Moments and limiting distribution of a portfolio of whole life annuity policies

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Abstract: A dual random model of a portfolio of variable amount whole life annuity is set with the mth moment of the present value of benefits, and the respective expressions of the moments under the assumption that the force of interest accumulation function is Wiener process or Ornstein-Uhlenbeck process. Furthermore, the limiting distribution of average cost of this portfolio is discussed with the expression of the limiting distribution under the assumption that the force of interest accumulation is an independent increment process.

Key words: Whole life annuity policy, Force of interest, Present value of benefit, Moment, Limiting distribution, Wiener process, Ornstein-Uhlenbeck process, Independent increment process
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INTRODUCTION

For life insurance and social pension insurance, interest rate and mortality are two very important random factors. But in traditional actuarial theory, only the mortality is random and the interest rate is not certain. In the mid-1970's, interest randomness began to be an assumption in actuarial science, and study of dual randomness (i.e. the mortality and interest rate are both random) affecting life insurance and social pension insurance gradually became an important field of actuarial science. (Wu, 1995; He, 1998).

The interest rate was usually modeled by random process under assumption of dual randomness. In some actuarial literature, the interest rate is modeled by the time series method, for example, white-noise process, AR process, ARIMA process. Boyle(1976), Panker & Bellhouse(1980; 1981), Giaccotto(1986), Dhaene (1989), Hürlimann(1992) used this method. Since the 1990's, the interest rate in some literature has been modeled by the perturbation method, which yielded a series of annuity results certain with dual randomness. In 1990 and 1991, Beekman and Fuelling(1990; 1991) obtained separately the first and second moments of some annuities under the assumption that the force of interest is modeled by Ornstein-Uhlenbeck process or Wiener process. In 1993, they obtained the first and second moments of the present value of benefit of whole life insurance under the assumption that the force of interest is modeled by Ornstein-Uhlenbeck process or Wiener process. In 1992, De Schepper et al. (1992a; 1992b; 1992c). obtained the moment generating function, distribution function and Laplace transform of annuity under the assumption that the force of interest is modeled by Wiener process.

Under assumption of dual randomness, calculating the moments of the present value of benefit is an important task, and exploring the distribution of the present value of benefit is a more important task. Gray Parker (1994a; 1994b; 1994c; 1994d) discussed the present value of benefit of a portfolio of term life policies and a portfolio of endowment policies established their dual random model, and obtained the first three moments of the two kinds of polices, and also discussed the limit of average cost of policies when the number of the policies approach infinity, and obtained the recurrence formula of the approximate distribution function of the limiting random variable.

He Wenjiong and Jiang Qingrong (1998) discussed increasing life insurance, where the death benefit is paid as soon as the insured dies, and established its dual random model where the force of interest accumulation function is a general Gauss process, and obtained the moments of the present value of the benefits. Since then He (2001) discussed a general kind of increasing annuity based on its force of interest accumulation function as a general random process; established the dual random model of the present value of the benefits of the increasing annuity; and then calculated moments under certain conditions.

Here we conduct a dual random model of a portfolio of whole life annuity policies; calculate the moments of the present value of its benefits, and discuss the limiting distribution of the average present value of benefits. The model is useful for clarifying the theory of "dual randomness", and is also as a powerful tool for calculating the government debt during the transformation of social pension insurance in China. The later will be discussed in a separate paper.

MODEL OF PRESENT VALUE OF BENEFITS

Annuity insurance is an important kind of commercial life insurance. Technically, social pension insurance is also a kind of annuity insurance. So annuity insurance is important for actuarial science. Because variable annuity insurance is increasing in China and the adjustment mechanism in social pension insurance has already been established, the whole life annuity discussed in this paper yields variable benefits.

Given n insured same sex, age x, people getting the same amount of pension in a year. The annuity benefits are paid at the beginning of the year if the insured is living. The first benefit is B, and the kth benefit is $B \cdot c(k)$, where $c(k)$ is a positive function of k, and $c(0)=1$.

We use the symbols of actuarial science with (x) indicating a person at the age of x, and T_l indicating the future lifetime of person l. In the model of interest randomness, we can use force of interest function $\delta(t)$ (the interest at time t) or force of interest accumulation function $u(t) = \int_o^t \delta(s)\mathrm{d}s$ to indicate interest randomness. Here the force of interest accumulation function is used to indicate interest randomness.

Therefore, we can set the dual randomness model for the portfolio of variable amount whole life annuity policies whose number is n:

$$S_n = B\sum_{l=1}^{n}\sum_{k=0}^{[T_l]} c(k) \cdot \mathrm{e}^{-u(k)} \tag{1a}$$

where $u(t)=\delta \cdot t + y(t)$, and $y(t)$ is a random process. δ is a nonnegative random variable or constant. $\mathrm{e}^{-u(t)}$ is a discount function, which indicates discount value of 1 unit from t to 0.

In model Eq.(1a), the following assumption is reasonable,

Assumption 1° $T_l(l=1,2,\cdots,n)$ have independent and identical distribution; their density functions are the same i.e. $f(t)$;

Assumption 2° $y(t)$ is independent of $\{T_l, l=1,2,\cdots,n\}$.

Model Eq.(1a) can also be written as

$$S_n = B\sum_{l=1}^{n} Y_l \tag{1b}$$

Where $Y_l = \sum_{k=o}^{[T_l]} c(k) \cdot \mathrm{e}^{-u(k)}$. It is evident that the random variables $T_l(l=1,2,\cdots,n)$ have the same distribution, but that they are not independent. In fact, they have a common random process $u(t)$ which indicates the force of interest accumulation function.

We notice that, if $c(k)\equiv 1$, the variable amount annuity becomes equal amount annuity. Correspondingly,

$$S_n = B\sum_{l=1}^{n}\sum_{k=0}^{[T_l]} \mathrm{e}^{-u(k)} \tag{2}$$

Without loss of generality, let $B=1$ in model Eqs.(1a) or (1b), and Eq.(2).

MOMENTS OF PRESENT VALUE OF BENEFITS

Now we study the mth moment of the present value of benefits, and obtain their expressions under certain conditions.

Theorem 1 In model Eqs.(1a) or (1b), if δ is a nonnegative constant, and $y(t)$ is independent of $\{T_l, l=1,2,\cdots,n\}$, and $\{T_l\}$ have independent and identical distribution and their density functions are $f(t)$, then

$$ES_n = nEY_1 \tag{3}$$

$$ES_n^2 = nEY_1^2 + n(n-1)EY_1Y_2 \tag{4}$$

$$ES_n^3 = n(n-1)(n-2)EY_1Y_2Y_3 + 3n(n-1)EY_1^2Y_2 + nEY_1^3 \tag{5}$$

$$ES_n^m = \sum_{r_1+r_2+\cdots+r_n=m} \frac{n!}{r_1!r_2!\cdots r_n!} EY_1^{r_1}Y_2^{r_2}\cdots Y_n^{r_n} \tag{6}$$

Where

$$EY_1^{r_1}Y_2^{r_2}\cdots Y_n^{r_n} = \int_0^\infty\int_0^\infty\cdots\int_0^\infty \sum_{j_1=0}^{[t_1]}\sum_{j_2=0}^{[t_2]}\sum_{j_n=0}^{[t_n]} c^{r_1}(j_1)c^{r_2}(j_2)\cdots c^{r_n}(j_n)\mathrm{e}^{-\delta(r_1j_1+r_2j_2+\cdots+r_nj_n)} \cdot$$
$$E\mathrm{e}^{-\{r_1y(j_1)+r_2y(j_2)+\cdots+r_ny(j_n)\}} f(t_1)f(t_2)\cdots f(t_n)\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n \tag{7}$$

Here, different expressions can be obtained when m and r_1, r_2, $\cdots$, r_n take different values; for example,

$$EY_1^2Y_2 = \int_0^\infty\int_0^\infty \sum_{k=0}^{[t_1]}\sum_{j=0}^{[t_1]}\sum_{i=0}^{[t_2]} c(k)c(j)c(i)\mathrm{e}^{-\delta(k+j+i)}E\mathrm{e}^{-(y(k)+y(j)+y(i))} f(t_1)f(t_2)\mathrm{d}t_1\mathrm{d}t_2 \tag{8}$$

$$EY_1Y_2Y_3 = \int_0^\infty\int_0^\infty\int_0^\infty \sum_{k=0}^{[t_1]}\sum_{j=0}^{[t_2]}\sum_{i=0}^{[t_3]} c(k)c(j)c(i)\mathrm{e}^{-\delta(k+j+i)}E\mathrm{e}^{-(y(k)+y(j)+y(i))} \cdot$$
$$f(t_1)f(t_2)f(t_3)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3 \tag{9}$$

Proof Theorem 1 can be directly obtained through assumptions 1° and 2° by simple calculation.

Now suppose $y(t)$ comprise some particular random processes. We will get the concrete form of moments in Theorem 1.

1°. If $y(t)$ is Wiener process with parameter σ, $y(t)=W(t)$, and $y(0)=0$, $j_{(1)}\leqslant j_{(2)}\leqslant\cdots\leqslant j_{(i)}\leqslant\cdots\leqslant j_{(n)}$ indicates the decreasing sort of j_1, j_2, $\cdots$, j_n, We have,

$$E\mathrm{e}^{-\{r_1y(j_1)+r_2y(j_2)+\cdots+r_ny(j_n)\}} = E\mathrm{e}^{-\{r_{(1)}y(j_{(1)})+r_{(2)}y(j_{(2)})+\cdots+r_{(n)}y(j_{(n)})\}} =$$
$$E\mathrm{e}^{-\{r_{(1)}\sum_{i=1}^{j(1)}[y(i)-y(i-1)]\}}\mathrm{e}^{-\{r_{(2)}\sum_{i=1}^{j(2)}[y(i)-y(i-1)]\}}\cdots\mathrm{e}^{-\{r_{(n)}\sum_{i=1}^{j(n)}[y(i)-y(i-1)]\}} =$$
$$E\mathrm{e}^{-\{[r_{(1)}+r_{(2)}+\cdots+r_{(n)}]\sum_{i=1}^{j(1)}[y(i)-y(i-1)]\}}\mathrm{e}^{-\{[r_{(2)}+\cdots+r_{(n)}]\sum_{i=j(1)+1}^{j(2)}[y(i)-y(i-1)]\}}\cdots\mathrm{e}^{-\{r_{(n)}\sum_{i=j(n-1)+1}^{j(n)}[y(i)-y(i-1)]\}} =$$
$$E\mathrm{e}^{-\{[r_{(1)}+r_{(2)}+\cdots+r_{(n)}]\sum_{i=1}^{j(1)}[y(i)-y(i-1)]\}}E\mathrm{e}^{-\{[r_{(2)}+\cdots+r_{(n)}]\sum_{i=j(1)+1}^{j(2)}[y(i)-y(i-1)]\}}\cdots E\mathrm{e}^{-\{r_{(n)}\sum_{i=j(n-1)+1}^{j(n)}[y(i)-y(i-1)]\}} =$$
$$\mathrm{e}^{\frac{1}{2}\sigma^2 j_{(1)}(r_{(1)}+r_{(2)}+\cdots+r_{(n)})^2}\mathrm{e}^{\frac{1}{2}\sigma^2(j_{(2)}-j_{(1)})(r_{(2)}+\cdots+r_{(n)})^2}\cdots\mathrm{e}^{\frac{1}{2}\sigma^2(j_{(n)}-j_{(n-1)})(r_{(n)})^2} =$$
$$\prod_{i=1}^n \mathrm{e}^{\frac{1}{2}\sigma^2(j_{(i)}-j_{(i-1)})(\sum_{k=i}^n r_{(k)})^2} \tag{10}$$

$$EY_1^{r_1}Y_2^{r_2}\cdots Y_n^{r_n} = \int_0^\infty\int_0^\infty\cdots\int_0^\infty \sum_{j_1=0}^{[t_1]}\sum_{j_2=0}^{[t_2]}\cdots\sum_{j_n=0}^{[t_n]} c^{r_1}(j_1)c^{r_2}(j_2)\cdots c^{r_n}(j_n)\mathrm{e}^{-\delta(r_1j_1+r_2j_2+\cdots+r_nj_n)} \cdot$$
$$\prod_{i=1}^n \mathrm{e}^{\frac{1}{2}\sigma^2(j_{(i)}-j_{(i-1)})(\sum_{k=i}^n r_{(k)})^2} f(t_1)f(t_2)\cdots f(t_n)\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n \tag{11}$$

Different expressions can be obtained when m and r_1, r_2, $\cdots$, r_n take different values, for example,

$$EY_1^2Y_2 = \int_0^\infty\int_0^\infty \sum_{k=0}^{[t_1]}\sum_{j=0}^{[t_1]}\sum_{i=0}^{[t_2]} c(k)c(j)c(i)\mathrm{e}^{-\delta(k+j+i)+\frac{1}{2}\sigma^2\{k+j+i+2\min(k,j)+2\min(j,i)+2\min(k,i)\}} \cdot$$
$$f(t_1)f(t_2)\mathrm{d}t_1\mathrm{d}t_2 \tag{12}$$

$$EY_1Y_2Y_3 = \int_0^\infty\int_0^\infty\int_0^\infty \sum_{k=0}^{[t_1]}\sum_{j=0}^{[t_2]}\sum_{i=0}^{[t_3]} c(k)c(j)c(i)\mathrm{e}^{-\delta(k+j+i)+\frac{1}{2}\sigma^2\{k+j+i+2\min(k,j)+2\min(j,i)+2\min(k,i)\}} \cdot$$

$$f(t_1)f(t_2)f(t_3)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3 \tag{13}$$

2°. If $y(t)$ is Ornstein-Uhlenbeck process, i. e. $y(t)=X(t)$, and satisfies

$$\begin{cases}\mathrm{d}X(t) = -\alpha X(t)\mathrm{d}t + \sigma \mathrm{d}W(t) \\ X(0) = 0\end{cases} \tag{14}$$

where $W(t)$ is Wiener process, then

$Ey(t)=0$, $Cov(y(t), y(s)) = \rho^2(\mathrm{e}^{-\alpha(t-s)} + \mathrm{e}^{-\alpha(t+s)})$, $s \leqslant t$, Where: $\rho^2 = \sigma^2/2\alpha$. Therefore

$$E\mathrm{e}^{-\{r_1y(j_1)+r_2y(j_2)+\cdots+r_ny(j_n)\}} = \mathrm{e}^{\frac{1}{2}\{\sum_{i=1}^{n} r_i Dy(j_i) + \sum_{k\neq l} r_k r_l \mathrm{cov}(y(j_k), y(j_l))\}}$$

$$EY_1^{r_1}Y_2^{r_2}\cdots Y_n^{r_n} = \int_0^\infty\int_0^\infty\cdots\int_0^\infty \sum_{j_1=0}^{[t_1]}\sum_{j_2=0}^{[t_2]}\cdots\sum_{j_n=0}^{[t_n]} c^{r_1}(j_1)c^{r_2}(j_2)\cdots c^{r_n}(j_n)\mathrm{e}^{-\delta(r_1j_1+r_2j_2+\cdots+r_nj_n)} \cdot \mathrm{e}^{\frac{1}{2}\{\sum_{i=1}^{n} r_i Dy(j_i) + \sum_{k\neq l} r_k r_l \mathrm{cov}(y(j_k), y(j_l))\}} f(t_1)f(t_2)\cdots f(t_n)\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n$$

LIMITING DISTRIBUTION OF $\frac{S_n}{n}$

The limiting distribution of $\frac{S_n}{n}$ is an important property of S_n and reflects the average cost of a portfolio of variable amount whole life annuity policies when the number of policies is large enough. About this, we have the following theorem.

Theorem 2 In Theorem1, if the moment generating function $E\mathrm{e}^{\tau y(t)}$ of random process $y(t)$ is finite, $\forall \tau \in R^1$, then,

$$\frac{S_n}{n} - \sum_{k=0}^{\omega} c(k)\mathrm{e}^{-\delta k - y(k)} \cdot p_k \xrightarrow{P} 0 \quad (n \to \infty) \tag{15}$$

where ω indicates limit age, $p_k = P_r(T_1 \geqslant k)$.

To prove this theorem, we give two lemmas,

Lemma 1 (Kolmoglov large number law) If random variables X_1, X_2, $\cdots$, X_n have independent and identical distribution, and EX_1 is finite, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} EX_1 \quad (n \to \infty)$$

Lemma 2 1° If the expectation and variance of X is finite, $Y_n \xrightarrow{P} 0$ $(n \to \infty)$, *then*

$$X \cdot Y_n \xrightarrow{P} O \quad (n \to \infty)$$

2° If $Y_n^i \xrightarrow{P} 0$ $(n \to \infty)$, $i = 1, 2, \cdots, c$ (c is a constant), then

$$\sum_{i=1}^{c} Y_n^i \xrightarrow{P} O \quad (n \to \infty)$$

Proof 1° Its proof is simple.

2° From $P_r\left(\left|\sum_{i=1}^{c} Y_n^i\right| \geqslant \varepsilon\right) \leqslant \sum_{i=1}^{c} P_r\left(|Y_n^i| \geqslant \frac{\varepsilon}{c}\right)$, we can obtain 2° simply.

Let us prove Theorem 2.

Proof $S_n = \sum_{l=1}^{n}\sum_{k=0}^{[T_l]} c(k)\mathrm{e}^{-\delta k - y(k)} = \sum_{l=1}^{n}\left\{\sum_{k=0}^{[T_l]} c(k)\mathrm{e}^{-\delta k - y(k)} \cdot I_{[T_l \geqslant 0]}\right\} =$

$$\sum_{l=1}^{n}\left\{\sum_{k=0}^{[T_l]} c(k)e^{-\delta k-y(k)}\cdot\sum_{\tau_l=0}^{\omega} I_{[\tau_l\leqslant T_l<\tau_l+1]}\right\}=$$
$$\sum_{l=1}^{n}\left\{\sum_{\tau_l=0}^{\omega}\sum_{k=0}^{\tau_l} c(k)e^{-\delta k-y(k)}\cdot I_{[\tau_l\leqslant T_l<\tau_l+1]}\right\}=$$
$$\sum_{l=1}^{n}\left\{\sum_{k=0}^{\omega} c(k)e^{-\delta k-y(k)}\cdot\sum_{\tau_l=k}^{\omega} I_{[\tau_l\leqslant T_l<\tau_l+1]}\right\}=$$
$$\sum_{l=1}^{n}\left\{\sum_{k=0}^{\omega} c(k)e^{-\delta k-y(k)}\cdot I_{[T_l\geqslant k]}\right\}=$$
$$\sum_{k=0}^{\omega} c(k)e^{-\delta k-y(k)}\left(\sum_{l=1}^{n} I_{[T_l\geqslant k]}\right)$$

Because the T_l's have independent and identical distribution, from Lemma 1, we have

$$\frac{1}{n}\sum_{l=1}^{n} I_{[T_l\geqslant k]}\xrightarrow{P} EI_{[T_l\geqslant k]}\quad (n\to\infty)$$

Let $X_k=c(k)e^{-\delta k-y(k)}$, then

$$EX_k=c(k)e^{-\delta k}Ee^{-y(k)}<\infty$$
$$EX_k^2=c^2(k)e^{-2\delta k}Ee^{-2y(k)}$$
$$DX_k=c^2(k)e^{-2\delta k}(Ee^{-2y(k)}-(Ee^{-y(k)})^2)=c^2(k)e^{-2\delta k}\cdot De^{-y(k)}<\infty$$

From lemma 2, we have

$$\frac{S_n}{n}-\sum_{k=0}^{\omega} c(k)e^{-\delta k-y(k)}P_r(T_1\geqslant k)=$$
$$\sum_{k=0}^{\omega} c(k)e^{-\delta k-y(k)}\left(\frac{1}{n}\sum_{l=1}^{n} I[T_l\geqslant k]-P_r(T_1\geqslant k)\right)\xrightarrow{P} 0\quad (n\to\infty).$$

In particular situations, we can get the limiting distribution of $\frac{S_n}{n}$. Let

$$\zeta=\sum_{k=0}^{\omega} c(k)e^{-\delta k-y(k)}\cdot p_k \tag{16}$$

If $y(t)=X(t)$, $X(t)$ has stationary and independent increments process, then

$$P_r(\zeta\leqslant x)=P_r\left(\sum_{k=0}^{\omega} c(k)e^{-\delta k-y(k)}\cdot p_k\leqslant x\right)=$$
$$P_r\left(\sum_{k=1}^{\omega} c(k)e^{-\delta k-\sum_{j=0}^{k-1}(y(k+1)-y(k))}\cdot p_k\leqslant x-p_0\right)$$

Let $z_k=e^{-(y(k+1)-y(k))}$, then z_k has independent and identical distribution. We suppose the density function of z_k is $f(t)$.

Let $Y_k=a_k z_1 z_2\cdots z_k$, where $a_k=c(k)e^{-\delta k}p_k$, then the density of $(Y_1, Y_2,\cdots Y_\omega)$ can be expressed as follows,

$$p(y_1,y_2,\cdots,y_\omega)=\frac{1}{a_\omega y_1 y_2\cdots y_{\omega-1}}f\left(\frac{y_1}{a_1}\right)f\left(\frac{a_1y_2}{a_2y_1}\right)\cdots f\left(\frac{a_{\omega-1}y_\omega}{a_\omega y_{\omega-1}}\right) \tag{17}$$

According to the definition of Y_k, we get

$$\zeta=\sum_{k=1}^{\omega} Y_k+p_0$$

and

$$P_r(\zeta \leqslant x) = P_r\left(\sum_{k=1}^{\omega} Y_k \leqslant x - p_0\right) = \int\limits_{y_1+y_2+\cdots+y_\omega \leqslant x-p_0} \cdots \int p(y_1, y_2, \cdots, y_\omega) \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_\omega = \int\limits_{y_1+y_2+\cdots+y_\omega \leqslant x-p_0} \cdots \int \frac{1}{a_\omega y_1 y_2 \cdots y_{\omega-1}} f\left(\frac{y_1}{a_1}\right) f\left(\frac{a_1 y_2}{a_2 y_1}\right) \cdots f\left(\frac{a_{\omega-1} y_\omega}{a_\omega y_{\omega-1}}\right) \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_\omega$$

The model of this paper comes from a realistic problem related to calculation of the government debt during the transformation of social pension insurance in China. The model Eq. (1a) expresses the debt of "the old employees" of x years old; the sum of the debt can be calculated when x varies from the age of retirement in law to the age of the insured when he dies. Two properties of the present value of benefits were discussed in this paper. The expressions of the moments were calculated in Theorem 1 and the limit distribution of S_n/n were discussed in Theorem 2. It is obviously that the results in this paper can be regarded not only as a powerful tool to calculate the government debt during the transformation of social pension insurance in China, but also as important for premium calculation, reserve calculation, insurance product design, risk management, etc. in life insurance and social insurance.

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