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BACKWARD WELLPOSEDNESS OF NONUNIFORM TIMOSHENKO BEAM EQUATION*

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Abstract: In this paper, we consider the Timoshenko equation of a nonuniform beam, with clamped boundary condition at one end and with feedback controls at the other end. It is proved that the system is backward wellposedness when the feedback controls are weak enough.

Key words: backward wellposedness, nonuniform beam, Timoshenko equation, C_0-semigroup
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INTRODUCTION

The purpose of this work is to investigate the back wellposedness of a nonuniform beam equation with boundary controls. It is well known that if the cross-section dimensions of the beam is not negligible compared with its length, then it is necessary to consider the effect of the rotatory inertia and, moreover, if the deflection due to shear is not negligible either, the dynamic model of the beam is called a Timoshenko beam. Consider the initial-boundary value problem for the Timoshenko equation of a nonuniform beam:

$$\begin{cases} \rho w_{tt} - (Kw')' + (K\Psi')' = 0, \\ \quad \text{in } (0,1) \times R^+, \\ I_\rho \Psi_{tt} - (EI\Psi')' + K(\Psi - w') = 0, \\ \quad \text{in } (0,1) \times R^+, \\ w(0,t) = \Psi(0,t) = 0, \\ K(1)\Psi(1,t) - K(1)w'(1,t) = \alpha w_t(1,t), \\ -EI(1)\Psi'(1,t) = \beta\Psi_t(1,t), \\ w(0,t) = w_1^0,\ w_t(x,0) = w_2^0, \\ \Psi(x,0) = \Psi_1^0,\ \Psi_t(x,0) = \Psi_2^0 \end{cases} \tag{1}$$

where the prime represents the derivative with respect to the spacial variable x, $w(x,t)$, $\Psi(x,t)$, $\rho(x)$, $K(x)$, $I_\rho(x)$, $EI(x)$ are the transversal displacement, rotation angle, density, shear elasticity modulus, rotatory inertia, flexural rigidity modulus. α, β are positive constants.

In recent years, the boundary feedback stabilization of large space flexible structures has attracted much attention of many authors (Chen et al., 1987; Rao, 1996). Feng et al. (1998), Kim and Renardy (1987) and Shi et al. (1998) considered the boundary feedback stabilization of a uniform Timoshenko beam. This paper is concerned with a nonuniform Timoshenko beam with boundary controls. Backward wellposedness is not certainly true for general wave equations. Using semigroup (Pazy, 1983) and multiplier method (Komornik, 1994), on conditon α and β are positive constants small enough, we will apply one important result in Liu et al. (1998) to prove that the system is backward wellposedness.

PRELIMINARIES

Let $W = \{w \in H^1(0,1) \mid w(0) = 0\}$, where $H^1(0,1)$ is the Sobolev space of order 1 (Adams, 1975). Let

$$H = L_\rho^2(0,1) \times L_{I_\rho}^2(0,1),$$

$$\| v_1, v_2 \|_H = \left(\int_0^1 (\rho \mid v_1 \mid^2 + I_\rho \mid v_2 \mid^2) \mathrm{d}x\right)^{\frac{1}{2}},$$

and

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$V = W \times W$,

$$\| v_1, v_2 \|_V = \left(\int_0^1 (K \mid v_2 - v'_1 \mid^2 + EI \mid v'_2 \mid^2) \mathrm{d}x\right)^{\frac{1}{2}}.$$

Then, both H and V are (complex) Hilbert spaces with the deduced inner product $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_V$. Define $\Theta = V \times H$ with the norm

$$\| (u_1, u_2, v_1, v_2) \|_\Theta = (\| (u_1, u_2) \|_V^2 + \| (v_1, v_2) \|_H^2)^{\frac{1}{2}}$$

Then Θ is a (complex) Hilbert space-the finite energy state space with the deduced inner product $\langle \cdot, \cdot \rangle$.

The energy of the solution to (1) at time t is

$$E(t) = \frac{1}{2}\int_0^1 (K \mid \Psi - w' \mid^2 + EI \mid \Psi' \mid^2 + \rho \mid w_t \mid^2 + I_\rho \mid \Psi_t \mid^2) \mathrm{d}x. \quad (2)$$

Hereafter, we omit the symbol of transposition and use a row vector to denote the actual column vector. Define in Θ

$$D(\Lambda) = \left\{ (w_1, \Psi_1, w_2, \Psi_2) \cdot \left| \begin{array}{l} w_1, \Psi_1, w_2, \Psi_2 \in W \\ Kw'_1, EI\Psi'_1 \in H^1(0,1) \\ K(1)\Psi_1(1) - K(1)w'_1(1) = \alpha w_2(1) \\ - EI(1)\Psi'_1 = \beta \Psi_2(1) \end{array} \right. \right\} \quad (3)$$

$$\Lambda(w_1, \Psi_1, w_2, \Psi_2) = \left(w_2, \Psi_2, \frac{1}{\rho}[(Kw'_1)' - (K\Psi_1)'], \frac{1}{I_\rho}[(EI\Psi'_1)' - K(\Psi_1 - w'_1)]\right) \quad (4)$$

Then the system (1) can be written as an abstract evolution equation in Θ

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \Lambda z, \ z(0) = z_0 \quad (5)$$

where $z = (w_1, \Psi_1, w_2, \Psi_2)$, $w_2 = w_{1t}$, $\Psi_2 = \Psi_{1t}$, $z_0 = (w_1^0, \Psi_1^0, w_2^0, \Psi_2^0)$.

The following lemma can be directly verified. The proof is omitted.

Lemma 2.1 Λ generates a C_0-semigroup, $e^{t\Lambda}$, of contractions on Θ, and Λ has compact resolvent.

Then (1) admits the finite energy solution

$$(w(\cdot, t), \Psi(\cdot, t), w_t(\cdot, t), \Psi_t(\cdot, t)) = e^{t\Lambda}(w_1^0, \Psi_1^0, w_2^0, \Psi_2^0).$$

for every initial value $(w_1^0, \Psi_1^0, w_2^0, \Psi_2^0) \in \Theta$.

In order to prove our result, we will employ the following lemmas

Lemma 2.2 The following conditions are equivalent (Zabczyk, 1976):

(1) The system (Λ, B) is exactly controllable.

(2) There is a $T > 0$ such that for every y_0, $y_1 \in \Theta$ there exists $u(\cdot) \in L^2(0, T; U)$ for which $y(u, 0) = y_0$, $y(u, T) = y_1$, where U is a Hilbert space.

(3) (observability inequality) There exist T, $\delta > 0$ such that

$$\int_0^T \| B^* e^{t\Lambda} y \|_U^2 \mathrm{d}t \geq \delta \| y \|_\Theta^2 \quad \forall y \in \Theta.$$

Lemma 2.3 If the system (Λ, B) with some bounded B is exactly controllable and Λ has compact resolvent, then Λ generates a C_0 group on Θ (Liu and Russell, 1998).

BACKWARD WELLPOSEDNESS

We have the following result.

Theorem 3.1 Under the conditions of ρ, I_ρ, K, $EI \in C^1[0,1]$; ρ, I_ρ, K, $EI \geq c > 0$; α and β are positive constants small enough, the system (1) is backward wellposedness, i.e., Λ generate a C_0 group.

Proof We note that $e^{t\Lambda}$ extends to a C_0 group if and only if $e^{t\Lambda^*}$ does. Thus, from Lemma 2.2 and Lemma 2.3, we need to establish the observability inequality

$$\int_0^T \| e^{t\Lambda}(w_1^0, \Psi_1^0, w_2^0, \Psi_2^0) \|_\Theta^2 \mathrm{d}t \geq \delta \| (w_1^0, \Psi_1^0, w_2^0, \Psi_2^0) \|_\Theta^2,$$

i.e.

$$\int_0^T E(t) \mathrm{d}t \geq \delta E(0). \quad (6)$$

for some T, $\delta > 0$ and all initial values $(w_1^0, \Psi_1^0, w_2^0, \Psi_2^0) \in \Theta$. Since $D(\Lambda)$ is dense in Θ, it suffices to establish (6) for all $(w_1^0, \Psi_1^0, w_2^0,$

$\Psi_2^0) \in D(\Lambda)$. For such an initial value, the solution to (1) has the following regularity:
$w(\cdot,t), \Psi(\cdot,t) \in C^2((0,\infty);L^2(0,1)) \cap C^1([0,\infty);W) \cap C([0,\infty);H^2(0,1))$

Moreover, we may assume without loss of generality that the solution $(w(x,t),\Psi(x,t))$ to (1) is real-valued because both $(\mathrm{Re}w(x,t), \mathrm{Re}\Psi(x,t))$ and $(\mathrm{Im}\, w(x,t), \mathrm{Im}\Psi(x,t))$ are solutions to (1) whenever $(w(x,t),\Psi(x,t))$ is.

From (1) and (2), we have

$$E_t = -w_t(K\Psi - Kw')\,|_0^1 + EI\Psi_t\Psi'\,|_0^1 = -\alpha w_t^2(1,t) - \beta\Psi_t^2(1,t),$$

$$E(0) - E(t) = \int_0^t [\alpha w_t^2(1,s) + \beta\Psi_t^2(1,s)]\mathrm{d}s.$$

It follows that

$$TE(0) - \int_0^T E(t)\mathrm{d}t = \int_0^T (T-t)[\alpha w_t^2(1,t) + \beta\Psi_t^2(1,t)]\mathrm{d}t. \tag{7}$$

Suppose (6) is false, then there is a sequence of initial values $\theta_n = (w_{1n}^0, \Psi_{1n}^0, w_{2n}^0, \Psi_{2n}^0)$, $n = 1,2,\cdots$, and the corresponding solutions $(w_n(\cdot,t), \Psi_n(\cdot,t), w_{nt}(\cdot,t), \Psi_{nt}(\cdot,t))$, such that

$$\|\theta_n\|_\Theta = 1,\ \int_0^T E_n(t)\mathrm{d}t \to 0. \tag{8}$$

From (7) and (8), we have

$$\lim_{n\to\infty}\int_0^T (T-t)[\alpha w_{nt}^2(1,t) + \beta\Psi_{nt}^2(1,t)]\mathrm{d}t = \frac{T}{2} \tag{9}$$

We now multiply the first equation of (1) by $(T-t)xw'$, then integrate by parts

$$\int_0^T\int_0^1 \rho w_{tt}(T-t)xw'\mathrm{d}x\mathrm{d}t = -T\int_0^1 \rho x w_1^{0\prime} w_2^0\mathrm{d}x + \int_0^T\int_0^1[\rho x w_t w' - \frac{1}{2}(T-t)\rho x(w_t^2)']\mathrm{d}x\mathrm{d}t =$$
$$-T\int_0^1 \rho x w_1^{0\prime} w_2^0\mathrm{d}x - \frac{1}{2}\int_0^T (T-t)\rho(1)w_t^2(1,t)\mathrm{d}t + \int_0^T\int_0^1[\rho x w_t w' + \frac{1}{2}(T-t)(\rho+\rho' x)w_t^2]\mathrm{d}x\mathrm{d}t \tag{10}$$

By direct computations, we have

$$-\int_0^1[(Kw')' + (K\Psi)']xw'\mathrm{d}x = -\int_0^1 Kxw''w'\mathrm{d}x - \int_0^1 K'xw'^2\mathrm{d}x + \int_0^1 (K\Psi)'xw'\mathrm{d}x =$$
$$-\frac{1}{2}K(1)w'^2(1,t) + \frac{1}{2}\int_0^1 (K - K'x)w'^2\mathrm{d}x + \int_0^1 Kxw'\Psi'\mathrm{d}x + \int_0^1 K'xw'\Psi\mathrm{d}x, \tag{11}$$

Similarly, we obtain

$$\int_0^T\int_0^1 I_\rho\Psi_{tt}(T-t)x\Psi'\mathrm{d}x\mathrm{d}t = -T\int_0^1 I_\rho x\Psi_1^{0\prime}\Psi_2^0\mathrm{d}x - \frac{1}{2}\int_0^T (T-t)I_\rho(1)\Psi_t^2(1,t)\mathrm{d}t + \int_0^T\int_0^1[I_\rho x\Psi_t\Psi' + \frac{1}{2}(T-t)(I_\rho + I'_\rho x)\Psi_t^2]\mathrm{d}x\mathrm{d}t \tag{12}$$

$$\int_0^1[-(EI\Psi')' + K(\Psi - w')]x\Psi'\mathrm{d}x = -\frac{1}{2}EI(1)\Psi'^2(1,t) + \frac{1}{2}\int_0^1(EI - EI'x)\Psi'^2\mathrm{d}x + \int_0^1 Kx\Psi\Psi'\mathrm{d}x - \int_0^1 Kxw'\Psi'\mathrm{d}x. \tag{13}$$

From (10) – (13), we have

$$\frac{1}{2}\int_0^T (T-t)[\rho(1)w_t^2(1,t) + I_\rho(1)\Psi_t^2(1,t)]\mathrm{d}t \leqslant -T\int_0^1 \rho x w_1^{0\prime} w_2^0\mathrm{d}x - T\int_0^T I_\rho x\Psi_1^{0\prime}\Psi_2^0\mathrm{d}x + C_2\int_0^T E(t)\mathrm{d}t \leqslant C_1 TE(0) + C_2\int_0^T E(t)\mathrm{d}t, \tag{14}$$

where C_1 and C_2 are positive constants independent of $(w_1^0, \Psi_1^0, w_2^0, \Psi_2^0)$. Therefore, we can deduce that

$$\frac{1}{2}\int_0^T (T-t)[\rho(1)w_{nt}^2(1,t) + I_\rho(1)\Psi_{nt}^2(1,t)]\mathrm{d}t \leqslant C_1 TE_n(0) + C_2\int_0^T E_n(t)\mathrm{d}t = \frac{1}{2}C_1 T + C_2\int_0^T E_n(t)\mathrm{d}t,$$

$$\lim_{n\to\infty}\int_0^T (T-t)[\rho(1)w_{nt}^2(1,t) + I_\rho(1)\cdot\Psi_{nt}^2(1,t)]\mathrm{d}x \leqslant \frac{1}{2}C_1 T \tag{15}$$

It is easy to see that when $\max(\alpha,\beta) < \frac{1}{C_1}\min(\rho(1), I_\rho(1))$, (15) is contradictive with (9).

Thus we have proved the above theorem.

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