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ALMOST SURE CONTINUITY OF INFINITE SERIES OF INDEPENDENT TWO-PARAMETER ORNSTEIN-UHLENBECK PROCESSES

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Abstract: This study on the conditions for the convergence and almost-sure continuity of infinite series of independent two-parameter Ornstein-Uhlenbeck processes yielded sufficient conditions similar to that for independent one-parameter Ornstein-Uhlenbeck processes.

Key words: two-parameter Ornstein-Uhlenbeck processes，infinite series，almost-sure continuity
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INTRODUCTION

Let $\{Y(t); -\infty < t < \infty\} = \{X_k(t); -\infty < t < \infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck（OU）processes with coefficients $\gamma_k \geqslant 0$ and $\lambda_k > 0$，i.e.，$X_k(\cdot)$ is a stationary，mean zero Gaussian process with $EX_k(s)X_k(t) = (\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|)$，$k = 1,2,\cdots$.

The process $Y(\cdot)$ was first introduced by Dawson（1972）as the stationary solution of the infinite array of stochastic differential equations

$$dX_i(t) = -\lambda_i X_i(t)dt + (2\gamma_i)^{1/2}dW_i(t) \quad (i = 1,2,\cdots), \tag{1}$$

where $\{W_i(t); -\infty < t < \infty\}$ are independent Wiener processes（cf. also Dawson，1975 and Walsh，1981）. Such processes have been extensively studied since they were first introduced by Dawson（1972）.

A two-parameter Ornstein-Uhlenbeck process（OUP_2）is an extension of this one-parameter process. A sequence of independent two-parameter Ornstein-Uhlenbeck processes $\{X_k(t,v); t \geqslant 0, v \geqslant 0\}$ is defined by

$$X_k(t,v) = e^{-\alpha_k t - \beta_k v}\{X_k + \sigma_k\int_0^t\int_0^v e^{\alpha_k x + \beta_k y}dW_k(x,y)\}, \quad t \geqslant 0, v \geqslant 0, \tag{2}$$

where $\alpha_k > 0$ and $\beta_k > 0$ are two coefficients，$\{W_k(\cdot,\cdot)\}_{k=1}^{\infty}$ is a sequence of independent two-parameter Wiener process，$\{X_k; k = 1,2,\cdots\}$ is a sequence of independent random variables and independent of $\{W_k(\cdot,\cdot)\}_{k=1}^{\infty}$. This definition was introduced by Wang（1983）. If X_k is a normal variable，then $X_k(\cdot,\cdot)$ is a Gaussian process for any $k \geqslant 1$.

Wang（1983）investigated some Markov properties of OUP_2. Chen（1989）studied the sample path properties of OUP_2 by giving the Hausdoff dimension of the graph and image sets of this process. Lin（1995a，b，c）obtained some direct depictions of sample path properties of this process by establishing its Lévy's moduli of continuity and limit theorems on its large increments.

Walsh（1981）presented a mathematical model for neural response and investigated many analytic properties of processes. One of the processes of interest in his study was the infinite series of independent OU coordinate processes of $Y(\cdot)$，namely the process $X(\cdot)$ defined by

$$\{X(t); -\infty < t < \infty\} = \{\sum_{k=1}^{\infty}X_k(t); -\infty < t < \infty\} \tag{3}$$

where the $X_k(\cdot)$ are the Ornstein-Uhlenbeck components of $Y(\cdot)$.

Csáki，Csörgö，Lin and Révész（1991）studied the properties of the process $X(\cdot)$. We present the results on the continuity of $X(\cdot)$ here.

Theorem A Let $X(\cdot)$ be defined as in Eq.（3）and

$\{X(t,n);\ -\infty<t<\infty,\ n=1,2,\cdots\}=\{\sum_{k=1}^{n}X_k(t);\ -\infty<t<\infty,\ n=1,2,\cdots\}$. (4)

If

$$\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}<\infty, \tag{5}$$

then for any fixed t, $X(t)$ is an r.v. with mean zero and variance $\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}$. Moreover assume that for some $\delta>0$

$$\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(\log(\lambda_k\vee e))^{1+\delta}<\infty. \tag{6}$$

Then $X(t,n)\to X(t)$ uniformly in t over any finite interval with probability one, i.e. for any $\varepsilon>0$, $T>0$ and for almost all $\omega\in\Omega$, there exists an integer $n_0=n_0(\varepsilon,T,\omega)$ such that

$$\sup_{|t|\leqslant T}|X(t,n,\omega)-X(t,\omega)|\leqslant\varepsilon \tag{7}$$

whenever $n\geqslant n_0$. As a consequence, the Gaussian process $\{X(t);\ -\infty<t<\infty\}$ is continuous with probability one.

It can be shown that the OUP_2 from Eq.(2) is a process whose increments are neither independent nor stationary. What about the results similar to Theorem A? We will discuss this problem in the next section.

RESULT AND PROOF

Throughout the paper, c or C denotes a positive constant, which may take different values at different places. $\|X\|_2$ denotes $(EX^2)^{1/2}$. Let $X_k(\cdot,\cdot)$ be defined as in Eq.(2). The convergence of $\sum_{k=1}^{\infty}e^{-\alpha_k t-\beta_k v}X_k$ can be easily discussed. For examples

1) If $\sum_{k=1}^{\infty}\frac{1}{\alpha_k\beta_k}<\infty$, then $\sum_{k=1}^{\infty}e^{-\alpha_k t-\beta_k v}X_k$ converges uniformly in $t\geqslant t_0$, $v\geqslant v_0$ for any $t_0>0$, $v_0>0$ with probability one.

2) If $\sum_{k=1}^{\infty}EX_k^2<\infty$, then $\sum_{k=1}^{\infty}e^{-\alpha_k t-\beta_k v}X_k$ converges uniformly in t, v over any bounded rectangle with probabilily one.

Therefore we assume $X_k=0\ (k=1,2,\cdots)$ in this section. Now letting

$$X(t,v,n)=\sum_{k=1}^{n}X_k(t,v),\quad t\geqslant0,v\geqslant0,n\geqslant1, \tag{8}$$

$$X(t,v)=\sum_{k=1}^{\infty}X_k(t,v),\quad t\geqslant0,v\geqslant0.$$

Theorem 1 Assume

$$\sum_{k=1}^{\infty}\frac{\sigma_k^2}{\alpha_k\beta_k}<\infty, \tag{9}$$

then for any fixed $t\geqslant0$, $v\geqslant0$, $X(t,v)$ is a normal r.v. with mean zero and variance not greater than $\sum_{k=1}^{\infty}\frac{\sigma_k^2}{4\alpha_k\beta_k}$. Moreover, assume that for some $\delta>0$

$$\sum_{k=1}^{\infty}\frac{\sigma_k^2}{\alpha_k\beta_k}(\log(\alpha_k\beta_k\vee e))^{1+\delta}<\infty. \tag{10}$$

Then $X(t,v,n)\to X(t,v)$ uniformly in t, v over any bounded rectangle with probability one, i.e. for any $\varepsilon>0$, $T>0$ and for almost all $\omega\in\Omega$, there exists an integer $n_0=n_0(\varepsilon,T,\omega)$ such that

$$\sup_{0\leqslant t,v\leqslant T}|X(t,v,n,\omega)-X(t,v,\omega)|\leqslant\varepsilon, \tag{11}$$

whenever $n\geqslant n_0$. As a consequence, the process $\{X(t,v);\ t\geqslant0,v\geqslant0\}$ is continuous with probability one.

The proof of Theorem 1 depends on the following lemma of Fernique (1964).

Lemma 2 Let $T=[0,1]^k$ and assume that X is a centered Gaussian process with covariance function Γ. For any $(s,t)\in T\times T$ define $d(s,t)=|s-t|=\sup_{1\leqslant i\leqslant k}|s_i-t_i|$. Let $\varphi:[0,1]\to R_+$ be a function defined as

$$\varphi(h)=\sup_{\substack{(s,t)\in T\times T\\ d(s,t)\leqslant h}}\|X_s-X_t\|_2. \tag{12}$$

Then there exists a constant K such that

$$E\sup_{t\in T}|X_t|\leqslant K\{\sup_{t\in T}\|X_t\|_2+\int_1^{\infty}\varphi(e^{-x^2})dx\}. \tag{13}$$

Furthermore, if $\int_1^{\infty}\varphi(e^{-x^2})dx<\infty$, then X is uniformly continuous almost surely (here, continuity is to mean d-continuity).

Proof of Theorem 1 Without lossing generality, we can assume $T=1$. From Eqs.(2), (3), it is easy to verify that for any $t_1\geqslant0$, $v_1\geqslant0$, $t_2\geqslant0$, $v_2\geqslant0$, $t\geqslant0$, $v\geqslant0$,

$$EX_k(t_1,v_1)X_k(t_2,v_2)=e^{-\alpha_k(t_1+t_2)-\beta_k(v_1+v_2)}\cdot\frac{\sigma_k^2(e^{2\alpha_k(t_1\wedge t_2)}-1)(e^{2\beta_k(v_1\wedge v_2)}-1)}{4\alpha_k\beta_k} \tag{14}$$

$$EX^2(t,v) = \sum_{k=1}^{\infty} e^{-2\alpha_k t - 2\beta_k v} \frac{\sigma_k^2 (e^{2\alpha_k t} - 1)(e^{2\beta_k v} - 1)}{4\alpha_k\beta_k} = \sum_{k=1}^{\infty} \frac{\sigma_k^2 (1 - e^{-2\alpha_k t})(1 - e^{-2\beta_k v})}{4\alpha_k\beta_k}. \tag{15}$$

It follows that for any $t \geqslant 0$, $v \geqslant 0$, from Eqs. (9) and (15) we have

$$EX^2(t,v) \leqslant \sum_{k=1}^{\infty} \frac{\sigma_k^2}{4\alpha_k\beta_k} < \infty,$$

then $\sum_{k=1}^{\infty} X_k(t,v)$ converges with probability 1. Therefore the first part of Theorem 1 has been obtained.

In order to verify Eq. (11), on account of Itô-Nisio's theorem, (cf. Theorem 6.1 of Ledoux and Talagrand 1991), it suffices to prove that

$$\sup_{0 \leqslant t, v \leqslant T} |X(t,v,n) - X(t,v)| = \sup_{0 \leqslant t, v \leqslant T} \left| \sum_{k=n+1}^{\infty} X_k(t,v) \right|.$$

converges to zero in probability as $n \to \infty$. Thus we want to show that for any $\varepsilon > 0$

$$\lim_{n \to \infty} P\{\sup_{0 \leqslant t, v \leqslant T} \left| \sum_{k=n+1}^{\infty} X_k(t,v) \right| > \varepsilon\} = 0.$$

The latter, in turn, will be established by showing that, under the condition Eq. (10), for any $\varepsilon > 0$ and $0 < \eta < 1$ there exists $n_0 = n_0(\varepsilon, \eta)$ such that

$$P\{\sup_{0 \leqslant t, v \leqslant T} \left| \sum_{k=n+1}^{m} X_k(t,v) \right| > \varepsilon\} \leqslant \eta, \tag{16}$$

whenever $m > n \geqslant n_0$. Let

$$X_{m,n}(t,v) = X(t,v,m) - X(t,v,n) = \sum_{k=n+1}^{m} X_k(t,v). \tag{17}$$

Consider the increments

$$X_k(t+s,v) - X_k(t,v) = \sigma_k e^{-\alpha_k(t+s) - \beta_k v}(1 - e^{\alpha_k s}) \cdot \int_0^{t+s}\int_0^v e^{\alpha_k x + \beta_k y} \mathrm{d}W_k(x,y) + \sigma_k e^{-\alpha_k t - \beta_k v} \int_t^{t+s}\int_0^v e^{\alpha_k x + \beta_k y} \mathrm{d}W_k(x,y).$$

Hence

$$E(X_k(t+s,v) - X_k(t,v))^2 = \frac{\sigma_k^2}{4\alpha_k\beta_k}(1 - e^{-2\beta_k v})\{(1 - e^{-2\alpha_k t}) \cdot (1 - e^{-\alpha_k s})^2 + 1 - e^{-2\alpha_k s}\} \leqslant \frac{\sigma_k^2}{2\alpha_k\beta_k}(1 - e^{-2\alpha_k s}),$$

$$E(X_k(t,v+u) - X_k(t,v))^2 = \frac{\sigma_k^2}{4\alpha_k\beta_k}(1 - e^{-2\alpha_k t})\{(1 - e^{-2\beta_k v}) \cdot (1 - e^{-\beta_k u})^2 + 1 - e^{-2\beta_k u}\} \leqslant \frac{\sigma_k^2}{2\alpha_k\beta_k}(1 - e^{-2\beta_k u}), \tag{18}$$

and

$$E(X_k(t+s,v+u) - X_k(t,v))^2 \leqslant 2E(X_k(t+s,v+u) - X_k(t+s,v))^2 + 2E(X_k(t+s,v) - X_k(t,v))^2 \leqslant \frac{\sigma_k^2}{\alpha_k\beta_k}[(1 - e^{-2\alpha_k s}) + (1 - e^{-2\beta_k u})]. \tag{19}$$

Now let

$$\varphi(h) = \sup_{(t,v)\in[0,1]^2, 0 \leqslant u \vee s \leqslant h} \| X_{m,n}(t+s,v+u) - X_{m,n}(t,v) \|_2$$

and $K_1 = \{k: \alpha_k < e^{\frac{u^2}{2}}\}$, $K_2 = \{k: \beta_k < e^{\frac{u^2}{2}}\}$. It is easy to see that

$$\varphi^2(h) \leqslant \sum_{k=n+1}^{m} \frac{\sigma_k^2}{\alpha_k\beta_k}[(1 - \exp(-2\alpha_k h)) + (1 - \exp(-2\beta_k h))],$$

then

$$\int_1^{\infty} \left(\sum_{k=n+1, k \in K_1 \cap K_2}^{m} \frac{\sigma_k^2}{\alpha_k\beta_k}((1 - \exp(-2\alpha_k e^{-u^2})) + (1 - \exp(-2\beta_k e^{-u^2}))) \right)^{1/2} \mathrm{d}u \leqslant$$

$$\int_1^{\infty} \left(\sum_{k=n+1}^{m} \frac{2\sigma_k^2}{\alpha_k\beta_k}(\alpha_k e^{-u^2} I\{\alpha_k < e^{u^2/2}\} + \beta_k e^{-u^2} I\{\beta_k < e^{u^2/2}\}) \right)^{1/2} \mathrm{d}u \leqslant$$

$$\left(\sum_{k=n+1}^{m} \frac{2\sigma_k^2}{\alpha_k\beta_k} \right)^{1/2} \int_1^{\infty} 2e^{-\frac{1}{2}u^2} \mathrm{d}u \leqslant c\left(\sum_{k=n+1}^{m} \frac{\sigma_k^2}{\alpha_k\beta_k} \right)^{1/2}, \text{ and}$$

$$\int_1^{\infty} \left(\sum_{k=n+1, k \in \overline{K_1 \cap K_2}}^{m} \frac{\sigma_k^2}{\alpha_k\beta_k}((1 - \exp(-2\alpha_k e^{-u^2})) + (1 - \exp(-2\beta_k e^{-u^2}))) \right)^{1/2} \mathrm{d}u \leqslant$$

$$\int_1^{\infty} \left(\sum_{k=n+1}^{m} \frac{2\sigma_k^2}{\alpha_k\beta_k}(I\{\alpha_k \geqslant e^{u^2/2}\} + \right.$$

$$I\{\beta_k \geqslant e^{u^2/2}\})\Big)^{1/2} du \leqslant$$

$$\int_1^\infty \left(\sum_{k=n+1}^m \frac{2\sigma_k^2}{\alpha_k\beta_k}((\log^+ \alpha_k \frac{2}{u^2})^{1+\delta} + (\log^+ \beta_k \frac{2}{u^2})^{1+\delta})\right)^{1/2} du \leqslant$$

$$\int_1^\infty \left(\sum_{k=n+1}^m \frac{2\sigma_k^2}{\alpha_k\beta_k}(\log^+ \alpha_k\beta_k)^{1+\delta}\left(\frac{2}{u^2}\right)^{1+\delta}\right)^{1/2} du \leqslant$$

$$c\left(\sum_{k=n+1}^m \frac{\sigma_k^2}{\alpha_k\beta_k}\log^+ \alpha_k\beta_k\right)^{1/2}.$$

It follows that

$$\int_1^\infty \varphi(e^{-u^2})du \leqslant c\left\{ \sum_{k=n+1}^m \frac{\sigma_k^2}{\alpha_k\beta_k}(\log(\alpha_k\beta_k \vee e))^{1+\delta}\right\}^{1/2} \qquad (20)$$

which together with Eq. (13) of Lemma 2 implies that

$$E \sup_{0\leqslant t, v\leqslant 1} |X_{m,n}(t,v)| \leqslant C\left\{ \sum_{k=n+1}^m \frac{\sigma_k^2}{\alpha_k\beta_k}(\log(\alpha_k\beta_k \vee e))^{1+\delta}\right\}^{1/2} =: C\eta_{m,n}.$$

Then

$$P(\sup_{0\leqslant t, v\leqslant 1} |X_{m,n}(t,v)| > \epsilon) \leqslant \frac{C}{\epsilon}\eta_{m,n} < \eta \qquad (21)$$

whenever $m > n \geqslant n_0$, since $\eta_{m,n} \to 0 (n \to \infty)$ by condition (10). As a consequence, the process $\{X(t,v); t \geqslant 0, v \geqslant 0\}$ is continuous with probability one.

On the other hand, we can use Lemma 2 directly on the continuity of $X(t,v)$. Putting

$$\psi(h) = \sup_{(t,v)\in[0,1]^2, 0\leqslant u \vee s \leqslant h} \| X(t+s, v+u) - X(t,v) \|_2,$$

With the same way of proving Eq. (20), we have

$$\int_1^\infty \psi(e^{-u^2})du \leqslant C\left\{ \sum_{k=1}^\infty \frac{\sigma_k^2}{\alpha_k\beta_k}(\log\alpha_k\beta_k \vee e))^{1+\delta}\right\}^{1/2} < \infty,$$

we thus obtain that $X(t,v)$ is almost surely uniformly continuous by using Lemma 2, and Theorem 1 is proved.

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References

Chen, X., 1989. The Hausdoff dimension of image and graph sets of a two-parameter Ornstein-Uhlenbeck process. *Acta Math. Sinica*, **32**: 433 – 438 (in Chinese, with English abstract).

Csáki, E., Csörgö, M., Lin, Z. Y., et al., 1991. On the infinite series of independent Ornstein-Uhlenbeck processes. *Stochastic Processes their Appl.*, **39**: 25 – 44.

Dawson, D. A., 1972. Stochastic evolution equations. *Math. Biosciences*, **15**: 287 – 316.

Dawson, D. A., 1975. Stochastic evolutions and related measure processes. *J. Multivariate Anal.*, **5**: 1 – 52.

Fernique, X., 1964. Continuité des processus gaussiens. *C. R. Acad. Sci. Paris*, **258**: 6059 – 6069.

Ledoux, M., Talagrand, M., 1991. Probability in Banach Spaces-Isoperimetry and processes. Springer Verlag, New York.

Lin, Z. Y., 1984. On how big are the increments of a two-parameter Wiener process. *Science in China (Ser. A)*, **12**: 1065 – 1073 (in Chinese).

Lin, Z. Y., 1991. Two-parameter Gaussian processes with kernel. *Acta Math. Sinica*, **34**: 12 – 26 (in Chinese).

Lin, Z. Y., 1995a. On moduli of continuity for a two-parameter Ornstein-Uhlenbeck process. *Acta Math. Hungar*, **66**: 345 – 360.

Lin, Z. Y., 1995b. On large increments of a two-parameter O-U process. *Acta Math. Sinica, New Series*, **11** (Special Issue): 59 – 68.

Lin, Z. Y., 1995c. On large increments of infinite series of Ornstein-Uhlenbeck processes. *Stochastic Processes their Appl.*, **60**: 161 – 169.

Walsh, J. B., 1981. A stochastic model of neural response. *Adv. Appl. Probab.*, **13**: 231 – 281.

Wang, Z. K., 1983. The two-parameter Ornstein-Uhlenbeck Process. *Acta Math. Scientia*, **3**: 395 – 406 (in Chinese).

Zhang, L. X., 1995. The exact moduli of non-differentiability of the infinite series of Ornstein-Uhlenbeck processes. *Chin. Ann. Math.*, **16A**: 263 – 268 (in Chinese).