{44} G_1^T}\Lambda\mathrm{Im}[f(z_0)]$$

$$w_m = \frac{2(G^P P_z + G^Q Q)}{G_1^T - G_2^T}\sum_{j=1}^{3} s_j k_{mj}\zeta_j \cdot \left[\left(1 - \frac{G_2^T}{G_1^T}\right)\frac{\partial}{\partial z_j}Re[f(z_j)] - \frac{G_2^T}{G_1^T a}\sin^{-1}\left(\frac{a}{l_{2j}}\right)\right] + \sum_{j=1}^{3}(\alpha_j P_z + \beta_j Q)s_j k_{mj}\frac{1}{R_j} \quad (22)$$

where $l_{2j} = l_2(a, \rho, z_j)$. The expressions for various derivatives of $f(z)$ which will be needed are as follows:

$$\frac{\partial f(z)}{\partial z} = -\frac{1}{a^2 - b^2} \cdot \left[aE\left(\sin^{-1}\left(\frac{a}{l_2}\right), \frac{b}{a}\right) - \frac{b^2(l_2^2 - a^2)^{1/2}}{l_2(l_2^2 - b^2)^{1/2}}\right]$$

$$\Lambda f(z) = -\frac{1}{g}\left[1 - \frac{(l_2^2 - \rho^2)^{1/2}}{(l_2^2 - b^2)^{1/2}}\right]$$

$$\Lambda f_0(z) = -\frac{e^{i\phi}}{\rho}\left[1 - \frac{(a^2 - l_1^2)^{1/2}}{a}\right] \quad (23)$$

2. One-sided loading of a penny-shaped crack

Consider the case when a point charge Q and a normal point force P_z are applied to the positive side of the crack at the point $(\rho_0, \phi_0, 0)$ while the negative side is free. The complete solution to this problem can be obtained by the superposition of the symmetric loading solution (Chen and Shioya, 1999) and the antisymmetric one presented here (Eq. (22)).

$$U = \frac{1}{2\pi^2}\sum_{j=1}^{3}(a_j P_z - b_j Q)f_1(z_j) + \frac{(G^P P_z + G^Q Q)}{G_1^T - G_2^T}\sum_{j=1}^{3}\zeta_j \cdot \left[\left(1 - \frac{G_2^T}{G_1^T}\right)\Lambda\mathrm{Re}[f(z_j)] + \frac{G_2^T}{G_1^T a}\Lambda f_0(z_j)\right] + \frac{1}{2}\sum_{j=1}^{3}(\alpha_j P_z + \beta_j Q)\frac{g}{R_j R_j^*} +$$

$$\frac{i(G^P P_z + G^Q Q)}{2\pi s_0 c_{44} G_1^T}\Lambda\mathrm{Im}[f(z_0)]]$$

$$w_m = \frac{1}{2\pi^2}\sum_{j=1}^{3}(a_j P_z - b_j Q)s_j k_{mj} f_2(z_j) + \frac{(G^P P_z + G^Q Q)}{G_1^T - G_2^T}\sum_{j=1}^{3} s_j k_{mj}\zeta_j \cdot \left[\left(1 - \frac{G_2^T}{G_1^T}\right)\frac{\partial}{\partial z_j}\mathrm{Re}[f(z_j)] - \frac{G_2^T}{G_1^T}\sin^{-1}\left(\frac{a}{l_{2j}}\right)\right] + \frac{1}{2}\sum_{j=1}^{3}(\alpha_j P_z + \beta_j Q)s_j k_{mj}\frac{1}{R_j} \quad (24)$$

where

$$a_j = \gamma_1\delta_j + \gamma_3\lambda_j,\ b_j = \gamma_2\delta_j + \gamma_4\lambda_j$$
$$\gamma_1 = \frac{\eta_4}{\eta_1\eta_4 - \eta_2\eta_3},\ \gamma_2 = \frac{\eta_2}{\eta_1\eta_4 - \eta_2\eta_3}$$
$$\gamma_3 = \frac{\eta_3}{\eta_2\eta_3 - \eta_1\eta_4},\ \gamma_4 = \frac{\eta_1}{\eta_2\eta_3 - \eta_1\eta_4}$$
$$\eta_1 = \sum_{j=1}^{3}\omega_{1j}\delta_j,\ \eta_2 = \sum_{j=1}^{3}\omega_{1j}\lambda_j$$
$$\eta_3 = -\sum_{j=1}^{3}\omega_{2j}\delta_j,\ \eta_4 = -\sum_{j=1}^{3}\omega_{2j}\lambda_j$$
$$\begin{Bmatrix}\delta_1\\ \delta_2\\ \delta_3\end{Bmatrix} = -\frac{1}{2\pi}\begin{bmatrix} s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13}\\ s_1 k_{11} & s_2 k_{12} & s_3 k_{13}\\ s_1 k_{21} & s_1 k_{22} & s_3 k_{23}\end{bmatrix}^{-1}\begin{Bmatrix}0\\1\\0\end{Bmatrix}$$
$$\begin{Bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{Bmatrix} = -\frac{1}{2\pi}\begin{bmatrix} s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13}\\ s_1 k_{11} & s_2 k_{12} & s_3 k_{13}\\ s_1 k_{21} & s_1 k_{22} & s_3 k_{23}\end{bmatrix}^{-1}\begin{Bmatrix}0\\0\\1\end{Bmatrix} \quad (25)$$

and

$$f_1(z) = \frac{1}{g}\left[\frac{(a^2 - \rho_0^2)^{1/2}}{\bar{s}}\tan^{-1}\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}} - \frac{z}{R_0}\tan^{-1}\frac{h}{R_0}\right]$$
$$f_2(z) = \frac{1}{R_0}\tan^{-1}\frac{h}{R_0} \quad (26)$$

where g is defined in Eq. (14), and

$$h = (a^2 - l_1^2)^{1/2}(a^2 - \rho_0^2)^{1/2}/a$$
$$s = (a^2 - \rho\rho_0 e^{i(\phi-\phi_0)})^{1/2}$$
$$R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2]^{1/2} \quad (27)$$

THE INTERACTIONS BETWEEN POINT FORCES AND POINT CHARGE WITH A PENNY-SHAPED CRACK

1. Interactions between point forces and point charge with a penny-shaped crack

Consider two systems in equilibrium (Fig. 2). Let three point forces P_x, P_y and P_z be applied at an arbitrary point (ρ, ϕ, z) in the positive x, y and z directions, respectively. A point charge Q is applied at the same point too. The crack faces are free. In the second system, a normal point force F is applied to the positive crack face in the positive z direction at the point with cylindrical coordinates $(\rho_0, \phi_0, 0^+)$.

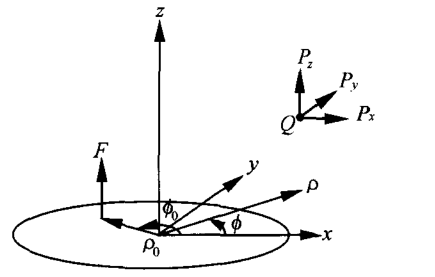

Fig. 2 Dual systems of loads (P_x, P_y, P_z, Q) and F used in the reciprocity theorem

Denote the normal displacement of the positive crack face at point $(\rho_0, \phi_0, 0^+)$ by w_{P_x}, w_{P_y}, w_{P_z} and w_Q due to the point forces P_x, P_y, P_z and point charge Q, respectively. The displacement in the positive x, y and z directions and electric potential at point (ρ, ϕ, z) are defined as u_F, v_F, w_F and Φ_F due to point forces F. Application of the reciprocity theorem (Hou, 2000) to the two systems yields:

$$Fw_{P_x} = P_x u_F,\ Fw_{P_y} = P_y v_F$$
$$Fw_{P_z} = P_z w_F,\ Fw_Q = -Q\Phi_F \quad (28)$$

which gives the normal displacement of the positive penny-crack face with the use of Eq. (24).

$$w_{P_x} = P_x\mathrm{Re}\left\{\frac{1}{2\pi^2}\sum_{j=1}^{3}a_j f_1(z_j) + \frac{G^P}{G_1^T - G_2^T}\sum_{j=1}^{3}\zeta_1 \cdot \left[\left(1 - \frac{G_2^T}{G_1^T}\right)\Lambda\mathrm{Re}[f(z_j)] + \frac{G_2^T}{G_1^T}\Lambda f_0(z_j)\right] + \frac{1}{2}\sum_{j=1}^{3}a_j\frac{g}{R_j R_j^*} + \frac{iG^P}{2\pi s_0 c_{44} G_1^T}\Lambda\mathrm{Im}[f(z_0)]\right\}$$

$$w_{P_y} = P_y\mathrm{Im}\left\{\frac{1}{2\pi^2}\sum_{j=1}^{3}a_j f_1(z_j) + \frac{G^P}{G_1^T - G_2^T}\sum_{j=1}^{3}\zeta_1 \cdot \left[\left(1 - \frac{G_2^T}{G_1^T}\right)\Lambda\mathrm{Re}[f(z_j)] + \frac{G_2^T}{G_1^T}\Lambda f_0(z_j)\right] + \right.$$

$$\frac{1}{2}\sum_{j=1}^{3}\alpha_j\frac{g}{R_jR_j^*} + \frac{iG^P}{2\pi s_0 c_{44} G_1^T}\Lambda\mathrm{Im}[f(z_0)]\Big]\Big\}$$

$$w_{P_z} = P_z\Big\{\frac{1}{2\pi^2}\sum_{j=1}^{3}s_jk_{1j}a_jf_1(z_j) + \frac{G^P}{G_1^T - G_2^T}\sum_{j=1}^{3}s_jk_{1j}\zeta_1\cdot \Big[\Big(1-\frac{G_2^T}{G_1^T}\Big)\frac{\partial}{\partial z_j}\mathrm{Re}[f(z_j)] - \frac{G_2^T}{G_1^T a}\sin^{-1}\Big(\frac{a}{l_{2j}}\Big)\Big] + \frac{1}{2}\sum_{j=1}^{3}s_jk_{1j}\alpha_j\frac{1}{R_j}\Big\}$$

$$w_Q = -Q\Big\{\frac{1}{2\pi^2}\sum_{j=1}^{3}s_jk_{2j}a_jf_2(z_j) + \frac{G^P}{G_1^T - G_2^T}\sum_{j=1}^{3}s_jk_{2j}\zeta_1\cdot \Big[\Big(1-\frac{G_2^T}{G_1^T}\Big)\frac{\partial}{\partial z_j}\mathrm{Re}[f(z_j)] - \frac{G_2^T}{G_1^T a}\sin^{-1}\Big(\frac{a}{l_{2j}}\Big)\Big] + \frac{1}{2}\sum_{j=1}^{3}s_jk_{2j}\alpha_j\frac{1}{R_j}\Big\} \tag{29}$$

where $f_1(z_j)$ and $f_2(z_j)$ are defined in Eq. (26). The derivatives of $f(z)$ and $f_0(z)$ are listed in Eq. (23).

The normal displacement of the negative face of the penny-shaped crack can be found in a similar manner.

2. Numerical result

Based on Fabrikant (1991) and the results obtained here, the real shapes of a penny-shaped crack due to an arbitrary point force P_z are compared in Fig. 3 between an infinite PZT-4 body (the material constants are given in Hou, 2000) and a corresponding infinite elastic body PZT-4 (E), which has the same elastic constants as PZT-4. Meanwhile, the deformation due to point charge Q for a penny-shaped crack in an infinite PZT-4 body is shown in Fig. 4.

From the figures above, we can see that the opening displacements of a penny-shaped crack in an infinite PZT-4 body are always less than those in an infinite PZT-4(E) body. So that the crack in piezoelectric materials will experience a less serious situation than that in the pure elasticity case. This is known as the crack arrest phenomenon. The contribution of charge Q to the opening displacements of a circular crack lies on the amount, character and location of the charge, so we can protect the piezoelectric components by means of controlling these charge parameters. In addition, since the crack opening displacement is zero for $z = 0$, and tends to zero for $z \to \infty$, there should be a location for external loading where they produce maximum crack opening.

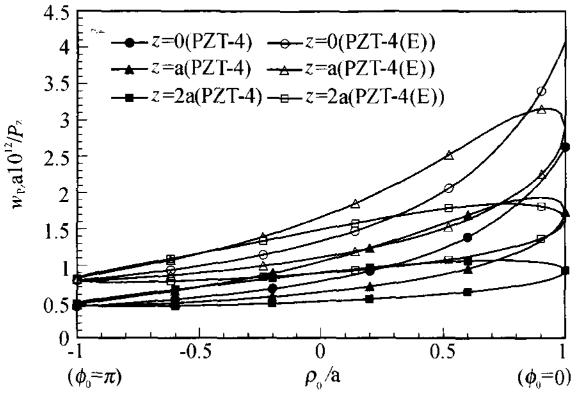

Fig.3 The crack shape with point force P_z applied at $(1.5a, 0, z)$

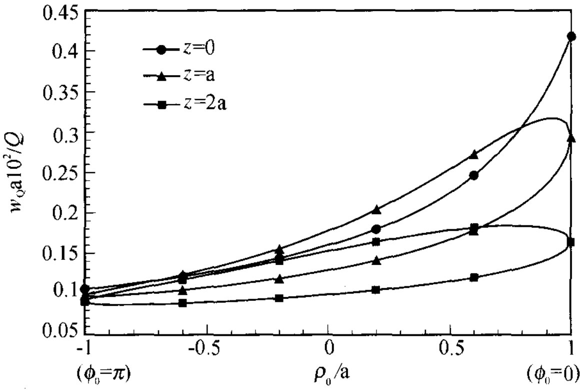

Fig.4 The crack shape with point charge-Q applied at $(1.5a, 0, z)$

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