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Gauss-Radau and Gauss-Lobatto formulae for the Jacobi weight and Gori-Micchelli weight functions*

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Abstract: The main purpose of this work is to find for any non-negative measure，the relations between the Gauss-Radau and Gauss-Lobatto formula and Gauss formulae for the same measure. As applications，the author obtained the explicit Gauss-Radau and Gauss-Lobatto formulae for the Jacobi weight and the Gori-Micchelli weight.

Key words: Quadrature，Gauss-Radau formula，Gauss-Lobatto formula，Gori-Micchelli weight
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INTRODUCTION

Throughout this paper，let N be the set of the natural numbers and P_n the set of all polynomials of degree $\leqslant n$，$n \in N$. For any nonnegative measure $d\mu$ supported on the interval $[a, b]$，with a a finite real number and its moments of all orders to exist，then it admits the Gauss formula of the form

$$\int_a^b f(x)d\mu(x) = \sum_{i=1}^{n} w_i f(x_i) + E_{2n-1}(f), \qquad (1)$$

as well as a quadrature rule of the type

$$\int_a^b f(x)d\mu(x) = w_0^R(d\mu)f(a) + \sum_{i=1}^{n} w_i^R(d\mu)f(x_i^R) + E_{2n}(f), \qquad (2)$$

which is exact for polynomials of degree $\leqslant 2n$，that is，

$$E_{2n}(f) = 0, \quad \forall f \in P_{2n}. \qquad (3)$$

The latter is called the $(n+1)$-point Gauss-Radau rule for the measure $d\mu$ (There is also an analogous Gauss-Radau formula，if $d\mu$ is bounded from the right，when a is replaced by b in the above formula. But the basic theory remains true except for the obvious changes in symbol，so the details are omitted). Its interior nodes x_i^R's are known to be the zeros of $p_n(\cdot; d\mu_a)$，the polynomial of degree n orthogonal relative to the modified measure $d\mu_a = (x-a)d\mu(x)$. The weights $w_i^R = w_i^R(d\mu)$ are obtainable by interpolation at the nodes $a, x_1^R, x_2^R, \cdots, x_n^R$. A more elegant construction procedure proposed by Golub (1973).

The first goal here is to show that for any measure $d\mu$，supported on the interval $[a, b]$，the Gauss-Radau formula is characterized by the following

Theorem 1 Let $d\mu$ be any nonnegative measure supported on the interval $[a, b]$ and x_i^R，$i = 1, 2, \cdots, n$，be the zeros of the nth degree orthogonal polynomial associated with the measure $d\mu_a = (x-a)d\mu$. Suppose further that the measure $d\mu$ assumes the following formula of Gauss-Radau type

$$\int_a^b f(x)d\mu(x) = w_0^R f(a) + \sum_{i=1}^{n} w_i^R f(x_i^R) + E_{2n}(f). \qquad (4)$$

We then have for the weights

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$$w_0^R = \frac{1}{p_n(a;d\mu_a)}\int_a^b p_n(x;d\mu_a)d\mu(x),$$
$$w_i^R = \frac{1}{x_i - a}w_i(d\mu_a),\ i = 1,\cdots,n, \tag{5}$$

where $w_i(d\mu_a)$, $i=1,\cdots,n$, are the Christoffel numbers for the modified measure $d\mu_a$.

If further, the nonnegative measure $d\mu$ is also bounded from the right, then there exists a so-called Gauss-Lobatto quadrature rule of the following form

$$\int_a^b f(x)d\mu(x) = w_0^L f(a) + \sum_{i=1}^n w_i^L f(x_i^L) + w_{n+1}^L f(b) + E_{2n+1}(f), \tag{6}$$

which is exact whenever f is a polynomial of degree at most $2n+1$,

$$E_{2n+1}(f) = 0,\ f \in P_{2n+1}. \tag{7}$$

The interior points x_i^L are the zeros of $p_n(\cdot;d\mu_{a,b})$, the nth degree polynomial of orthogonal with respect to the modified measure $d\mu_{a,b}=(x-a)(b-x)d\mu(x)$, and the weights w_i^L are obtainable by interpolation at the points a, w_1^L, $\cdots$, w_n^L, b.

There is an analogous eigenvalue-eigenvector procedure also proposed by Golub(1973). Gauss-Lobatto formulae are then computable by the QR algorithm in light of Golub's theorem. This works quite well as long as n is not too large. When n becomes large, one encounters solving a singular system. For the Legendre measure, this happens starting with $n=79$ in single precision, and beginning with $n=543$ in double precision. This phenomenon was reported in Gautschi (2000b).

One way to correct this problem is to re-scale the orthogonal polynomials. Accuracy, however, is improved if the weights are computed separately. We shall develop explicit expressions for the weights in terms of n, α, β, $P_n^{(\alpha+1,\beta+1)\prime}(x_i)$, which enable us to obviate the need of computing eigenvectors. This is somewhat analogous to that of the Radau case.

The second purpose here is to prove:

Theorem 2 Let $d\mu$ be any nonnegative measure supported on the interval [a, b] and x_i^L, $i=1,2,\cdots,n$, be the zeros of the nth degree orthogonal polynomial associated with the modified measure $d\mu_{a,b}=(x-a)(b-x)d\mu$. Suppose further that the measure $d\mu$ assumes the following formula of Gauss-Lobatto type

$$\int_a^b f(x)d\mu(x) = w_0^L f(a) + \sum_{i=1}^n w_i^L f(x_i^L) + w_{n+1}^L f(b) + E_{2n+1}(f). \tag{8}$$

We then have for the weights

$$w_0^L = \frac{1}{(b-a)p_n(a;d\mu_{a,b})}\cdot \int_a^b (b-x)p_n(x;d\mu_{a,b})d\mu(x),$$
$$w_i^L = \frac{1}{(x_i-a)(b-x_i)}w_i(d\mu_{a,b}),$$
$$i = 1,\cdots,n,$$
$$w_{n+1}^L = \frac{1}{(b-a)p_n(b;d\mu_{a,b})}\cdot \int_a^b (x-a)p_n(x;d\mu_{a,b})d\mu(x), \tag{9}$$

where $w_i(d\mu_{a,b})$, $i=1,\cdots,n$ are the Christoffel numbers for the measure $d\mu_{a,b}$.

Gori and Micchelli(1996) introduced the following weight function class to consist of all nonnegative integrable functions w on $[-1,1]$ such that

$$w\sqrt{1-x^2} = \sum_{l=0}^{\infty}{}' \rho_l(w)T_{2ln}(x),\ n \in N, \tag{10}$$

where the prime on the summation indicates that the term corresponding to $l=0$ is halved. They proved in the same paper that for any weight $w \in W_n$, the nth orthogonal polynomial relative to the weight w is $T_n(x)$, the Chebyshev polynomial of the first kind. In the sequel, we shall call it the *Gori-Micchelli class* and its element the *Gori-Micchelli weight*. See Gori and Micchelli (Goriand Micchelli, 1996) and Yang and Wang (Yang and Wang, 2002) for the Guass-Turan quadrature relative to it.

The last goal is to apply Theorems 1-2 to the Jacobi and Gori-Micchelli weight functions to find explicit expressions for the weights w_i^R, $i=0,1,\cdots,n$ and w_i^L, $i=0,1,\cdots,n,n+1$ in the Gauss-Radau and Gauss-Lobatto formulae respectively. It should be mentioned that explicit expressions of the Gauss-Radau and Gauss-Lobatto formulae for the Jacobi and Laguerre weight can be found in Gautschi (2000a), Gautschi (2000b) and Davis and Rabinowitz

(1975). But our method is simpler than theirs. Besides, explicit Gauss-Radau and Gauss-Lobatto formulae for the Gori-Micchelli weight to our best knowledge are new.

PROOFS OF THEOREMS.

Proof of Theorem 1 It is easy to check that the fundamental interpolation functions at nodes $a, x_1, \cdots, x_n$ are

$$l_i(x) = \begin{cases} \dfrac{p_n(x; d\mu_a)}{p_n(a; d\mu_a)}, & \text{if } i = 0, \\ \dfrac{(x-a)p_n(x; d\mu_a)}{(x_i - a)p_n{}'(x_i; d\mu_a)}, & \text{if } i \neq 0. \end{cases} \quad (11)$$

So it is trivially true for the first equality in Eq. (5). For the second, we have by interpolation theory

$$w_i^R = \int_a^b \frac{(x-a)p_n(x; d\mu_a)}{(x_i - a)p_n{}'(x_i; d\mu_a)} d\mu(x) = \frac{1}{x_i - a}\int_a^b \frac{p_n(x; d\mu_a)}{(x_i - a)p_n{}'(x_i; d\mu_a)} d\mu_a(x) = \frac{1}{x_i - a} w_i(d\mu_a).$$

This proves the theorem.

Proof of Theorem 2 It is easy to see that the node polynomial in this situation is

$$(x-a)(x-b)\prod_{i=1}^{n}(x - x_i) = (x-a)(x-b)p_n(x; d\mu_{a,b}),$$

up to a multiplicative factor. Thus, the fundamental interpolation functions are

$$l_0(x; d\mu_{a,b}) = \frac{(b-x)p_n(x; d\mu_{a,b})}{(b-a)p_n(a; d\mu_{a,b})},$$

$$l_i(x; d\mu_{a,b}) = \frac{(x-a)(b-x)p_n(x; d\mu_{a,b})}{(x_i - a)(b - x_i)(x - x_i)p_n{}'(x_i; d\mu_{a,b})}, \ i = 1, \cdots, n,$$

$$l_{n+1}(x; d\mu_{a,b}) = \frac{(x-a)p_n(x; d\mu_{a,b})}{(b-a)p_n(b; d\mu_{a,b})},$$

The proof is finished after a direct calculation using $w_i^L = \int_a^b l_i(x; d\mu_{a,b}) d\mu(x)$, $i = 0, 1, \cdots, n, n+1$.

APPLICATIONS OF THE THEOREMS

Theorem 1 is simple but very useful for deriving explicit expressions for the weight when the measure is $d\mu = w^{(\alpha,\beta)}dx$, the Jacobi weight. More precisely, the Gauss-Radau weight coefficient w_i^R, defined in Eq.(2), is related with the Gauss weight(i.e. the Christoffel number).

$$w_i^R = \frac{1}{1 + x_i} w_i^{(\alpha,\beta+1)}, \ i = 1, 2, \cdots, n. \quad (12)$$

In order to compute w_0^R, we have by Theorem 1

$$w_0^R = \frac{1}{P_n^{(\alpha,\beta+1)}(-1)} \cdot \int_{-1}^{1} P_n^{(\alpha,\beta+1)}(x) w^{(\alpha,\beta)}(x) dx,$$

where $P_n^{(\alpha,\beta+1)}(x)$ is the nth Jacobi polynomials relative to $w^{(\alpha,\beta+1)}(x)$. Gautschi (2000a) obtained

$$\int_{-1}^{1} P_n^{(\alpha,\beta+1)}(x) w^{(\alpha,\beta)}(x) dx = (-1)^n 2^{\alpha+\beta+1}\Gamma(\beta+1)\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)}. \quad (13)$$

And it is well known from Szegö (1938) that

$$P_n^{(\alpha,\beta+1)}(-1) = (-1)^n \binom{n+\beta}{n}. \quad (14)$$

Thus,

$$w_0^R = \frac{2^{\alpha+\beta+1}\Gamma(\beta+1)\Gamma(n+\alpha+1)}{\binom{n+\beta+1}{n}\Gamma(n+\alpha+\beta+2)}. \quad (15)$$

Now explicit expressions can be found for w_i^R, $i = 0, 1, \cdots, n$, since the Christoffel numbers $w_i^{(\alpha,\beta+1)}$, $i = 1, \cdots, n$, are exactly the coefficients for the Gaussian quadrature rule relative to the weight function $w^{(\alpha,\beta+1)}(x)$. From Eq. 15.3.1 (with β replaced by $\beta + 1$) in Szegö (1938), we have

$$w_i^R = \frac{2^{\alpha+\beta+2}\Gamma(n+\alpha+1)\Gamma(n+\beta+2)}{\Gamma(n+1)\Gamma(n+\alpha+2)(1-x_i)(1+x_i)^2[P_n^{(\alpha,\beta+1)\prime}(x_i)]^2} = \frac{2^{\alpha+\beta}(2n+\alpha+\beta+3)^2\Gamma(n+\alpha+1)\Gamma(n+\beta+2)(1-x_i)}{(n+1)(n+\alpha+\beta+2)\Gamma(n+2)\Gamma(n+\alpha+\beta+3)[P_{n+1}^{(\alpha,\beta+1)}(x_i)]^2},\ i=1,\cdots,n, \tag{16}$$

since $(2n+\alpha+\beta+3)(1-x_i^2)P_n^{(\alpha,\beta+1)\prime}(x_i) = -2(n+1)(n+\alpha+\beta+2)P_{n+1}^{(\alpha,\beta+1)}(x_i)$ by the second relation in Formula Eq. 4.5.7 of Szegö (1938). This, together with Eq.(15), completely gives explicit expressions for the coefficients in Gauss-Radau quadrature rule with respect to the Jacobi weight. Other possible explicit expressions, such as in terms of n, α, β, $P_n^{(\alpha,\beta+1)\prime}(x_i)$, $P_{n-1}^{(\alpha,\beta+1)}(x_i)$ can be found in Gautschi(2000a). There, Gautschi uses a different method to get explicit expressions for w_i^R, $i=0,1,\cdots,n$.

From Theorem 2 to the Jacobi weight, we obtain

Corollary 1 If $w_i^{(\alpha,\beta)}$ is the i-th Gauss weight for the Jacobi measure $w^{(\alpha,\beta)}(x)\mathrm{d}x$, then the Gauss-Lobatto weight w_i^L for the same measure can be obtained via

$$w_0^L = w_{n+1}^L = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+3)}\cdot\frac{\binom{n+\alpha+1}{n}}{\binom{n+\beta+1}{n}\binom{n+\alpha+\beta+2}{n}},$$

$$w_i^L = \frac{1}{1-x_i^2}w_i^{(\alpha+1,\beta+1)},\ i=1,2,\cdots,n. \tag{17}$$

Proof It is easy to see that $w_0^L = w_{n+1}^L$ since the change of variable $x\to -x$ converts $w^{(\alpha,\beta)}(x)$ into $w^{(\beta,\alpha)}(x)$. Using Eq.(13)(with α replaced by $\alpha+1$), one easily sees by Theorem 2

$$\begin{aligned} w_0^L &= \frac{1}{2P_n^{(\alpha+1,\beta+1)}(-1)}\int_{-1}^{1}P_n^{(\alpha+1,\beta+1)}(x)w^{(\alpha+1,\beta)}(x)\mathrm{d}x \\ &= \frac{(-1)^n}{2P_n^{(\alpha+1,\beta+1)}(-1)}2^{\alpha+\beta+2}\Gamma(\beta+1)\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+\beta+2)} \\ &= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+3)}\frac{\binom{n+\alpha+1}{n}}{\binom{n+\beta+1}{n}\binom{n+\alpha+\beta+2}{n}}. \end{aligned} \tag{18}$$

The last equality holds because of Eq.(14). The second equality in Eq.(17) is a straightforward consequence of Eq.(9).

This corollary, together with the explicit expression for the Gauss weight (Eq. 15.3.1 in Szegö(1938) with α replaced by $\alpha+1$, β by $\beta+1$ respectively), gives

$$w_0^L = w_{n+1}^L = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+3)}\frac{\binom{n+\alpha+1}{n}}{\binom{n+\beta+1}{n}\binom{n+\alpha+\beta+2}{n}},$$

$$w_i^L = \frac{2^{\alpha+\beta+3}\Gamma(\alpha+n+2)\Gamma(\beta+n+2)}{\Gamma(n+1)\Gamma(\alpha+\beta+n+3)[(1-x_i^2)P_n^{(\alpha+1,\beta+1)\prime}(x_i)]^2},\ i=1,\cdots,n, \tag{19}$$

This allows us to compute the Gauss-Jacobi-Lobatto weights directly and obviate the need of computing eigenvectors. A different but somewhat more complicated method to derive explicit expressions for w_i^L, $i=1,2,\cdots,n$, were proposed by Gautschi (2000b).

Gauss-Radau and Gauss-Lobatto formulae for the Gori-Micchelli weight

We first consider the Gauss-Radau formula of the form

$$\int_{-1}^{1} f(x) w \mathrm{d}x \approx \lambda_0^R(w) f(-1) + \sum_{i=1}^{n-1} \lambda_i^R(w) f(x_i). \quad (20)$$

which is exact for $f \in P_{2n-2}$ and where the interior nodes x_i, $i=1,\cdots,n-1$, are the zeros of the $(n-1)$-th orthogonal polynomial $p_{n-1}(x;(1+x)w)$ relative to the modified weight $(1+x)w$. We have

Lemma 1 For any weight $w \in W_n$, $n \in N$,

$$p_{n-1}(x;(1+x)w) = V_{n-1}(x), \quad (21)$$

up to a constant factor, where $V_{n-1}(x)$ is the $(n-1)$-th Chebyshev polynomial of the third kind.

Proof For any $q \in P_{n-2}$, we obtain from Eq. (20) and orthogonality that

$$\int_{-1}^{1} q(x) V_{n-1}(x)(1+x) w \mathrm{d}x = \sum_{l=0}^{\infty}{}' \rho_l(w) \cdot \int_{-1}^{1} q(x) V_{n-1}(x) T_{2ln}(x)(1+x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{1}{2}\rho_0(w) \cdot \int_{-1}^{1} q(x) V_{n-1}(x)(1+x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} + \rho_1(w) \int_{-1}^{1} (1+x) q(x) V_{n-1}(x) T_{2n}(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{1}{2}\rho_0(w) \int_{-1}^{1} q(x) V_{n-1}(x) \sqrt{\frac{1+x}{1-x}} \mathrm{d}x = 0. \quad (22)$$

The third equality holds since $(1+x) q V_{n-1} \in P_{2n-2}$ and orthogonality.

We now derive the coefficients for Eq. (20). The coefficients $\lambda_i^R(w)$, $i=0,1,\cdots,n-1$, in formula Eq. (20) for any weight $w \in W_n$ are equal to $\rho_0(w)$ times those for the weight $\frac{1}{\sqrt{1-x^2}}$. More specifically, we have

Lemma 2 Let $w \in W_n$ and $\lambda_i^R(w)$, $i=0,1,\cdots,n-1$, be the weights of Gauss-Radau formula relative to it. We have

$$\lambda_i^R(w) = \frac{1}{2}\rho_0(w) w_i^R, \quad i = 0,1,\cdots,n-1, \quad (23)$$

where w_i^R are Gauss-Radau coefficients for the weight $\frac{1}{\sqrt{1-x^2}}$.

Proof For any $f \in P_{2n-2}$, it is easy to check that

$$\int_{-1}^{1} f(x) w \mathrm{d}x = \sum_{l=0}^{\infty}{}' \rho_l(w) \int_{-1}^{1} f(x) T_{2ln}(x) \cdot \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{1}{2}\rho_0(w) \int_{-1}^{1} f(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} + \sum_{l=1}^{\infty}{}' \rho_l(w) \int_{-1}^{1} f(x) T_{2ln}(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{1}{2}\rho_0(w) \left\{ w_0^R f(-1) + \sum_{i=1}^{n-1} w_i^R f(x_i) \right\}.$$

This finishes the proof. Next, we consider the Gauss-Lobatto formula of the form

$$\int_{-1}^{1} f(x) w \mathrm{d}x \approx \lambda_0^L(w) f(-1) + \sum_{i=1}^{n-1} \lambda_i^L(w) f(x_i) + \lambda_n^L(w) f(1) \quad (24)$$

for any Gori-Micchelli weight w. Here x_i, $i=1,\cdots,n-1$, are the zeros of the $(n-1)$ degree orthogonal polynomial with respect to the modified weight $(1-x^2)w(x)$. First, similar to the Gauss-Radau case just treated, we have

Lemma 3 For any weight $w \in W_n$, $n \in N$,

$$p_{n-1}(x;(1-x^2)w) = U_{n-1}(x), \quad (25)$$

up to a constant factor, where $U_{n-1}(x)$ is the $(n-1)$-th Chebyshev polynomial of the second kind.

Proof The proof is straightforward, so we leave out the details.

Analogous to Lemma 2, we have

Lemma 4 Let $w \in W_n$ and $\lambda_i^L(w)$, $i=0,1,\cdots,n$, be the weights of Gauss-Lobatto formula relative to it. We have

$$\lambda_i^L(w) = \frac{1}{2}\rho_0(w) w_i^L, \quad i=0,1,\cdots,n, \quad (26)$$

where w_i^L are Gauss-Lobatto coefficients for the weight $\frac{1}{\sqrt{1-x^2}}$.

This lemma in combination with Corollary 1 (with n replaced by $n-1$ and $\alpha=\beta=-1/2$) leads to the following

$$\lambda_0^L(w) = \lambda_n^L(w) = \frac{\pi}{4n}\rho_0(w),$$

$$\lambda_i^L(w) = \frac{\pi}{2n}\rho_0(w), \quad i = 1,\cdots,n-1. \quad (27)$$

Therefore, we conclude with the following.

Theorem 3 For any weight $w \in W_n$, $n \in N$, the Gauss-Lobatto formula takes the following form

$$\int_{-1}^{1} f(x) w \mathrm{d}x = \frac{\pi}{n}\rho_0(w)\left\{\frac{1}{2}f(-1) + \sum_{i=1}^{n-1} f\left(\cos\frac{i\pi}{n}\right) + \frac{1}{2}f(1)\right\} + E_{2n-1}(f),$$

which has algebraic degree of precision $2n-1$.

We end this section with some examples of Gauss-Lobatto formulae relative to weights in the Gori and Micchelli class. The first and the simpliest is the Chebyshev weight of the first kind. It is trivial to see ($\rho_0(w)=2$ in this case) that

$$\int_{-1}^{1} f(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{\pi}{n}\left\{\frac{1}{2}f(-1) + \sum_{i=1}^{n-1} f\left(\cos\frac{i\pi}{n}\right) + \frac{1}{2}f(1)\right\} + E_{2n-1}(f). \quad (28)$$

The second example is when the weight $w(x) = \frac{T_n^{2m}(x)\mathrm{d}x}{\sqrt{1-x^2}}$, $n \in N \cup \{0\}$, $n \in N$. It is easy to check that $\rho_0(w) = \frac{1}{4^m}\binom{2m}{m}$, so the Gauss-Lobatto formula associated with this weight assumes the following

$$\int_{-1}^{1} f(x)\frac{T_n^{2m}(x)\mathrm{d}x}{\sqrt{1-x^2}} = \frac{\pi}{2^{2m}n}\binom{2m}{m}\left\{\frac{1}{2}f(-1) + \sum_{i=1}^{n-1} f\left(\cos\frac{i\pi}{n}\right) + \frac{1}{2}f(1)\right\} + E_{2n-1}(f).$$

When $m=0$, we get Eq.(28) again.

The last example is with the generalized Gegenbauer weight (see Gori and Micchelli (1996) and references therein) $|U_{n-1}(x)/n|^{2\lambda+1}(1-x^2)^\lambda$, $x \in [-1,1]$. It is easy to see that

$$\rho_0(w) = \frac{2}{n^{2\lambda+1}\pi}\int_{-1}^{1}(1-x^2)^\lambda \mathrm{d}x = \frac{4^{\lambda+1}}{n^{2\lambda+1}\pi}B(\lambda+1,\lambda+1),$$

where $B(\cdot,\cdot)$ denotes the beta function. Now we obtain the Gauss-Lobatto formula for this weight.

$$\int_{-1}^{1} f(x)\,|U_{n-1}(x)/n|^{2\lambda+1}(1-x^2)^\lambda \mathrm{d}x = \frac{2^{2\lambda+2}}{n}\frac{B(\lambda+1,\lambda+1)}{2}\left\{\frac{1}{2}f(-1) + \sum_{i=1}^{n-1} f\left(\cos\frac{i\pi}{n}\right) + \frac{1}{2}f(1)\right\} + E_{2n-1}(f),$$

which is exact for $f \in P_{2n-1}$.

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