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On the approximate zero of Newton method*

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Abstract: A judgment criterion to guarantee a point to be a Chen's approximate zero of Newton method for solving nonlinear equation is sought by dominating sequence techniques. The criterion is based on the fact that the dominating function may have only one simple positive zero, assuming that the operator is weak Lipschitz continuous, which is much more relaxed and can be checked much more easily than Lipschitz continuous in practice. It is demonstrated that a Chen's approximate zero may not be a Smale's approximate zero. The error estimate obtained indicated the convergent order when we use $|f(x)| < \varepsilon$ to stop computation in software. The result can also be applied for solving partial derivative and integration equations.

Key words: Approximate zero, Newton method, Generalized Kantorovich Condition
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INTRODUCTION

Many nonlinear problems, such as nonlinear elliptic boundary value conditions, integration equation in radical transfer, optimal solution in operator theory, can be deduced to solve the equation

$$f(x) = 0, \qquad (1)$$

where $f: X \to Y$ is a Frechet differentiable nonlinear operator which maps Banach space X into Banach space Y (Rheinboldt, 1998; Tsuchiya, 1999). The well-known iterative method for solving the equation Eq.(1) is the Newton method defined by

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n), \quad n \geqslant 0, \qquad (2)$$

provided that $f'(x_n)^{-1}$ exists at each step, and its variant methods. (Traub, 1963; Ostrowski, 1973; Rheinboldt, 1998 and references therein).

The notion of an approximate zero of Eq.(2), a point where fast convergence to a zero of f starts immediately under iteration, plays an important role in Smale's computational complexity theory (Shub et al., 1996).

Definition (Smale's Approximate Zero) We call $x_0 \in X$ an approximate zero of Eq.(2) for f, if there exists a positive number q $(0 < q < 1)$ such that the iterative Eq.(2) starting from x_0 satisfies

$$\| x_{n+1} - x_n \| \leqslant q^{2^n - 1} \| x_1 - x_0 \|, \quad n = 0, 1, 2, \cdots. \qquad (3)$$

Definition (Chen's Approximate Zero) We call $x_0 \in X$ an approximate zero of Eq.(2) for f, if there exist positive numbers $a_0, a_1, \cdots, a_n, \cdots$ satisfying $a_0 < 1$ and $a_{n+1} \leqslant a_n^2$ such that for all $k \geqslant 0$,

$$\| f'(x_0)^{-1} f(x_{n+k}) \| \leqslant a_n^{2^k - 1} \| f'(x_0)^{-1} \cdot f(x_n) \|, \quad n \geqslant 0, k \geqslant 0. \qquad (4)$$

Several versions of the definition of the approximate zero and many more studies of it exist for the iteration Eq.(2) and its higher-order variant methods (Blum et al., 1997; Chen, 1994; Curry, 1989; Kim, 1988; Shub and Smale, 1985, 1986; Smale, 1981, 1985, 1986, 1987; Wang 1997, 1998a, 1998b, 1999, 2000a, 2000b; Wang and Han, 1990, 1997; Wang and Li 2001; Wang and Xuan 1987).

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Let $\alpha(x,f)=\beta(x,f)\gamma(x,f)$ for f being an analytic operator defined on X, where $\beta(x,f)=\|f'(x)^{-1}f(x)\|$, $\gamma(x,f)=\sup_{k\geq 2}\|\frac{1}{k!}f'(x)^{-1}f^{(k)}(x)\|^{\frac{1}{k-1}}$. It was proved in Wang and Han (1990) that x_0 is a Smale's approximate zero of Eq.(2) for f with $q=\frac{1}{2}$, if $\alpha(x_0,f)<\alpha_0\simeq 0.157671\cdots$, which improves the result in Smale (1986). Chen (1994) showed that x_0 is a second kind approximate zero of Newton method for any analytic function f in complex plane, if $\alpha(x_0,f)+\beta(x_0,f)<\alpha_0\simeq 0.142301\cdots$. It tells us that under this criterion x_0 is not only a Chen's but also a Smale's approximate zero of Eq.(2) for f because $\alpha(x_0,f)<0.142301\cdots<0.157671\cdots$. Furthermore, it can be proved that x_0 is so also if x_0 satisfies a Kantorovich condition in a region, or more generally, satisfies the weak Lipschitz condition in Wang (1999). Since Kantorovich condition had been successfully applied to partial derivative equation (Tsuchiya, 1999), discussions of approximate zero is practical. A question is whether a Chen's approximate zero is a Smale's approximate zero. Example 3 shows that a Chen's approximate zero may not be a Smale's approximate zero. A problem arises. Which condition can guarantee a point to be a Chen's approximate zero of Eq.(2) for an analytic operator f in a Banach space X?

Question Which condition can guarantee a point to be a Chen's approximate zero of Eq.(2) for a Frechet differentiable operator f which maps Banach space X into Banach space Y?

Based on the weak Lipschtiz continuous in Wang (1999; 2000a), we will answer the question under the Generalized Kantorovich Condition defined in Section 2, which takes the famous Kantorovich condition and Smale condition as special cases. From Theorem 1, we can see that in order to verify a point to be a Chen's approximate zero, what we do is to determine that Eq.(5) has a positive zero. Example 3 shows that the condition obtained is checked easily and is much more relaxed than the Kantorovich-type and Smale-type condition, because we not need get stricter Lipschitz condition and Smale's-type relation in a region. We will state and prove theorems in Section 2, and give several application examples in Section 3.

THEOREMS AND THEIR PROOFS

Let $z\in X$ and t be a positive number. Define

$$B(z,t)=\{y\in X \mid \|y-z\|<t\},$$
$$\overline{B(z,t)}=\{z\in X \mid \|y-z\|\leq t\}.$$

Definition (Generalized Kantorovich Condition) We say $z\in D\subset X$ satisfies Generalized Kantorovich Condition in a convex subset D for a nonlinear Frechet differentiable operator f, if

$$\|f'(z)^{-1}[f'(x)-f'(y)]\|\leq\int_{\rho(\|x-z\|)}^{\rho(\|x-y\|+\|x-z\|)}L(u)\mathrm{d}u,\ \forall x,y\in D$$

holds, and $\overline{B(z,t^*)}\subset D$, where t^* is the minimum positive simple zero of

$$h(t)=\int_0^{\rho(t)}L(u)(t-\rho^{-1}(u))\mathrm{d}u-t+\beta(z,f),\tag{5}$$

where $L(u)$ increases continuously satisfying $L(u)\geq 0$ on $[0,R)$, $\rho(0)=0$, and $\rho'(t)>0$ is bounded on $[0,R)$.

Theorem 1 If $x_0\in D\subset X$ satisfies the Generalized Kantorovich Condition in a convex subset D for an operator f, then x_0 is a Chen's approximate zero of Eq.(2) for f.

If $h(t)$ has another positive zero, then we can write a_n in Eq.(4) much more precisely.

Theorem 2 If $x_0\in D\subset X$ satisfies the Generalized Kantorovich Condition in a convex subset D for an operator f, and $h(t)$ has another zero $t^{**}<R$, and $\rho'(t)$ increases on $[0, R)$, then x_0 is a Chen's approximate zero of Eq.(2) for f with $a_n=q^{2n}$, $q\leq\frac{t^*}{t^{**}}$.

The following Lemmas are needed.

Lemma 1 Let $h(t)$ be defined in Eq.(5), then $h'(t)$ increases and is negative on $[0, t^*]$, and $h(t)$ has at most two positive zeros on $(0, R)$.

Lemma 2 Newton sequence $\{t_n\}_0^\infty$ defined by

$$t_{n+1}=t_n-\frac{h(t_n)}{h'(t_n)},\quad n=0,1,\cdots,t_0=0$$

satisfies

$$0 = t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < t^*,$$
$$\lim_{t\to\infty} t_n = t^*, \tag{6}$$

and

$$t^* - t_{n+1} = \mu_n (t^* - t_n)^2, \tag{7a}$$

where $\mu_n = \dfrac{\int_0^1 \int_{\rho(t_n)}^{\rho(t_n+\theta(t^*-t_n))} L(u)\,du\,d\theta}{h'(t_n)(t^* - t_n)}$ is a bounded sequence.

If $h(t)$ has another positive zero $t^{**} < R$, then

$$t^* - t_{n+1} = \frac{(t^{**} - t*)q(n)}{1 - q(n)\delta}\delta, \tag{7b}$$

where

$$\delta = \frac{t^*}{t^{**}},\ q(n) = \prod_{i=0}^{n-1} (\lambda_i \delta)^{2^{n-i-1}},$$

$$\lambda_i = \frac{(t^{**} - t_i)\int_0^1 \int_{\rho(t_i)}^{\rho(t_i+\theta(t^*-t_i))} L(u)\,du\,d\theta}{(t^* - t_i)\int_0^1 \int_{\rho(t_i)}^{\rho(t_i+\theta(t^{**}-t_i))} L(u)\,du\,d\theta}.$$

Proof By induction, Eq.(6) and Eq.(7a) follow immediately from the fact that the function $t - h(t)/h'(t)$ increases on $[0, t^*]$, and

$$t^* - t_{n+1} = -\frac{1}{h'(t_n)} \cdot \int_0^1 \int_{\rho(t_n)}^{\rho(t_n+\theta(t^*-t_n))} L(u)\,du\,d\theta (t^* - t_n) = \mu_n (t^* - t_n)^2,$$

where $\mu_n = \mu(t_n)$ and

$$\mu(t) = -\frac{\int_0^1 \int_{\rho(t)}^{\rho(t+\theta(t^*-t))} L(u)\,du\,d\theta}{h'(t)(t^* - t)} = -\frac{\int_0^1 L(\xi_\theta)\rho'(\eta_\theta)\theta\,d\theta}{h'(t)} > 0$$

is bounded on $[0, t^*)$, where $0 < \xi_\theta, \eta_\theta < t + \theta(t^* - t)$.

If $h(t)$ has another positive zero $t^{**} < R$, then

$$t^{**} - t_{n+1} = -\frac{1}{h'(t_n)} \cdot \int_0^1 \int_{\rho(t_n)}^{\rho(t_n+\theta(t^{**}-t_n))} L(u)\,du\,d\theta (t^{**} - t_n).$$

So,

$$\frac{t^* - t_{n+1}}{t^{**} - t_{n+1}} = \lambda_n \left(\frac{t^* - t_n}{t^{**} - t_n}\right)^2 = q(n)\delta.$$

Eq.(7b) follows.

Lemma 3 If $x_0 \in D$ satisfies the Generalized Kantorovich Condition in D, then $\{x_n\}$ defined by Eq.(2) converges to a solution of Eq.(1), and

$$\| x_{n+1} - x_n \| \leqslant t_{n+1} - t_n, \tag{8}$$

$$\frac{\| f'(x_0)^{-1} f(x_{n+1}) \|}{\| f'(x_0)^{-1} f(x_n) \|} \leqslant \frac{h(t_{n+1})}{h(t_n)}. \tag{9}$$

Proof Eq.(8) and Eq.(9) hold for $n = 0$. Suppose they hold for $i \leqslant k$, then

$$\| x_{k+1} - x_0 \| \leqslant \sum_{i=0}^{k} \| x_{i+1} - x_i \| \leqslant \sum_{i=0}^{k} (t_{i+1} - t_i) = t_{k+1} < t^*.$$

By

$$f(x_{k+1}) = f(x_{k+1}) - f(x_k) - f'(x_k)(x_{k+1} - x_k) = \int_0^1 (f'(x_k + \theta(x_{k+1} - x_k)) - f'(x_k))d\theta(x_{k+1} - x_k), \tag{10}$$

and Banach Lemma, we have

$$\| f'(x_0)^{-1} f(x_{k+1}) \| \leqslant h(t_{k+1}). \tag{11}$$

and

$$\| f'(x_{k+1})^{-1} f(x_0) \| \leqslant \frac{1}{1 - (1 + h'(t_{k+1}))} = -h'(t_{k+1})^{-1}.$$

So,

$$\| x_{k+2} - x_{k+1} \| \leqslant \| -f'(x_{k+1})^{-1} f'(x_0) \| \cdot \| f'(x_0)^{-1} f(x_{k+1}) \| \leqslant -h'(t_{k+1})^{-1} h(t_{k+1}) = t_{k+2} - t_{k+1},$$

which proves Eq.(8) by induction. From Eq.(11) and Lemma 2, $\exists x^* \in D \subset X$, s.t. $f(x^*) = 0$ and $\lim_{n\to\infty} x_n = x^*$. By Eq.(10), we have

$$\| f'(x_0)^{-1} f(x_{k+1}) \| \leqslant \frac{h(t_{k+1})}{h(t_k)} \| f'(x_0)^{-1} f(x_k) \|,$$

which completes the proof of Eq.(9) by induction.

Proof of Theorem 1 By Lemma 2, for all $n \geqslant 1$,

$$t^* - t_n = \mu_{n-1}(t^* - t_{n-1})^2, \text{ and } \frac{t^* - t_n}{t^* - t_{n-1}} = \mu_{n-1}(t^* - t_{n-1}) < 1. \quad (12)$$

Since $\{\mu_n\}$ is bounded and $t^* - t_n \to 0$ as $n \to \infty$, so $\exists\, n_0 > 0$, $0 < q < 1$ such that

$$0 \leqslant \mu_{n_0+k}(t^* - t_{n_0}) \leqslant q < 1 \quad \forall k \geqslant 0,$$

from which it can be derived that

$$\frac{t^* - t_{n_0+k+1}}{t^* - t_{n_0+k}} = \mu_{n_0+k}(t^* - t_{n_0}) \prod_{i=0}^{k-1} [\mu_{n_0+i}(t^* - t_{n_0})]^{2^{k-i-1}} \leqslant q^{2^k}, \quad \forall k \geqslant 0. \quad (13)$$

Define $\{a_n\}$ as

$$a_n = \begin{cases} q^{2^{n-n_0}} & n \geqslant n_0 \\ \max\{a_{n+1}^{\frac{1}{2}}, \mu_n(t^* - t_n)\} & n < n_0, \end{cases}$$

then $0 < a_{n+1} \leqslant a_n^2 < 1$ follows from Eq. (12) and Eq. (13). Since $\frac{h(t)}{t^* - t}$ decreases on $[0, t^*)$, we obtain

$$\frac{h(t_{n+1})}{h(t_n)} = \frac{\dfrac{h(t_{n+1})}{t^* - t_{n+1}}}{\dfrac{h(t_n)}{t^* - t_n}} \cdot \frac{t^* - t_{n+1}}{t^* - t_n} \leqslant \frac{t^* - t_{n+1}}{t^* - t_n} \leqslant a_n,$$

from which it can be deduced that

$$\| f'(x_0)^{-1} f(x_{n+1}) \| \leqslant a_n \| f'(x_0)^{-1} f(x_n) \|, \quad \forall n \geqslant 0$$

and

$$\| f'(x_0)^{-1} f(x_{n+k}) \| \leqslant a_n^{2^k - 1} \| f'(x_0)^{-1} f(x_n) \|, \forall n \geqslant 0, k > 1$$

by Lemma 2 and induction. That is, x_0 is a Chen's approximate zero.

Proof of Theorem 2 Let $\lambda = \max_n \{\lambda_n\}$, $q = \lambda\delta$, where λ_n, δ is defined in Lemma 2. From

$$\left(\frac{\int_{\rho(v)}^{\rho(t+v)} L(u) \mathrm{d}u}{t} \right)' = \frac{1}{t^2} \left[L(\rho(t+v))\rho'(t+v)t - \int_{\rho(v)}^{\rho(t+v)} L(u)\mathrm{d}u \right]$$
$$= \frac{1}{t}[L(\rho(t+v))\rho'(t+v) - L(\rho(\tau))\rho'(\eta)] \geqslant 0$$
$$v < \tau, \eta < t + v, \forall v \geqslant 0, t \geqslant 0,$$

we get $0 < \lambda_n < 1$ for all $n \geqslant 0$, $q(n)$ decreases as n increases and $0 < q \leqslant \delta < 1$. Thus, we have

$$\frac{h(t_{n+1})}{h(t_n)} \leqslant \frac{t^* - t_{n+1}}{t^* - t_n} = \frac{1 - q(n)\delta}{1 - q(n+1)\delta} \cdot \frac{q(n+1)}{q(n)} < q^{2^n},$$

which implies that

$$\| f'(x_0)^{-1} f(x_{n+k}) \| \leqslant a_n^{2^k - 1} \| f'(x_0)^{-1} f(x_n) \|, \quad \forall n \geqslant 0, k > 1$$

by Lemma 2 and induction, where $a_n = q^{2^n}$. That is, x_0 is a Chen's approximate zero.

EXAMPLES

In this section, we give some application examples of the Generalized Kantorovich Condition.

Example 1 If x_0 satisfies the Kantorovich condition in D, a convex subset of X, then it satisfies the Generalized Kantorovich Condition in D. Therefore x_0 is a Chen's approximate zero.

Proof Given $x_0 \in D$ satisfy the Kantorovich condition, that is, there exists a constant $K > 0$ such that $K\beta(x_0, f) < \frac{1}{2}$,

$$\| f'(x_0)^{-1}[f'(x) - f'(y)] \| \leqslant K \| x - y \|, \quad \forall x, y \in D,$$

and $\overline{B(x_0, 1 - \sqrt{1 - 2K\beta(x_0, f)})} \subset D$. We have x_0 satisfies the Generalized Kantorovich Condition in D with $L(u) = K$, $\rho(t) = t$, and $h(t) = \frac{1}{2}Kt^2 - t + \beta(x_0, f)$, which has two positive zeros $t^* = 1 - \sqrt{1 - 2K\beta(x_0, f)}$, $t^{**} = 1 + \sqrt{1 - 2K\beta(x_0, f)}$. So, x_0 is a Chen's approximate point of Eq.(2) for f with $a_n = q^{2^n}$, $q = \frac{t^*}{t^{**}} = \frac{1-\sqrt{1-2K\beta(x_0,f)}}{1+\sqrt{1-2K\beta(x_0,f)}}$ by Theorem 2.

Example 2 There is a constant $\alpha_0 = 3-2\sqrt{2}$, such that x_0 is a Chen's approximate zero of Eq.(2) for f, if $\alpha(x_0,f) < \alpha_0$, where f is an analytic operator defined on X.

Proof When $\alpha(x_0,f) < 3-2\sqrt{2}$, we have

$$\| f'(x_0)^{-1}[f'(x) - f'(y)] \| \leqslant \int_{\rho(\|x-x_0\|)}^{\rho(\|x-x_0\|+\|x-y\|)} L(u)\mathrm{d}u$$

$$\forall x \in B\left(x_0, \frac{1-\frac{\sqrt{2}}{2}}{\gamma(x_0,f)}\right)$$

with $L(u) = \frac{2\gamma(x_0,f)}{(1-\gamma(x_0,f)u)^3}$, $\rho(t) = t$, and function $h(t) = \beta(x_0,f) - t + \frac{\gamma(x_0,f)t^2}{1-\gamma(x_0,f)t}$ has two positive zeros $0 < t^* < t^{**} < \frac{1-\frac{\sqrt{2}}{2}}{\gamma(x_0,f)}$. It follows that x_0 satisfies Generalized Kantorovich Condition in $B\left(x_0, \frac{1-\frac{\sqrt{2}}{2}}{\gamma(x_0,f)}\right)$. So, x_0 is a Chen's approximate zero for analytic operator f with $a_n = q^{2^n}$ and $q = \frac{t^*}{t^{**}}$ by Theorem 2.

Example 3 For a large enough positive constant c, $t_0 = 0$ is a Chen's but is not a Smale's approximate zero of Eq.(2) for the function defined by

$$f(t) = \left(\frac{1}{4}\arctan t - 1\right)t - \frac{1}{8}\ln(1+t^2) + c \quad t \in (-\infty, +\infty)$$

Proof It can be verified that $h(t) = f(t)$ with $L(u) = \frac{1}{4}$, $\rho(t) = \arctan t$. Since

$$h'(t) = \frac{\arctan t}{4} - 1 < 0 \quad \forall t \in (-\infty, +\infty),$$

$h(0) = c > 0$, $\lim_{t\to+\infty} h(t) = -\infty$, so, $h(t)$ has only one positive zero t^*. Therefore, t_0 satisfies the Generalized Kantorovich Condition in $(-\infty, +\infty)$. As a result of Theorem 1, $\forall c > 0$, t_0 is a Chen's approximate zero of Eq.(2) for the f defined.

Suppose $\{t_n\}$ is the iterative sequence starting from $t_0 = 0$ defined by Eq.(2) for f.

Since $\lim_{c\to+\infty} c(\arctan c - 1) - \frac{1}{4}\ln(1+c^2) = +\infty$, so for a large enough c

$$\frac{1}{4}c \arctan c - \frac{1}{8}ln(1+c^2) > c\left(1 - \frac{1}{4}\arctan c\right).$$

It follows that

$$t_2 - t_1 = \frac{\frac{1}{4}c\arctan c - \frac{1}{8}\ln(1+c^2)}{\left(1-\frac{1}{4}\arctan c\right)} > c = t_1 - t_0,$$

which implies that Eq.(3) does not holds for $n = 1$, namely, t_0 is not a Smale's approximate zero of Eq.(2) for the f defined.

On the other hand, since $|h'(x) - h'(y)| \leqslant K|x-y|$ with $K = \frac{1}{4}$, for all $x, y \in (-\infty, +\infty)$, if $t_0 = 0$ satisfies the Kantorovich condition, then $Kc \leqslant 1/2$, or $c \leqslant 2$.

If $0 < h'(y) - h'(x) = \int_x^y \frac{1}{1+u^2}\mathrm{d}u \leqslant \int_x^y \frac{2\gamma}{(1-\gamma u)^3}\mathrm{d}u$, $\forall -\infty < x < y < +\infty$ with a constant $\gamma > 0$, then $\left.\frac{1}{1+u^2}\right|_{u=0} \leqslant \left.\frac{2\gamma}{(1-\gamma u)^3}\right|_{u=0}$, or $\frac{1}{\gamma} \leqslant 2$. When $t_0 = 0$ satisfies the Smale's condition, then $c \leqslant 2(3-2\sqrt{2})$. So, It is failure when we use Kantorovich condition or Smale condition to judge $t_0 = 0$ to be a Smale's approximate zero for $c > 2$, in other words, a point much more closer to zero is needed, which shows that the condition obtained in this paper is easier to check and is much more relaxed than that before.

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