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Reliability analysis of DOOF for Weibull distribution*

CHEN Wen-hua(陈文华)[1†], CUI Jie(崔杰)[1], FAN Xiao-yan(樊晓燕)[1]
LU Xian-biao(卢献彪)[2], XIANG Ping(相平)[2]

([1] *The State Key Laboratory of Fluid Power Transmission and Control, Zhejiang University, Hangzhou 310027, China*; [2] *Hangzhou Electrical Connector Factory, Hangzhou 310015, China*)

[†]E-mail: chen-wenhua@sohu.com

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Abstract: Hierarchical Bayesian method for estimating the failure probability p_i under DOOF by taking the quasi-Beta distribution $B(p_{i-1}, 1, 1, b)$ as the prior distribution is proposed in this paper. The weighted Least Squares Estimate method was used to obtain the formula for computing reliability distribution parameters and estimating the reliability characteristic values under DOOF. Taking one type of aerospace electrical connector as an example, the correctness of the above method through statistical analysis of electrical connector accelerated life test data was verified.

Key words: DOOF(Data only one failure) data, Hierarchical Bayesian estimate, Reliability analysis
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INTRODUCTION

Statistical analysis of reliability test data showed that when the failure number exceeds 2, there are many tested methods for processing this problem (Zhang *et al*. 1989). However, in the reliability test of product, with the appearances of high reliability units, even in the accelerated life test, it may occur that none of the test units fail or only one unit fail before the predetermined test time is reached. Such phenomenon is common especially in the case of small samples. Most papers (Liu *et al*., 2001; Anderman *et al*., 1997; Pugh, 1993) only mentioned Zero-Failure problems. Sandoh *et al*. (1991) and Bailey (1997) proposed some theories and methods using statistical analysis to deal with problems of Zero-Failure. However, none of the reliability statistical analysis methods of DOOF (Data Only One Failure) were considered. In order to take full advantage of information on the product failure and increase the precision of reliability evaluation, it is necessary to study the reliability statistical analysis methods of DOOF. In this paper, the authors propose the hierarchical Bayesian method for estimating failure probability by taking the quasi-Beta distribution as the prior distribution on the basis of zero-failure data analysis method and validate this method through statistical analysis of testing data.

DATA MODEL AND STATISTICAL ANALYSIS METHODS OF DOOF

1. DOOF model

We assume that specimens are put on time censored life test inspected interval with a predetermined time. The inspected time is noted as $0 < \tau_1 < \tau_2 < \cdots \tau_k$. Such test can yield DOOF defined by (s_i, r_i, τ_i) when only one item failed in the interval of (τ_{m-1}, τ_m) and the rest of items are in good condition. Here s_i is the specimen number and r_i is the failure number at time τ_i. If $i \leqslant m-1$, then $r_i = 0$, or else if $i > m-1$, then $r_i = 1$; and $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_k$.

We can obtain the following information from the above DOOF model:

(1) Product failure probability $p_0 = 0$ when

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$\tau_0 = 0$.

(2) The failure probability is defined as $p_i = P(T < \tau_i)$. It is reasonable that we accept $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$ for $0 < \tau_1 < \tau_2 < \cdots \tau_k$. The larger s_k is, the smaller $p_i (i = 1, 2, \cdots, k)$ are.

(3) The life of all products is not longer than τ_m because one item fails in time period (τ_{m-1}, τ_m).

2. Feasible statistical analysis methods of DOOF

At present, statistical analysis methods in the case of DOOF include minimum χ^2 method, equivalent number of failures method and weighted Least Squares Estimate method. Practical information indicates that the estimates of reliability obtained by the χ^2 method and the equivalent number of failures method are relatively low (Mao *et al*., 1993). The reliability estimate by minimum χ^2 method improved by Bayesian χ^2 method tends to be higher. Compared with minimum χ^2 method, equivalent number of failures method is better; but its estimate of reliability is prone to be higher. Furthermore, the calculation of this method is laborious and the judging of data type is necessary. It is difficult to analyze reliability under DOOF because of the limitation of the equivalent number of failures method.

The weighted Least Squares Estimate method is a reliability analysis method that reaches the reliability target by means of fitting the curve of distribution. In comparison with the χ^2 method and equivalent number of failures method, the weighted Least Squares Estimate method, because of its simplicity and estimate precision, has become the most commonly used method for treating DOOF. Because the weighted Least Squares Estimate method is based on the reliability analysis method under failure case, it can directly be used in resolving the problem under DOOF. The steps are as follows: (1) Estimate of failure probability $p_i = P(T < \tau_i)$, $i = 1, 2, \cdots, k$, at time τ_i. (2) Fit a curve of distribution to the points $(\tau_i, \hat{p}_i)$, by using the weighted least squares theory. (3) Estimate the reliability from the fitted distribution curve. The second and third steps are not difficult. The crucial point is the first step, how to estimate the probability $p_i = P(T < \tau_i)$.

BAYESIAN ESTIMATE OF FAILURE POSSIBILITY p_i UNDER DOOF

Among the methods for estimating failure probability p_i, the classical estimate method is simple in calculation. However, it yields the estimate of p_i, defined by $\hat{p}_i = 0.5/(s_i + 1)$ $(i = 1, 2, \cdots, k)$, which is derived from the estimate method under failure data and the result is lower precision and cannot truly reflect the reliability level of the product. So, the Bayesian method is more attractive. Bayesian analysis involves expression of subjective knowledge or degree-of-belief about model parameter values as a prior distribution for them. This distribution is then mathematically combined with observed data to yield the posterior distribution of the parameter values. The posterior distribution is narrower than the prior one, thereby reflecting the added information from the data. The posterior distribution yields a Bayesian estimate and probability for the true parameter values and their functions. Generally speaking, the estimate performance of the Bayesian method is better. However, we prefer to use the classical method instead of the Bayesian method, if the prior distribution is not chosen properly. Therefore, the choice of proper prior distribution of p_i is essential.

1. Establishment of prior distribution

The fact that in the case of DOOF, none of the samples fail in the time period $(0, \tau_i)$ showed that the reliability in the time period $(0, \tau_i)$ might be very high. So at time τ_i, the possibility that failure probability p_i is small is very large and that p_i is large is very small. In practice, quasi-Beta distribution can be taken as the prior distribution of p_i. Its density function is

$$f(p_i; \theta_1, \theta_2, a, b) = \frac{(p_i - \theta_1)^{a-1}(\theta_2 - p_i)^{b-1}}{B(a, b)(\theta_2 - \theta_1)^{a+b-1}},$$
$$0 < \theta_1 < p_i < \theta_2 \leqslant 1 \qquad (1)$$

From the above function, it can be seen that it is a strictly monotonously decreasing function for p_i when a and b are constant. It can not only satisfy the requirement that the possibility that failure probability p_i is small is very high and that

p_i is large is very small, but also can deal with most testing requirements by changing the integral interval of (θ_1, θ_2). Especially when the number of sample running in the test does not vary, Bayesian estimate of $p_i (i = 1, 2, \cdots, k)$ can also meet the requirement that $\hat{p}_1 < \hat{p}_2 < \cdots < \hat{p}_k$. Thus, it is reasonable to take quasi-Beta distribution $B(p_{i-1}, 1, 1, b)$ as the prior distribution under zero-failure data.

In the case of DOOF, though choosing quasi-Beta distribution $B(p_{i-1}, 1, 1, b)$ as the prior distribution will easily lose the important information that one item has failed when $i > m - 1$; the loss can be avoided by modifying the lower bound value p_i. So quasi-Beta distribution $B(p_{i-1}, 1, 1, b)$ can still be chosen as the prior distribution under DOOF.

2. Choosing hyperparameter

The density function of quasi-Beta distribution is an exact monotonously decreasing function of p_i when $a \leqslant 1$, $b > 1$. Such characteristic corresponds to the prior information that failure probability p_i is small is very large and that p_i is large is very small. In this way, we can get a rough scope of a and b. That is, $a \leqslant 1, b > 1$. From Fig. 1, it can be seen that, when b is constant and $a \leqslant 1$, the smaller a is, the thinner is the right tail of the quasi-Beta distribution density function. According to Berger's (1985) viewpoint on the robustness of the Bayesian estimate, the thin tail distribution often makes the Bayesian estimate have poor robustness. Hence, we choose hyperparameter a as 1.

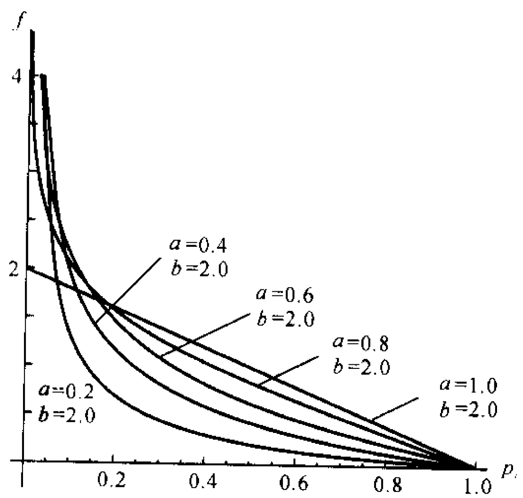

Fig. 1 Density function of quasi-Beta distribution when $0 \leqslant a < 1$, $b = 2$

It is difficult to confirm the exact value of b further. Though it can be decided on the basis of expert experience, cases may happen that the expert is not experienced enough to give a value for b. When a prior distribution contains a hyperparameter, giving another prior distribution to the hyperparameter may lead to a more robust result by two prior distributions than by one prior distribution. In our problem, the prior distribution of the hyperparameter b could be chosen as the uniform distribution $U(1, C)$, denoted as

$$\pi_{i2}(b) = U(1, C) \qquad (2)$$

Here C is constant. According to the characteristic of quasi-Beta distribution density function, the larger b is (under the condition of $a = 1$), the thinner is the right tail of the quasi-Beta distribution density function (see Fig. 2). Likewise, considering the robustness of the Bayesian estimate, C cannot be taken too large and it is suitable to take $3 - 7$ as C (Mao *et al*., 1996).

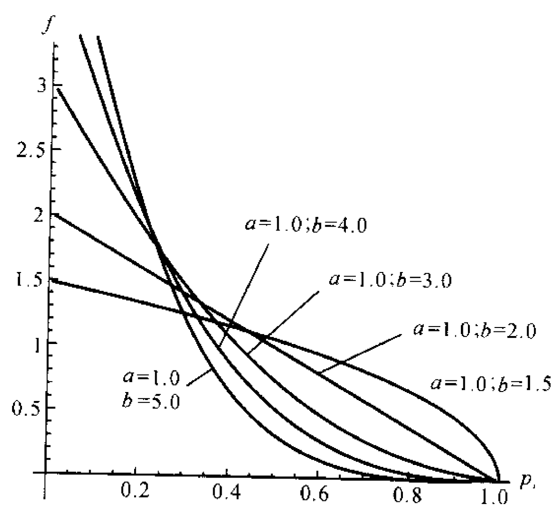

Fig. 2 Density function of quasi-Beta distribution when $a = 1, b > 1$

Because θ_1 and θ_2 represent the lower and upper bound of the interval p_i respectively, it is reasonable to take 1 as θ_2. That means the maximal failure probability is equal to 1. It would naturally be better with an acute upper bound of p_i through expert experience. The value of θ_1 should accord with the inequality $\theta_1 \geqslant \hat{p}_{i-1}$, which assures that no backhang phenomenon exists, so that $\hat{p}_1 \geqslant \hat{p}_2 \geqslant \cdots \geqslant \hat{p}_k$. Thus, using Eq. (1) we can obtain the first stage prior distribution at once as follows.

$$\pi_{i1}(p_i \mid \hat{p}_{i-1}, 1, 1, b) = \frac{(1-p_i)^{b-1}}{B(1, b)(1-\hat{p}_{i-1})^b} \tag{3}$$

It is not difficult to generate the prior distribution of p_i by using the Bayesian formula below:

$$\pi(p_i \mid \hat{p}_{i-1}) = \frac{1}{C-1}\int_1^C \frac{(1-p_i)^{b-1}}{B(1, b)(1-\hat{p}_{i-1})^b}\mathrm{d}b \tag{4}$$

It is obvious that the above formula is a monotonously decreasing function of p_i.

3. Bayesian estimate of failure probability p_i

Suppose we do a life test with s_i products, and that r_i products fail in the time period $(0, \tau_i)$. As the failure probability of each product is p_i, the likelihood function can be represented by the binomial distribution

$$L(r_i, p_i) = \binom{s_i}{r_i} p_i^{r_i}(1-p_i)^{s_i-r_i} \tag{5}$$

When $r_i = 0$ (under the condition of $i < m$), that means the likelihood function

$$L(0, p_i) = (1-p_i)^{s_i} \tag{6}$$

Under square error loss, combining likelihood function Eq.(6) with prior distribution Eq.(4), the Bayesian estimate of p_i can be obtained as follows.

$$\hat{p}_i = E(p_i \mid s_i) = \frac{\int_1^C\int_{\hat{p}_i-1}^1 \frac{p_i(1-p_i)^{b+s_i-1}}{B(1, b)(1-\hat{p}_{i-1})^b}\mathrm{d}p_i\,\mathrm{d}b}{\int_1^C\int_{\hat{p}_i-1}^1 \frac{(1-p_i)^{b+s_i-1}}{B(1, b)(1-\hat{p}_{i-1})^b}\mathrm{d}p_i\,\mathrm{d}b} = \hat{p}_{i-1} + (1-\hat{p}_{i-1})\left[(1+s_i)\ln\frac{s_i+C+1}{s_i+2} - s_i\ln\frac{s_i+C}{s_i+1}\right]\Big/\left(C-1-s_i\ln\frac{s_i+C}{s_i+1}\right) \tag{7}$$

If $r_i = 1$ (under the condition of $i \geqslant m$), it means one product fails with s_i products in the test within τ_i. According to Eq. (5), the likelihood function is

$$L(1, p_i) = s_i p_i(1-p_i)^{s_i-1} \tag{8}$$

Suppose the product fails at time τ_a, where $m-1 < \alpha < m$. In terms of gambling theory, some failure phenomena can occur to a certain extent at certain time while products have not failed at other time. Therefore, it is reasonable to suppose the failure probability p_α at time τ_α equals $\max(\hat{p}_{m-1}, 0.5)$. By using Eq. (1) we can obtain the prior distribution of p_m

$$p_m = \frac{1}{C-1}\int_1^C \frac{(1-p_m)^{b-1}}{B(1, b)(1-p_\alpha)^b}\mathrm{d}b \tag{9}$$

It is reasonable to take Eq. (9) as the prior distribution of p_m because it is a monotonously decreasing function of p_m.

Under square error loss, the Bayesian estimate of p_m is its posterior expectation, i.e.

$$\hat{p}_m = E(p_m \mid s_m, r_m = 1) = \frac{\int_1^C\int_{p_\alpha}^1 \frac{p_m^2(1-p_m)^{s_i+b-2}}{B(1, b)(1-p_\alpha)^b}\mathrm{d}p_m\,\mathrm{d}b}{\int_1^C\int_{p_\alpha}^1 \frac{p_m(1-p_m)^{s_i+b-2}}{B(1, b)(1-p_\alpha)^b}\mathrm{d}p_m\,\mathrm{d}b} = 1-(1-p_\alpha)\cdot \frac{(C-1)p_\alpha+(1-p_\alpha)(1+s_i)\ln\frac{s_i+C+1}{s_i+2}-s_i\ln\frac{s_i+C}{s_i+1}}{(C-1)p_\alpha+(1-p_\alpha)s_i\ln\frac{s_i+C}{s_i+1}-(s_i-1)\ln\frac{s_i+C-1}{s_i}} \tag{10}$$

After acquiring the Bayesian estimate of p_m, we can obtain the hierarchical Bayesian estimate of p_i ($i > m$) under DOOF on the condition of square error loss.

$$\hat{p}_i = E(p_i \mid s_i, r_i = 1) = \frac{\int_1^C\int_{\hat{p}_i-1}^1 \frac{p_i^2(1-p_i)^{s_i+b-2}}{B(1, b)(1-\hat{p}_{i-1})^b}\mathrm{d}p_i\,\mathrm{d}b}{\int_1^C\int_{\hat{p}_{i-1}}^1 \frac{p_i(1-p_i)^{s_i+b-2}}{B(1, b)(1-\hat{p}_{i-1})^b}\mathrm{d}p_i\,\mathrm{d}b} = 1-(1-\hat{p}_{i-1})\cdot \frac{(C-1)\hat{p}_{i-1}+(1-\hat{p}_{i-1})(1+s_i)\ln\frac{s_i+C+1}{s_i+2}-s_i\ln\frac{s_i+C}{s_i+1}}{(C-1)\hat{p}_{i-1}+(1-\hat{p}_{i-1})s_i\ln\frac{s_i+C}{s_i+1}-(s_i-1)\ln\frac{s_i+C-1}{s_i}} \tag{11}$$

RELIABILITY DISTRIBUTION PARAMETERS AND ESTIMATE OF RELIABILITY CHARACTERISTIC VALUES UNDER DOOF

After obtaining the Bayesian estimates of all the probability $p_i = P(T < \tau_i)$, we can estimate reliability distribution parameters and characteristic values using the weighted Least Squares Estimate method.

Suppose the product's life T follows Weibull distribution, whose Cumulative Distribution Function is

$$F(t) = 1 - \exp[-(t/\eta)^m] \quad (t > 0)$$

Here $m > 1$ is a shape parameter and $\eta > 0$ is the characteristic life.

Again, suppose that the product's failure probability is p_i, at the time point $t = \tau_i$, and that $\hat{p}_i$ is its estimate. Then we have

$$p_i = 1 - \exp[-(\tau_i/\eta)^m] \, (i = 1, 2, \cdots, k)$$

and

$$\ln\tau_i = \ln\eta + \frac{1}{m}\ln[-\ln(1 - p_i)] \quad (i = 1, 2, \cdots, k)$$

Let $y_i = \ln\tau_i$, $x_i = \ln[-\ln(1 - p_i)]$, then we have $y_i = \frac{1}{m}x_i + \ln\eta$

Replace $x_i = \ln[-\ln(1 - p_i)]$ in the above formula by $\hat{x}_i = \ln[-\ln(1 - \hat{p}_i)]$ to yield

$$y_i = \frac{1}{m}\hat{x}_i + \ln\eta + \varepsilon_i$$

Here ε_i is the deviation resulting from the replacement of p_i by $\hat{p}_i$.

In order to estimate the two parameters m and η of Weibull distribution, we can use the weighted least squares method. Hence we can obtain the weighted Least Squares Estimators $\hat{m}$, $\hat{\eta}$ of m and η in the following formula

$$\sum_{i=1}^{k} w_i\left(y_i - \frac{1}{m}x_i - \ln\eta\right)^2 \Rightarrow \min$$

from which we obtain

$$\hat{m} = \frac{B - A^2}{D - AC}, \quad \hat{\eta} = \exp\left(\frac{BC - AD}{B - A^2}\right) \quad (12)$$

Here

$A = \sum_{i=1}^{k} w_i x_i$, $B = \sum_{i=1}^{k} w_i x_i^2$, $C = \sum_{i=1}^{k} w_i y_i$, $D = \sum_{i=1}^{k} w_i x_i y_i$ and w_i is weight.

In the analysis of the calculating process of the weighted least squares method, it can be seen that it is not τ_i, but $\ln\tau_i$, that plays a role in the calculation. So the weights are chosen by

$$w_i = \frac{s_i \ln\tau_i}{s_1\ln\tau_1 + s_2\ln\tau_2 + \cdots + s_k\ln\tau_k} \quad (i = 1, 2, \cdots, k) \quad (13)$$

Again from the estimates of m and η, we can obtain the estimate of reliability at time t.

$$\hat{R}(t) = \exp[-(t/\hat{\eta})^{\hat{m}}] \quad (14)$$

EXAMPLE AND CONCLUSION

Now, take the model Y11X-2221 aerospace electrical connector accelerated constant-stress life test under vibration stress as an example. Four groups are independently put on time censored life test inspected interval with a predetermined time under different stress levels. Five samples were tested when the power spectrum density was 0.2 g^2/Hz, five when it was under 0.4 g^2/Hz, five when it was under 0.6 g^2/Hz and six when it was under 1.0 g^2/Hz. The testing result yielded complete failure data under stress levels of 0.4 g^2/Hz, 0.6 g^2/Hz and 1.0 g^2/Hz and DOOF under the stress level of 0.2 g^2/Hz. By the fixed-time of τ_k (τ_k = 85h), only one product failed because its contact resistance exceeded the criterion 3mΩ (GJB101-86, 1986). The data are shown in Table 1.

Because the life of the aerospace electrical connector followed two-parameter Weibull distribution whose shape parameter was $m > 1$ (Chen *et al*., 1997), we can treat the data with the above theory and method. In calculation, we consider C = 4, 5, 6. The estimates of failure probability p_i ($i = 1, 2, \cdots, k$) and the weights under different C value are also shown in Table 1.

Combing the data in Table 1 with Eq. (12) and calculating by computer program, we obtained estimates of the distribution parameters. When C = 4, we got $\hat{m}$ = 3.1609,

$\hat{\eta} = 72.5182$. When $C = 5$ we got $\hat{m} = 3.2289$, $\hat{\eta} = 74.8714$ and when $C = 6$ we got $\hat{m} = 3.2891$, $\hat{\eta} = 75.3594$.

Table 1 DOOF and the estimate of failure probability at time τ_i

Testing interval (h)	Failure number	$\ln\tau_i$	The sample number by the time τ_i	Weight	The estimate of failure probability		
					$C = 4$	$C = 5$	$C = 6$
0 – 38	0	3.6376	5	0.1433	0.1158	0.1091	0.1032
38 – 44	0	3.7842	5	0.1491	0.2171	0.2063	0.1957
44 – 61	0	4.1109	5	0.1620	0.3073	0.2929	0.2787
61 – 67	1	4.2047	4	0.1325	0.5831	0.5776	0.5725
67 – 73	0	4.2905	4	0.1352	0.6508	0.6418	0.6332
73 – 79	0	4.3694	4	0.1377	0.7066	0.6954	0.6846
79 – 85	0	4.4427	4	0.1400	0.7529	0.7405	0.7283

The results showed that different values of C had different, but not great influence on the estimate. It agrees with the fact that the possibility of probability p_i being large is small, that is, $\hat{p}_1 < \hat{p}_2 < \cdots < \hat{p}_k$. The result in probability estimate was robust and the estimate of the shape parameter m approximated that obtained under other stress levels (Wei, 2002). So we can say the result was credible.

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