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Vector refinement equation and subdivision schemes in L_p spaces*

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Abstract: In this paper we will first prove that the nontrivial L_p solutions of the vector refinement equation exist if and only if the corresponding subdivision scheme with a suitable initial function converges in L_p without assumption of the stability of the solutions. Then we obtain a characterization of the convergence of the subdivision scheme in terms of the mask. This gives a complete answer to the existence of L_p solutions of the refinement equation and the convergence of the corresponding subdivision schemes.

Key words: Refinement equations, Subdivision schemes, Joint spectral radius.
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INTRODUCTION

Let $\boldsymbol{\Phi}$ be a function vector $\boldsymbol{\Phi} = (\phi^1, \cdots, \phi^r)^T$ on $\boldsymbol{R}$, $\{a(\alpha)\}$ be a finitely supported sequence of $r \times r$ matrices. An equation of the form

$$\boldsymbol{\Phi} = \sum a(\alpha)\boldsymbol{\Phi}(2\cdot - \alpha) \qquad (1)$$

is called a vector refinement equation. The sequence a is called refinement mask.

We consider the solution of Eq. (1), which is called a refinable function vector. Let

$$M := \sum a(\alpha)/2$$

As usual, we assume that the $r \times r$ matrix M has the following form

$$M = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix} \qquad (2)$$

where $\lim_{n\to\infty} J^n = 0$.

It was proved by Heil and Colella (1996) that if M satisfies Eq. (2) there exists a unique vector $\boldsymbol{\Phi}$ of compactly supported distributions such that $\boldsymbol{\Phi}$ satisfies the refinement equation Eq. (1) and $\hat{\boldsymbol{\Phi}}(0) = (1, 0, \cdots, 0)^T$, where $\hat{\boldsymbol{\Phi}}(\xi) = (\hat{\phi}^1(\xi), \cdots, \hat{\phi}^r(\xi))^T$ denotes the Fourier transform. Such a solution is called the normalized solution of Eq. (1).

In order to study the solution of the vector refinement equation in $(L_p(\boldsymbol{R}))^r$ $(1 \leqslant p \leqslant \infty)$ we define a linear operator:

$$T_a f = \sum a(\alpha) f(2\cdot - \alpha) \quad f \in (L_p(\boldsymbol{R}))^r \qquad (3)$$

The operator T_a is called cascade operator, and the iteration process $T_a^n f$ is called a subdivision scheme associated with the mask a. Choose an initial function vector with compact support $f_0 \in (L_p(\boldsymbol{R}))^r$ satisfying the following moment conditions of order 1:

$$e_1^T \hat{f}_0(2k\pi) = \delta_{0k} \qquad k \in \boldsymbol{Z}$$

where $e_j (j = 1, \cdots, r)$ denote the jth column of $r \times r$ identity matrix. If the subdivision scheme $T_a^n f_0$ associated with the mask a converges, that is, there exists a function vector $\boldsymbol{\Phi} \in (L_p(\boldsymbol{R}))^r$ such that

$$\lim_{n\to\infty} \| T_a^n f_0 - \boldsymbol{\Phi} \|_p = 0,$$

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we obtain that the function vector $\boldsymbol{\Phi}$ is the L_p-solution of the refinement equation. Conversely, Jia, Riemenschneider and Zhou (1998) proved the following result on the convergence of the subdivision scheme:

Theorem A Let a be a finitely supported sequence of $r \times r$ matrices such that the matrix $M = \sum a(\alpha)/2$ satisfies Eq.(2), and let $\boldsymbol{\Phi} = (\phi^1, \cdots, \phi^r)^{\mathrm{T}}$ be the normalized solution of the refinement Eq.(1). If $\phi^1, \cdots, \phi^r$ lie in $L_p(\boldsymbol{R})(1 \leqslant p \leqslant \infty)$, ($\phi^1, \cdots, \phi^r$ are continuous in the case $p = \infty$), and the shifts of the solution are L_p stable, then the subdivision scheme associated with the mask a converges to $\boldsymbol{\Phi}$ in the L_p norm.

When the shifts of the solution are not stable, the situation is completely different. Even if in the simplest case $r = 1$ the following example is given by Cavaretta et al. (1991) to illustrate that the corresponding subdivision scheme may diverge if the shifts of the solution of the refinement equation are not stable.

Example: Consider

$$\boldsymbol{\Phi} = \sum_{j=0}^{7} a(j)\boldsymbol{\Phi}(2 \cdot - j)$$

The symbol of the mask a is

$$\tilde{a}(z) = 1 + z - z^3 - z^4 + z^6 + z^7.$$

It is known that $\boldsymbol{\Phi}(x) = (B_1(x) + B_1(x-3) + B_1(x-6))/3$ is its solution. Here B_1 is the B spline of order 1. It is easy to verify that the solution is not stable. It was shown that the corresponding subdivision scheme diverges. Refer to Cavaretta et al. (1991) and Jia (1995) for the details.

From the above theorem we know that the stability of the solution is crucial for the convergence of the subdivision scheme. Because of the importance of the subdivision schemes for studying the L_p solutions of the vector refinement equation, the relation between the existence of $L_p(1 \leqslant p \leqslant \infty)$ solutions of the refinement equation and the convergence of the corresponding subdivision scheme has attracted much attention (cf. Jia and Han, 1998; Jia, 1995; Jia, 1997; Jia et al., 1998). Most related works are based on assumption of the solution stability.

In this paper we will first prove that the nontrivial L_p solutions of the vector refinement equation exist if and only if the corresponding subdivision scheme with a suitable initial function converges in L_p without assumption of the stability of the solutions. This gives complete comprehension about the existence of L_p solutions of the refinement equation and the convergence of the corresponding subdivision schemes. Then we also obtain a characterization of the convergence of the subdivision scheme in terms of the mask.

First we introduce some notations. For $1 \leqslant p \leqslant \infty$, $(L_p(\boldsymbol{R}))^r$ denotes the linear space of vectors $\boldsymbol{\Phi} = (\phi^1, \cdots, \phi^r)^{\mathrm{T}}$ such that $\phi^i \in L_p(\boldsymbol{R})$ $(1 \leqslant i \leqslant r)$. The norm on $(L_p(\boldsymbol{R}))^r$ is defined by

$$\|\boldsymbol{\Phi}\|_p := \begin{cases} \left(\sum_{j=1}^{r} \|\phi^j\|_p^p\right)^{1/p}, & \text{for } 1 \leqslant p < \infty, \\ \max_{1 \leqslant j \leqslant r} \|\phi^j\|_\infty, & \text{for } p = \infty. \end{cases}$$

We say that the shifts of $\boldsymbol{\Phi} \in (L_p(\boldsymbol{R}))^r$ are L_p stable, if there exist two positive constants C_1 and C_2 such that

$$C_2 \sum_{j=1}^{r} \|b_j\|_p \geqslant \left\| \sum_{j=1}^{r} \sum_{\alpha} b_j(\alpha)\phi^j(\cdot - \alpha) \right\|_p \geqslant C_1 \sum_{j=1}^{r} \|b_j\|_p, \quad \forall\, b_j \in l_p(\boldsymbol{Z}),$$

Another concept related to the stability is the linearly independent shifts. We say that the shifts of $\boldsymbol{\Phi} = (\phi^1, \cdots, \phi^r)^{\mathrm{T}}$ are linearly independent if for any $d_j \in l(\boldsymbol{Z})(j = 1, \cdots, r)$

$$\sum_{j=1}^{r} \sum_{\alpha} d_j(\alpha)\phi^j(\cdot - \alpha) = 0, \Rightarrow d_j = 0 (j = 1, \cdots, r)$$

As usual, let $l(\boldsymbol{Z})$ denote the linear space of all sequences on $\boldsymbol{Z}$, and let $l_0(\boldsymbol{Z})$ denote the linear space of all finitely supported sequences on $\boldsymbol{Z}$. By $l_p(\boldsymbol{Z} \to C^r)$ we denote the linear space of all $u: \boldsymbol{Z} \to C^r$ such that $u(\alpha) = (u_1(\alpha), \cdots, u_r(\alpha))^{\mathrm{T}}$ for some $u_1, \cdots, u_r \in l_p(\boldsymbol{Z})$. Obviously we can identify $l_p(\boldsymbol{Z} \to C^r)$ with $(l_p(\boldsymbol{Z}))^r$ by a canonical isomorphism. The norm of $u = (u_1, \cdots, u_r)^{\mathrm{T}} \in (l_p(\boldsymbol{Z}))^r$ is defined by

$$\| u \|_p := (\sum_{j=1}^{r} \| u_j \|_p)^{1/p}.$$

We also denote by $l_0(\boldsymbol{Z} \to C^{r \times t})$ the linear space of all finitely supported sequences of $r \times t$ matrices and identify $l_0(\boldsymbol{Z} \to C^{r \times t})$ with $(l_0(\boldsymbol{Z}))^{r \times t}$. Similarly we identify $l_0(\boldsymbol{Z} \to C^{r \times 1}) = l_0(\boldsymbol{Z} \to C^r)$ with $(l_0(\boldsymbol{Z}))^r$.

Furthermore we denote by $l_p(\boldsymbol{Z} \to C^{r \times t})$ the linear space of all matrix sequences $b: \boldsymbol{Z} \to C^{r \times t}$ such that $b(\alpha) = (b_{jk}(\alpha))_{1 \leqslant j \leqslant r, 1 \leqslant k \leqslant t}$ where $b_{jk} \in l_p(\boldsymbol{Z})(1 \leqslant j \leqslant r, 1 \leqslant k \leqslant t)$. We also identify $l_p(\boldsymbol{Z} \to C^{r \times t})$ with $(l_p(\boldsymbol{Z}))^{r \times t}$. The norm of $b = (b_{jk})_{1 \leqslant j \leqslant r, 1 \leqslant k \leqslant t}$ in $(l_p(\boldsymbol{Z}))^{r \times t}$ is defined by

$$\| b \|_p := \{\sum_{j=1}^{r} \sum_{k=1}^{t} \| b_{jk} \|_p^p\}^{1/p}$$

Let b_k be the sequence of $r \times 1$ matrix $(b_{jl})_{1 \leqslant j \leqslant r, l=k}$ corresponding to the kth column of the matrix sequence $b = (b_{jl})_{1 \leqslant j \leqslant r, 1 \leqslant l \leqslant t}$. It is easy to verify that

$$\| b \|_p = \{\sum_{k=1}^{t} \| b_k \|_p^p\}^{1/p} \qquad (4)$$

Obviously we have

Lemma 1 Let $\boldsymbol{\Phi} = (\phi^1, \cdots, \phi^r)^{\mathrm{T}}$. If the shifts of $\boldsymbol{\Phi}$ are stable, then there exist two positive constants C_1 and C_2 such that

$$C_2 \| b \|_p \geqslant \| \sum b(\alpha) \boldsymbol{\Phi}(\cdot - \alpha) \|_p \geqslant C_1 \| b \|_p \quad \forall b \in (l_p(\boldsymbol{Z}))^{s \times r}$$

Throughout the paper the same constants may represent different values in different places.

EXISTENCE OF L_p SOLUTIONS

In this section we first study the relation between the existence of L_p solutions and the convergence of the subdivision scheme . The main idea is based on the structure of shift-invariant spaces. A linear space S of distributions on $\boldsymbol{R}$ is said to be shift-invariant if

$$f \in S \Rightarrow f(\cdot - j) \in S, \quad j \in \boldsymbol{Z}.$$

Let ϕ be a compactly supported distribution on $\boldsymbol{R}$, the semi-convolution of ϕ with a sequence is defined by

$$\phi *' a := \sum_j \phi(\cdot - j) a(j)$$

Given a finite collection Φ of compactly supported distributions on $\boldsymbol{R}$. By $S_0(\Phi)$ we denote the linear span of $\{\phi(\cdot - j): \phi \in \Phi; j \in \boldsymbol{Z}\}$, and by $S(\Phi)$ the linear space of all distributions of the form $\sum_{\phi \in \Phi} \phi *' a_\phi$ with a_ϕ being a sequence on Z for each $\phi \in \Phi$.

The structure of shift-invariant spaces was investigated by Jia (1997). We have the following lemma for our purpose.

Lemma 2 Let $\boldsymbol{\Phi}$ be the compactly supported normalized distribution solution of the Eq.(1). Then there exists a compactly supported distribution vector $\boldsymbol{\psi} = (\psi^1, \cdots, \psi^s)^{\mathrm{T}} (s \leqslant r)$ such that

1. the shifts of $\boldsymbol{\psi}$ are linearly independent;
2. there exists a finitely supported sequence of $r \times s$ matrices g such that $\boldsymbol{\Phi} = \sum g(\alpha) \boldsymbol{\psi}(\cdot - \alpha)$;
3. $S(\{\psi^1, \cdots, \psi^s\}) = S(\{\phi^1, \cdots, \phi^r\})$;
4. $\boldsymbol{\psi}$ is refinable function vector, that is, there exists a finitely supported sequence b of $s \times s$ matrices such that

$$\boldsymbol{\psi} = \sum b(\alpha) \boldsymbol{\psi}(2 \cdot - \alpha).$$

If, in addition , $\boldsymbol{\Phi} \in (L_p(\boldsymbol{R}))^r$ for some p $(1 \leqslant p \leqslant \infty)$, then $\boldsymbol{\psi}$ can be chosen to be in $(L_p(\boldsymbol{R}))^s$.

Proof We prove 4. The other conclusions can be obtained from Jia(1997) immediately. Since $S(\{\psi^1, \cdots, \psi^s\}) = S(\{\phi^1, \cdots, \phi^r\})$, there exists a sequence of $s \times r$ matrices $\{d(\alpha)\}$ such that

$$\boldsymbol{\psi} = \sum_\alpha d(\alpha) \boldsymbol{\Phi}(\cdot - \alpha).$$

Noting that both $\boldsymbol{\psi}$ and $\boldsymbol{\Phi}$ are compactly supported distribution vectors on $\boldsymbol{R}$, we have

$$\begin{aligned}\boldsymbol{\psi} &= \sum_\alpha d(\alpha) \boldsymbol{\Phi}(\cdot - \alpha) \\ &= \sum_\alpha d(\alpha) \sum_\beta a(\beta) \boldsymbol{\Phi}(2 \cdot - 2\alpha - \beta) \\ &= \sum_\beta (\sum_\alpha d(\alpha) a(\beta - 2\alpha)) \sum_k g(k) \cdot \\ &\quad \boldsymbol{\psi}(2 \cdot - \beta - k) \\ &= \sum_\alpha b(\alpha) \boldsymbol{\psi}(2 \cdot - \alpha) \qquad (5)\end{aligned}$$

where $b(\alpha) = \sum_\beta \sum_k d(k)a(\beta - 2k) \cdot g(\alpha - \beta)$.

It is obvious that $\{b(\alpha)\}$ is a sequence of $s \times s$ matrices. We prove that $\{b(\alpha)\}$ has finite support. In fact, since the shifts of $\boldsymbol{\psi}$ are linearly independent, to each ψ^j there is a test function $u_j \in C_c^\infty(\boldsymbol{R})$ such that

$$\langle \psi^i(\cdot - \alpha), u_j \rangle = \delta_{ij}\delta_{\alpha 0}, \quad \forall \alpha \in \boldsymbol{Z}.$$

(Ben-Artzi and Ron, 1990). From Eq. (5) we have

$$\boldsymbol{\psi}(\cdot /2) = \sum b(\alpha)\boldsymbol{\psi}(\cdot - \alpha).$$

Noting that both $\boldsymbol{\psi}$ and u_j have compact support, we know that $b(\alpha) = 0$ for α large enough.

Assume that $f^* = (f^1, \cdots, f^r)^{\mathrm{T}} \in (L_p(\boldsymbol{R}))^r$. When $s \leqslant r$, we define the operator $f = Pf^* = (f^1, \cdots, f^s)^{\mathrm{T}}$. Based on the notations of Lemma 2 we have

Theorem 1 Let the mask a of refinement Eq. (1) be a finitely supported se-quence of $r \times r$ matrices, and $M = \sum a(\alpha)/2$ satisfy Eq. (2). Let $\boldsymbol{\phi} = (\phi^1, \cdots, \phi^r)^{\mathrm{T}}$ be the normalized solution of refinement equation Eq. (1) and $1 \leqslant p \leqslant \infty$. Assume that $f_0^* = (f_0^1, \cdots, f_0^r)^{\mathrm{T}}$ in $(L_p(\boldsymbol{Z}))^r$ is a compactly supported function vector satisfying the moment conditions of order 1. Then $\boldsymbol{\phi} \in (L_p(\boldsymbol{R}))^r$ (continuous in the case $p = \infty$) if and only if the subdivision $T_a^n \phi_0$ converges to $\boldsymbol{\phi}$ in $(L_p(\boldsymbol{R}))^r$ where $\phi_0 = \sum_\alpha g(\alpha) f_0(\cdot - \alpha)$.

Proof Our proof is described for $1 \leqslant p < \infty$. The case $p = \infty$ can be treated similarly.

The sufficiency is obvious. Now we prove the necessity.

Suppose $\boldsymbol{\phi} \in (L_p(\boldsymbol{R}))^r$ is the compactly supported normalized solution of the equation. There exists $\boldsymbol{\psi} = (\psi^1, \cdots, \psi^s)^{\mathrm{T}}$ given in Lemma 2 such that

$$\boldsymbol{\psi} = \sum b(\alpha)\boldsymbol{\psi}(2 \cdot - \alpha), \qquad (6)$$

$$\boldsymbol{\phi} = \sum g(\alpha)\boldsymbol{\psi}(\cdot - \alpha). \qquad (7)$$

We define the symbol $\tilde{b}(z)$ for a sequence of matrices b as follows

$$\tilde{b}(z) = \sum b(k) z^k.$$

Taking Fourier transform in Eq. (1), Eqs. (6) and Eq. (7), we have

$$\hat{\boldsymbol{\phi}}(\xi) = \tilde{a}(\mathrm{e}^{-i\xi/2})\hat{\boldsymbol{\phi}}(\xi/2)/2, \qquad (8)$$

$$\hat{\boldsymbol{\psi}}(\xi) = \tilde{b}(\mathrm{e}^{-i\xi/2})\hat{\boldsymbol{\psi}}(\xi/2)/2, \qquad (9)$$

$$\hat{\boldsymbol{\phi}}(\xi) = \tilde{g}(\mathrm{e}^{-i\xi})\hat{\boldsymbol{\psi}}(\xi), \qquad (10)$$

where $\tilde{a}(z)$, $\tilde{b}(z)$, $\tilde{g}(z)$ are symbols of a, b, and g respectively.

Hence it follows that

$$\tilde{g}(\mathrm{e}^{-i\xi})\hat{\boldsymbol{\psi}}(\xi) = \tilde{a}(\mathrm{e}^{-i\xi/2})\tilde{g}(\mathrm{e}^{-i\xi/2})\hat{\boldsymbol{\psi}}(\xi/2)/2,$$

Further we have

$$(\tilde{g}(\mathrm{e}^{-i\xi})\tilde{b}(\mathrm{e}^{-i\xi/2}) - \tilde{a}(\mathrm{e}^{-i\xi/2})\tilde{g}(\mathrm{e}^{-i\xi/2}))\hat{\boldsymbol{\psi}}(\xi/2)/2 = 0. \qquad (11)$$

Since the shifts of $\boldsymbol{\psi}$ are linearly independent, we have

$$\tilde{g}(\mathrm{e}^{-i\xi})\tilde{b}(\mathrm{e}^{-i\xi/2}) = \tilde{a}(\mathrm{e}^{-i\xi/2})\tilde{g}(\mathrm{e}^{-i\xi/2}). \qquad (12)$$

It is well known that compactly supported $\boldsymbol{\psi} \in (L_p(\boldsymbol{R}))^s$ is L_p stable if the shifts of $\boldsymbol{\psi}$ are linearly independent. Now we consider the refinement equation with the mask b. By theorem A we have

$$\lim_{n \to \infty} \| T_b^n f_0 - \boldsymbol{\psi} \|_p = 0. \qquad (13)$$

Set

$$\phi_0 = \sum g(\alpha) f_0(\cdot - \alpha), \qquad (14)$$

choosing $\phi_0(x)$ as an initial function vector, we will prove

$$\lim_{n \to \infty} \| T_b^n \phi_0 - \boldsymbol{\phi} \|_p = 0.$$

Define

$$a_1(\alpha) = a(\alpha),$$

$$a_n(\alpha) = \sum_\alpha a_{n-1}(\beta) a(\alpha - 2\beta). \quad (n \geqslant 2)$$

Similarly define $b_n(\alpha)$. It is easy to verify

$$T_a^n \phi_0(x) = \sum_\alpha a_n(\alpha)\phi_0(2^n x - \alpha),$$

$$T_b^n f_0(x) = \sum_\alpha b_n(\alpha) f_0(2^n x - \alpha). \qquad (15)$$

Thus we have

$$\begin{aligned} T_a^n \phi_0(x) &= \sum_\alpha a_n(\alpha)\phi_0(2^n x - \alpha) \\ &= \sum_\alpha a_n(\alpha)\Big(\sum_\beta g(\beta) f_0(2^n x - \beta - \alpha)\Big) \\ &= \sum_\beta \Big(\sum_\alpha a_n(\alpha) g(\beta - \alpha)\Big) f_0(2^n x - \beta) \\ &= \sum_\beta d_n(\beta) f_0(2^n x - \beta), \end{aligned} \qquad (16)$$

where we write $d_n(\beta) = \sum_\alpha a_n(\alpha) g(\beta - \alpha)$. Using the symbols we have

$$\tilde{a}_n(z) = \tilde{a}_{n-1}(z^2)\tilde{a}(z),$$
$$\tilde{d}_n(z) = \tilde{a}_n(z)\tilde{g}(z).$$

Consequently

$$\tilde{a}_n(z) = \tilde{a}(z^{2^{n-1}})\cdots\tilde{a}(z^2)\tilde{a}(z). \quad (17)$$

Noting

$$\tilde{g}(z^2)\tilde{b}(z) = \tilde{a}(z)\tilde{g}(z), \quad z = e^{-i\xi} \quad (18)$$

we have

$$\begin{aligned}\tilde{d}_n(z) &= \tilde{a}(z^{2^{n-1}})\cdots\tilde{a}(z^2)\tilde{a}(z)\tilde{g}(z)\\ &= \tilde{a}(z^{2^{n-1}})\cdots\tilde{a}(z^2)\tilde{g}(z^2)\tilde{b}(z)\\ &= \tilde{g}(z^{2^n})\tilde{b}(z^{2^{n-1}})\cdots\tilde{b}(z)\\ &= \tilde{g}(z^{2^n})\tilde{b}_n(z).\end{aligned}$$

For $k \in \boldsymbol{Z}$ and $\alpha \in \boldsymbol{Z}$, set

$$g_n(k) = \begin{cases} g(\alpha), & k = 2^n\alpha \\ 0, & k \neq 2^n\alpha \end{cases}$$

we have

$$\tilde{g}(z^{2^n}) = \sum_\alpha g(\alpha) z^{2^n\alpha} = \sum_k g_n(k) z^k.$$

It follows

$$\begin{aligned}d_n(\beta) &= \sum_k g_n(k) b_n(\beta - k)\\ &= \sum_\alpha g(\alpha) b_n(\beta - 2^n\alpha).\end{aligned}$$

Hence we obtain

$$\begin{aligned}T_a^n\phi_0 &= \sum_\beta (\sum_\alpha g(\alpha) b_n(\beta - 2^n\alpha)) f_0(2^n x - \beta)\\ &= \sum_\alpha g(\alpha) \sum_\beta b_n(\beta - 2^n\alpha) f_0(2^n x - \beta)\\ &= \sum_\alpha g(\alpha) \sum_\beta b_n(\beta) f_0(2^n(x - \alpha) - \beta),\end{aligned} \quad (19)$$

For $\alpha \in \boldsymbol{Z}$ the $r \times s$ matrix $g(\alpha)$ corresponds to an operator $g_*(\alpha)$ from $(L_p(\boldsymbol{R}))^s$ to $(L_p(\boldsymbol{R}))^r$, the norm of which is denoted as $\| g_*(\alpha) \|$. It follows

$$\| T_a^n\phi_0 - \boldsymbol{\phi} \|_p \leqslant \sum_\alpha \| g_*(\alpha) \| \cdot$$
$$\| \sum_\beta b_n(\beta) f_0(2^n(\cdot - \alpha) - \beta) - \boldsymbol{\psi}(\cdot - \alpha) \|_p \quad (20)$$

Thus, combining Eq. (13), Eq. (15) and Eq. (20) we have

$$\lim_{n\to\infty} \| T_a^n\phi_0 - \boldsymbol{\phi} \|_p = 0$$

as desired.

CONVERGENCE OF SUBDIVISION SCHEMES

As shown above the key for studying the L_p solution of the vector refinement equation is to investigate the convergence of the corresponding subdivision scheme. In this section we study the convergence of the subdivision scheme in terms of the corresponding mask. The main tool is the joint spectral radius of some linear operators.

The p-norm joint spectral radius of some linear operators was defined by Jia (1995) as follows.

Let X be a finite-dimensional linear space equipped with a norm $\| \cdot \|$, and A be a linear operator on X. The norm of the operator is defined by

$$\| A \| = \sup_{\| x \| = 1} \| Ax \|.$$

Let $\mathscr{A}$ be finite collection of linear operators on X. For a positive integer n, set

$$\mathscr{A}^n := \{(A_1, \cdots, A_n) : A_1, \cdots, A_n \in \mathscr{A}\}.$$

For $1 \leqslant p < \infty$, let

$$\| \mathscr{A}^n \|_p := \Big(\sum_{(A_1,\cdots,A_n)\in\mathscr{A}^n} \| A_1, \cdots, A_n \|^p \Big)^{1/p}.$$

For $p = \infty$

$$\| \mathscr{A}^n \|_\infty := \max\{ \| A_1, \cdots, A_n \| : (A_1, \cdots, A_n) \in \mathscr{A}^n \}.$$

The p-norm joint spectral radius of $\mathscr{A}$ is defined by

$$\rho_p(\mathscr{A}) := \lim_{n\to\infty} \| \mathscr{A}^n \|_p^{1/n}.$$

It is well known that

$$\lim_{n\to\infty} \| \mathscr{A}^n \|_p^{1/n} = \inf_n \| \mathscr{A}^n \|_p^{1/n}. \quad (21)$$

For the mask $a \in (l_0(\boldsymbol{Z}))^{r\times r}$ with finite support we define linear operators $A_\varepsilon (\varepsilon = 0, 1)$ on $(l_0(\boldsymbol{Z}))^r$:

$$A_\varepsilon u(\alpha) = \sum_\beta a(\varepsilon + 2\alpha - \beta) u(\beta), \quad \alpha \in \boldsymbol{Z}, \quad u \in (l_0(\boldsymbol{Z}))^r.$$

Let u be a sequence in $(l_0(\mathbf{Z}))^r$ and $V(u)$ denote the invariant subspace of A_0 and A_1 generated by u. It is obvious that $V(u)$ is finite-dimensional.

Set $\mathscr{A} = \{A_0, A_1\}$. Given $v \in l_0(\mathbf{Z})$ we define

$$\| \mathscr{A}^n v \|_p := \Big(\sum_{\varepsilon_1, \cdots, \varepsilon_n \in \{0,1\}} \| A_{\varepsilon_n} \cdots A_{\varepsilon_1} v \|_p^p \Big)^{1/p}, \quad 1 \leqslant p < \infty$$

and for $p = \infty$

$$\| \mathscr{A}^n v \|_\infty := \max \{ \| A_{\varepsilon_n} \cdots A_{\varepsilon_1} v \|_\infty : \varepsilon_1, \cdots, \varepsilon_n \in \{0,1\} \}.$$

Consider the convergence of the iteration process $T_a^n \phi_0$ where ϕ_0 is given in the proof of Theorem 1.

$$T_a^{n+1}\phi_0(x) - T_a^n\phi_0(x) = \sum_\alpha a_{n+1}(\alpha)\phi_0(2^{n+1}x - \alpha) - \sum_\alpha a_n(\alpha)\phi_0(2^n x - \alpha)$$
$$= \sum_\alpha \sum_\beta a_n(\beta) a(\alpha - 2\beta)\phi_0(2^{n+1}x - \alpha) - \sum_\alpha a_n(\alpha)\phi_0(2^n x - \alpha)$$
$$= \sum_\beta a_n(\beta)\Big(\sum_\alpha a(\alpha)\phi_0(2^{n+1}x - 2\beta - \alpha) - \phi_0(2^n x - \beta)\Big).$$

Set

$$f(x) = \sum_\alpha a(\alpha)\phi_0(2x - \alpha) - \phi_0(x).$$

Without loss of generality we assume $f \neq 0$. Now we have

$$T_a^{n+1}\phi_0(x) - T_a^n\phi_0(x) = \sum_\beta a_n(\beta) f(2^n x - \beta).$$

Based on the result of Jia(1997) there exists compactly supported function vector $\boldsymbol{\theta} \in (L_p(\mathbf{R}))^t$ with linearly independent shifts such that

$$f = \sum \lambda(\alpha)\boldsymbol{\theta}(\cdot - \alpha),$$

where λ is a finitely supported sequence of $r \times t$ matrices.

Thus

$$T_a^{n+1}\phi_0(x) - T_a^n\phi_0(x) = \sum_\beta a_n(\beta)\Big(\sum_\alpha \lambda(\alpha)\boldsymbol{\theta}(2^n x - \alpha - \beta)\Big)$$
$$= \sum_\alpha \Big(\sum_\beta a_n(\beta)\lambda(\alpha - \beta)\Big)\boldsymbol{\theta}(2^n x - \alpha)$$
$$= \sum_\alpha (a_n * \lambda)(\alpha)\boldsymbol{\theta}(2^n x - \alpha)$$

Since the shifts of $\boldsymbol{\theta}$ are linearly independent, by Lemma 1, there are positive constants C_1 and C_2 such that

$$C_1 2^{-n/p} \| a_n * \lambda \|_p \leqslant \| T_a^{n+1}\phi_0 - T_a^n\phi_0 \|_p \leqslant C_2 2^{-n/p} \| a_n * \lambda \|_p \tag{22}$$

In order to estimate $\| a_n * \lambda \|_p$ we prove the following lemmas.

Lemma 3 Let $\mu \in l_p(\mathbf{Z} \to C^r)$, $\mathscr{A} = \{A_0, A_1\}$. We have

(1) $\| a_n * \mu \|_p = \| \mathscr{A}^n \mu \|_p$;

(2) There exist two positive constants L_1 and L_2 independent from n such that

$$L_1 \| \mathscr{A}^n |_{V(\mu)} \|_p \leqslant \| \mathscr{A}^n \mu \|_p \leqslant L_2 \| \mathscr{A}^n |_{V(\mu)} \|_p. \tag{23}$$

Proof In order to prove (1) we only need to prove

$$a_n * \mu(\alpha) = A_{\varepsilon_n} \cdots A_{\varepsilon_1} \mu(\gamma). \tag{24}$$

for any $\alpha = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma$, where $\varepsilon_1, \cdots, \varepsilon_n \in \{0,1\}$ and $\gamma \in \mathbf{Z}$.

We use induction on n. For $n = 1$ and $\alpha = \varepsilon_1 + 2\gamma$ we have

$$a_1 * \mu(\alpha) = \sum a(\beta)\mu(\alpha - \beta) = \sum a(\varepsilon_1 + 2\gamma - \beta)\mu(\beta) = A_{\varepsilon_1}\mu(\gamma).$$

Suppose Eq. (24) has been established for $n - 1$. Let $\alpha = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma$, where $\varepsilon_1, \cdots, \varepsilon_n \in \{0, 1\}$ and $\gamma \in \mathbf{Z}$. Write α_1 for $\varepsilon_2 + 2\varepsilon_3 + \cdots + 2^{n-2}\varepsilon_n + 2^{n-1}\gamma$.

Now for $\alpha = \varepsilon_1 + 2\alpha_1$ we have

$$a_n * \mu(\alpha) = \sum_\beta a_n(\beta)\mu(\alpha - \beta)$$
$$= \sum_\beta \Big(\sum_\gamma a_{n-1}(\gamma) a(\beta - 2\gamma)\Big)\mu(\alpha - \beta)$$
$$= \sum_\gamma a_{n-1}(\alpha_1 - \gamma) \sum_\beta a(\varepsilon_1 + 2\gamma - \beta)\mu(\beta)$$
$$= \sum_\gamma a_{n-1}(\alpha_1 - \gamma)(A_{\varepsilon_1}\mu)(\gamma)$$
$$= (a_{n-1} * A_{\varepsilon_1}\mu)(\alpha_1).$$

By induction hypothesis we have

$$a_n * \mu(\alpha) = A_{\varepsilon_n} \cdots A_{\varepsilon_1}\mu(\gamma)$$

as desired.

It is easy to prove 2.

Lemma 4 Let $\lambda = (\lambda_{jk})_{1\leqslant j\leqslant r, 1\leqslant k\leqslant t} \in l_p(\boldsymbol{Z} \to C^{r\times t})$ and $\lambda_k = (\lambda_{jk})_{1\leqslant j\leqslant r}$ we have

(1) $\| a_n * \lambda \|_p = (\| \mathscr{A}^n\lambda_1 \|_p^p + \cdots + \| \mathscr{A}^n\lambda_t \|_p^p)^{1/p}$.

(2) Let $V(\lambda)$ be the common invariant subspace of $\mathscr{A} = \{A_0, A_1\}$ generated by $\lambda_1, \cdots, \lambda_t$. Then there exists a positive constant L independent from n such that

$$\max_{1\leqslant i\leqslant t} \| \mathscr{A}^n|_{V(\lambda_i)} \|_p \leqslant \| \mathscr{A}^n|_{V(\lambda)} \|_p \leqslant L \max_{1\leqslant i\leqslant t} \| \mathscr{A}^n|_{V(\lambda_i)} \|_p. \quad (25)$$

Proof From Eq.(4) and Lemma 3 we obtain 1. Now we prove 2. Because $V(\lambda)$ is finite dimensional, there exist a positive integer l such that $V(\lambda)$ is spanned by the set

$$Y := \{A_1\cdots A_j\lambda_k : (A_1, \cdots, A_j) \in \mathscr{A}^j, \; j = 0, 1, \cdots, l, k = 1, \cdots, t\}.$$

Thus there exists a constant $C_1 > 0$ such that

$$\| \mathscr{A}^n y \|_p \leqslant C_1 \max_{1\leqslant i\leqslant t}(\| \mathscr{A}^n\lambda_i \|_p)$$

for any $y \in Y$ and all $n = 1, 2, \cdots$. Moreover there exists a positive constant C_2 such that

$$\| \mathscr{A}^n|_{V(\lambda)} \|_p \leqslant C_2 \max_{y\in Y}(\| \mathscr{A}^n y \|_p) \quad (n = 1, 2, \cdots).$$

Therefore, there exists a positive constant L such that

$$\| \mathscr{A}^n|_{V(\lambda)} \|_p \leqslant L \max_{1\leqslant i\leqslant t}(\| \mathscr{A}^n|_{V(\lambda_i)} \|_p.$$

It is obvious that

$$\| \mathscr{A}^n|_{V(\lambda_i)} \|_p \leqslant \| \mathscr{A}^n|_{V(\lambda)} \|_p.$$

Summarizing the above discussion we now obtain the characterization of the convergence of the subdivision scheme $T_a^n\phi_0$ in terms of the mask.

Theorem 2 The subdivision scheme $T_a^n\phi_0$ converges in $L_p(\boldsymbol{R})$ $(1\leqslant p\leqslant\infty)$ ($C(\boldsymbol{R})$ *if* $p=\infty$) if and only if

$$\rho_p(A_0|_{V(\lambda)}, A_1|_{V(\lambda)}) < 2^{1/p}.$$

Proof The proof is also described for $1\leqslant p < \infty$, the case $p = \infty$ can be treated similarly.

For convenience let ρ denote $\rho_p(A_0|_{V(\lambda)}, A_1|_{V(\lambda)})$. By Eq.(21) and Lemma 3, 4 it follows that there exists a positive C such that

$$\rho^n \leqslant \| \mathscr{A}^n|_{V(\lambda)} \|_p \leqslant C \| a_n * \lambda \|_p.$$

Thus

$$(2^{-1/p}\rho)^n \leqslant C 2^{-n/p} \| a_n * \lambda \|_p.$$

If $T_a^n\phi_0$ converges, from Eq.(22) we have

$$\lim_{n\to\infty} 2^{-n/p} \| a_n * \lambda \|_p = 0.$$

Hence

$$2^{-1/p}\rho < 1.$$

Conversely, suppose $\rho < 2^{1/p}$. Let r be a positive real number such that $\rho < r < 2^{1/p}$. We can find a positive integer N such that for $n > N$

$$\| \mathscr{A}^n|_{V(\lambda)} \|_p^{1/n} < r.$$

By Lemma 4 for $i = 1, 2, \cdots, t$ we have

$$\| \mathscr{A}^n|_{V(\lambda_i)} \|_p^{1/n} < r.$$

Thus by Lemma 4 there exists a positive constant K independent from n

$$2^{-n/p} \| a_n * \lambda \|_p \leqslant K(r2^{-1/p})^n, \quad n > N.$$

Noting $r2^{-1/p} < 1$, from Eq.(22) $T_a^n\phi_0$ converges in $(L_p(\boldsymbol{R}))^r$.

References

Ben-Artzi, A., Ron, A., 1990. On the integer translates of a compactly supported function: dual bases and linear projectors. *SIAM J. Math. Anal.*, **21**: 1550 – 1562.

Cavaretta, A. S., Dahmen, W., Micchelli, C. A., 1991. Stationary subdivision. *Memoirs of Amer. Math. Soc.*, **93**:

Han, B., Jia, R. Q., 1998. Multivariate refinement equations and convergence of subdivision schemes. *SIAM J. Math. Anal.*, **29**: 1177 – 1199.

Heil, C., Collela, D., 1996. Matrix refinement equations, existence and uniqueness. *J. Fourier Anal. Appl.*, **2**: 363 – 377.

Jia, R. Q., 1995. Subdivision schemes in L_p spaces. *Advances in computation mathematics*, **3**: 309 – 341.

Jia, R. Q., 1997. Shift-invariant spaces on the real line. *Proc. Amer. Math. Soc.*, **125**: 785 – 793.

Jia, R. Q., Riemenschneider, S. D., Zhou, D. X., 1998. Vector subdivision schemes and multiple wavelets. *Mathematics of computation*, **67**: 1533 – 1563.