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Construction of some hypergroups from combinatorial structures

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Abstract: Jajcay's studies (1993; 1994) on the automorphism groups of Cayley maps yielded a new product of groups, which he called, rotary product. Using this product, we define a hyperoperation $\odot$ on the group $Sym_e(G)$, the stabilizer of the identity $e \in G$ in the group $Sym(G)$.

We prove that $(Sym_e(G), \odot)$ is a hypergroup and characterize the subhypergroups of this hypergroup. Finally, we show that the set of all subhypergroups of $Sym_e(G)$ constitute a lattice under ordinary join and meet and that the minimal elements of order two of this lattice is a subgroup of $Aut(G)$.

Key words: Finite group, Rotary closed subgroup, Hypergroup, Sub-hypergroup, Combinatorial structures
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INTRODUCTION

We will first recall some algebraic definitions that will be used in the paper. A hyperstructure is a set H together with a function $\cdot : H \times H \to P^*(H)$ called hyperoperation, where $P^*(H)$ denotes the set of all non-empty subsets of H.

Following Marty (1934), we define a hypergroup as a hyperstructure $(H, .)$ such that the following axioms hold: (1) $(x.y).z = x.(y.z)$ for all x, y, z in H; (2) $x.H = H.x = H$ for all x in H, which is called the reproduction axiom.

Let G be a group, $Sym(G)$ be the group of all permutations on G and $Sym_e(G)$ be the stabilizer of the identity $e \in G$ in $Sym(G)$. Given two permutations ϕ and ψ from $Sym_e(G)$ and an element $g \in G$, we define a new permutation $\phi \odot_g \psi = L_{\phi(g)^{-1}} \phi L_g \psi$, where $L_{\phi(g)^{-1}}$, $L_g \in Sym(G)$ are left multiplications by the elements $\phi(g)^{-1}$ and g, respectively.

According to Jajcay's (1994), a subgroup H of $Sym_e(G)$ closed under taking products of this form is called rotary closed, i.e. $H \leqslant Sym_e(G)$ is called rotary closed provided that $\phi \odot_g \psi \in H$, for all ϕ, $\psi \in H$ and $g \in G$. A nice family of rotary closed subgroups of $Sym_e(G)$, for finite G's, comes from the theory of Cayley graphs and can be obtained in the following way. Let Ω be a set of generators for a finite group G not containing the identity element e but containing x^{-1} together with every x contained in Ω. The subgroup $Rot_\Omega(G)$ of $Sym_e(G)$ of all permutations preserving e and satisfying the condition $\phi(a)^{-1}\phi(ax) \in \Omega$, for every $a \in G$ and $x \in \Omega$, is rotary closed (Jajcay, 1993; 1994).

Suppose that $(G, \cdot)$ is a hypergroup. A non-empty subset K of G is called a sub-hypergroup of G, if $a \cdot K = K \cdot a = K$, for all $a \in K$. We now assume that X is a set. The map $\odot : G \times X \to P(X)^*$ is called a generalized action of G on X, if the following axioms hold:

1. For all $g, h \in G$ and $x \in X$, $(gh) \odot x \subseteq g \odot (h \odot x)$,
2. For all $g \in G$, $g \odot X = X$.

Where, for any $g \in G$ and $Y \subseteq X$, $g \odot Y = \bigcup_{x \in Y} g \odot x$, also for any $x \in X$ and $B \subseteq G$, $B \odot x = \bigcup_{b \in B} b \odot x$. If equality holds in the axiom (1) of definition, the generalized action is called strong (Madanshekaf et al., 1998).

Suppose that G is a group. A permutation f on G such that $f(xy) = f(y)f(x)$, for all x, $y \in G$, is called an anti-automorphism of the group G. Throughout this paper all groups considered are assumed to be finite groups. Our notation is standard and taken mainly from Biggs et al., 1979; Birkhoff, 1967; Corsini, 1993; and Vougiouklis, 1994.

MAIN RESULTS AND THEOREMS

In this section, we first introduce a hyperoperation $\odot$ on $Sym_e(G)$ and prove that $(Sym_e(G), \odot)$ is a hypergroup. Next, we characterize the sub-hypergroups of this hypergroup and construct some of its sub-hypergroups. To do this, assume that $\phi, \psi \in Sym_e(G)$, we define:

$$\phi \odot \psi = \{\phi \odot_g \psi \mid g \in G\}. \quad (1)$$

Proposition 1 $(Sym_e(G), \odot)$ is a hypergroup.

Proof Suppose ϕ, ψ and η are arbitrary permutations of $Sym_e(G)$. Then we have,

$$\begin{aligned}(\phi \odot \psi)\odot \eta &= \\ \{\phi \odot_g \psi \mid g \in G\}\odot \eta &= \\ \bigcup_{g\in G}(\phi \odot_g \psi)\odot \eta &= \\ \bigcup_{g\in G}\{(\phi \odot_g \psi)\odot_h \eta \mid h \in G\} &= \\ \{(\phi \odot_g \psi)\odot_h \eta \mid g, h \in G\}. \end{aligned} \quad (2)$$

Using similar argument as that above, we can show that

$$\phi \odot (\psi \odot \eta) = \{\phi \odot_g (\psi \odot_h \eta) \mid g, h \in G\}. \quad (3)$$

We now assume that $g, h \in G$, then we have,

$$\begin{aligned}&(\phi \odot_g \psi)\odot_h \eta = \\ &L_{\phi\odot_g\psi(h)^{-1}}\,\phi \odot_g \psi L_h \eta = \\ &L_{\phi(g\psi(h))^{-1}\phi(g)}\,\phi \odot_g \psi L_h \eta = \\ &L_{\phi(g\psi(h))^{-1}\phi(g)}\ L_{\phi(g)^{-1}}\,\phi L_g \psi L_h \eta = \\ &L_{\phi(g\psi(h))^{-1}}\,\phi L_g \psi L_h \eta = \\ &L_{\phi(g\psi(h))^{-1}}\,\phi L_{g\psi(h)}\ L_{\psi(h)^{-1}}\,\psi L_h \eta = \\ &\phi \odot_{g\psi(h)} (\psi \odot_h \eta) \in \phi \odot (\psi \odot \eta)\end{aligned} \quad (4)$$

Therefore, $\phi \odot (\psi \odot \eta) \subseteq (\phi \odot \psi) \odot \eta$. Using similar argument we have, $\phi \odot (\psi \odot \eta) \subseteq (\phi \odot \psi) \odot \eta$ and the associativity is valid. Next we assume that $\phi \in Sym_e(G)$ and we have,

$$\begin{aligned}\phi \odot Sym_e(G) &= \\ \bigcup_{\psi \in Sym_e(G)} \phi \odot \psi &= \\ \bigcup_{\psi \in Sym_e(G)} \{\phi \odot_g \psi \mid g \in G\}.\end{aligned} \quad (5)$$

Suppose $\delta \in Sym_e(G)$ is arbitrary and $\psi = L_g^{-1}\phi^{-1}L_{\phi(g)}\delta$. Then, $\phi \odot_g \psi = \delta$ and so $Sym_e(G) = \phi \odot Sym_e(G)$. Similarly, $Sym_e(G)\odot \phi = Sym_e(G)$, which completes the proof.

In what follows, we characterize the sub-hypergroups of the hypergroup $(Sym_e(G), \odot)$.

Proposition 2 Let G be a finite group and H be a non-empty subset of $Sym_e(G)$. H is a sub-hypergroup of $Sym_e(G)$ if and only if H is a rotary closed subgroup of $Sym_e(G)$.

Proof $(\Rightarrow)$ Suppose H is a sub-hypergroup of $Sym_e(G)$. We first show that H is a closed subset of $Sym_e(G)$. To do this, suppose ϕ and ψ are elements of H. Then $\phi\psi = \phi \odot_e \psi \in \phi \odot \psi \subseteq H$ and so $\phi\psi \in H$. Next, for $\phi, \psi \in H$ and $g \in G$, $\phi \odot_g \psi \in \phi \odot \psi \subseteq H$, as desired.

$(\Leftarrow)$ Suppose $H \leqslant Sym_e(G)$ is rotary closed and $\phi \in G$. Since H is rotary closed $\phi \odot H \subseteq H$. Suppose $\psi \in H$. Put $\eta = \phi^{-1}\psi$ and $g = e$. Then, $\phi \odot_g \eta = \phi\phi^{-1}\psi = \psi \in \phi \odot \eta$ and so $H = \phi \odot H$. Similar argument shows that $H \odot \phi = H$, proving the result.

It is a well-known fact that the set of all subgroups of a group G has a lattice structure under the ordinary operations of meet and join. In general, it is far from true that the set of all sub-hypergroups of a hypergroup has a lattice structure under these operations. In fact, the intersection of two sub-hypergroups of a hypergroup is not necessarily non-empty.

Let $\mathscr{L}(Sym_e(G))$ denote the set of all sub-hypergroups of the hypergroup $Sym_e(G)$. In what follows, we show that $Sym_e(G)$ has a lattice structure under the ordinary operations of join and meet.

Proposition 3 $\mathscr{L}(Sym_e(G))$ has a lattice structure under the ordinary operations of meet and join.

Proof It is easy to understand fact that $\{1_G\}$ and $Sym_e(G)$ are rotary closed. Suppose that H and K are two rotary closed subgroups of $Sym_e(G)$. It is clear that $H \cap K$ is rotary closed. We claim that $\langle H, K\rangle$ is also rotary closed. To do this, we assume that $\psi \in H$, $\phi \in K$ and $g \in G$. Then we have:

$$\begin{aligned}&\psi \odot_g \phi = \\ &L_{\psi(g)^{-1}}\,\psi L_g \phi = \\ &L_{\psi(g)^{-1}}\,\psi L_g \psi\psi^{-1}\phi = \\ &\psi \odot_g \psi\psi^{-1}\phi \in \langle H, K\rangle\end{aligned} \quad (6)$$

Also, for ψ_1, $\psi_2 \in H$, ϕ_1, $\phi_2 \in K$ and $g \in G$, we have

$$\psi_1\phi_1 \odot_g \psi_2\phi_2 =$$
$$L_{\psi_1\phi_1(g)^{-1}}\psi_1\phi_1 L_g\psi_2\phi_2 =$$
$$L_{(\psi_1(\phi_1(g)))^{-1}}\psi_1 L_{\phi_1(g)}\psi_1\psi_1^{-1}L_{\phi_1(g)^{-1}}\phi_1 Lg\psi_2\phi_2 =$$
$$(\psi_1 \odot_{\phi_1(g)} \psi_1)\psi_1^{-1}(\phi_1 \odot_g \psi_2\phi_2) \in$$
$$HK \subseteq \langle H, K\rangle. \quad (7)$$

Using similar argument as that above, we can show that $\langle H, K\rangle$ is a rotary closed subgroup of $Sym_e(G)$. This shows that $\mathscr{L}(Sym_e(G))$ has a lattice structure under ordinary operations of join and meet.

In the following simple lemma, we introduce a most important class of sub-hypergroups of $Sym_e(G)$.

Lemma 1 If $H \leqslant Aut(G)$, then H is a sub-hypergroup of $Sym_e(G)$.

Proof Suppose $\phi, \psi \in H$ and $g \in G$. Then $\phi \odot_g \psi = \phi\psi \in H$, so H is rotary closed and by Proposition 2, the proof is complete.

Lemma 2 Let $H \subseteq Sym_e(G)$ and $|H| = 2$. H is a sub-hypergroup of $Sym_e(G)$ if and only if $H \leqslant Aut(G)$.

Proof Suppose $H = \{\sigma, \xi\}$. By proposition 2, we can assume that $\xi = 1_G$. It is easy to understand fact that for every $g \in G$, $\sigma \odot_g 1_G = \sigma$. This shows that σ is an automorphism and so $H \leqslant Aut(G)$. For the converse, we apply the Lemma 1.

It is far from true that each sub-hypergroup of $Sym_e(G)$ is a subgroup of $Aut(G)$. In the following example, we obtain a sub-hypergroup of $Sym_e(G)$ which is not a subset of $Aut(G)$.

Example In this example we show that if G is a non-abelian group, then $A(G)$ is a sub-hypergroup of $Sym_e(G)$ and $Aut(G) \subset A(G)$. Let f be the mapping that maps each element onto its inverse. Then f is an anti-automorphism of G and that $f \in / Aut(G)$. Therefore, $A(G) \neq Aut(G)$. Suppose ϕ, $\psi \in A(G)$ and $g \in G$. If ϕ is an automorphism of G then $\phi \odot_g \psi = \phi\psi \in A(G)$. Otherwise, $\phi \odot_g \psi = \phi(g)^{-1}\phi\psi\phi(g) \in A(G)$. Thus, $\phi \odot_g A(G) \subseteq A(G)$. We now assume that ψ is an arbitrary element of $A(G)$. Set $\delta = \phi^{-1}\psi$. Then

$$\phi \odot_g \delta =$$
$$L_{\phi(g)^{-1}}\phi L_g \delta =$$
$$L_{\phi(g)^{-1}}\phi L_g \phi^{-1}\psi = \phi \odot_g \phi^{-1}\psi = \psi$$

and we can see that $\delta \in A(G)$. This shows that $A(G)$ is a sub-hypergroup of $Sym_e(G)$ and $Aut(G) < A(G)$.

Suppose $Alt_e(G)$ is the set of all even permutations of $Sym_e(G)$. In what follows, we characterize all finite groups G such that $Alt_e(G)$ is a sub-hypergrup of $Sym_e(G)$.

Proposition 4 Let G be a finite group, then $Alt_e(G)$ is a sub-hypergroup of $Sym_e(G)$ if and only if one of the following holds:

(a) G has odd order;

(b) G is a non-cyclic 2-group;

(c) $|G| = 2^n \cdot m$, n is a positive integer, $m > 1$ is odd and G does not have a cyclic Sylow 2-subgroup.

Proof ($\Longrightarrow$) Suppose $Alt_e(G)$ is a sub-hypergroup of $Sym_e(G)$. We can assume that $|G| = 2^n \cdot m$, for some integer $n \geqslant 1$ and odd m. We first assume that $m = 1$ and G is cyclic. Choose an element x of order 2^n, an element y of order 2^{n-1} and a different element $z \neq e$ of G. Suppose $\phi = (xyz)$ is a cycle of $Sym(G)$. Then $\phi \in Alt_e(G)$ and by assumption $\phi \odot_x \phi \in Alt_e(G)$, a contradiction. Next, we assume that $m > 1$, p is a prime divisor of m and a Sylow 2-subgroup of G is cyclic. Choose an element x of order 2^n, an element y of order p and an element $z \neq e$ different from x, y of G. Suppose $\phi = (xyz)$ is a cycle of $Sym(G)$. Then $\phi \in Alt_e(G)$ and by assumption $\phi \odot_x \phi \in Alt_e(G)$, which is a contradiction.

($\Longleftarrow$) Suppose $|G| = n$, $x \in G$, $o(x) = m$ and $\frac{n}{m} = k$. Then it is easy to see that L_x is a product of k cycle of length m. We first assume that G has odd order, then L_x is an even permutation, for all $x \in G$. This shows that for every ϕ, $\psi \in Alt_e(G)$ and $g \in G$, we have $\phi \odot_g \psi \in Alt_e(G)$. Using similar argument, in the case when G is a non-cyclic 2-subgroup or a Sylow 2-subgroup of G is not cyclic, $Alt_e(G)$ is a rotary closed subgroup of $Sym_e(G)$.

In the following example, we define a generalized action of $Sym_e(G)$ on the group G.

Example 2 Suppose $\square$: $Sym_e(G) \times G \rightarrow P^*$

(G) sends (ϕ, g) to $\phi(\langle g\rangle)$. We show that $\square$ is a generalized action of $Sym_e(G)$ on G. Then we have,

$$\phi\square(\psi\square g) = \\ \phi\square\psi(\langle g\rangle) = \\ \bigcup_{i\in Z}\phi\square\psi(g^i) = \\ \bigcup_{i\in Z}\phi(\langle\psi(g^i)\rangle) = \\ \{\phi(\psi(g^i))^j \mid i, j\in Z\}$$

and $\phi\psi\square g = \phi\psi(\langle g\rangle) = \{\phi(\psi(g^i)) \mid i\in Z\}$. This shows that $\phi\psi\square g \subseteq \phi\square(\psi\square g)$. On the other hand, $\phi\square G = \bigcup_{g\in G}\phi\square g = \bigcup_{g\in G}\phi(\langle g\rangle) = G$, which shows that $\square$ is a generalized action of $Sym_e(G)$ on the group G.

We end this paper with the following open problem:

Question When is this generalized action strong?

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References

Biggs, N. and White, A.T., 1979. Permutation Groups and Combinatorial Structures. Mathematical Society Lecture Notes **33**, Cambridge University Press, Cambridge.

Birkhoff, G., 1967. Lattice Theory (3rd. ed.), AMS Coll. Publ. Providence.

Corsini, P., 1993, Prolegomena of Hypergroup Theory. Second Edition, Aviani Edittore.

Jajcay, R., 1993, Automorphism groups of Cayley maps. *J. of Comb. Theory, Ser. B*, **59**: 297 – 310.

Jajcay, R., 1994, On a new product of groups. *Europ. J. Combinatorics*, **15**: 251 – 252.

Madanshekaf, A. and Ashrafi, A. R., 1998. Generalized Action of a hypergroup on a set. *Italian J. of Pure and Appl. Math.*, (3): 127 – 135.

Marty, F., 1934. Sur une generalization de lanotion de groupe, 8[iem] Congress Math. Scandinaves, Stockholm. p.45 – 49.

Vougiouklis, T., 1994. Hyperstructures and their Representations, Hadronic Press, Inc..