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Nonlinear predator-prey singularly perturbed Robin Problems for reaction diffusion systems*

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Abstract: The nonlinear predator-prey reaction diffusion systems for singularly perturbed Robin Problems are considered. Under suitable conditions, the theory of differential inequalities can be used to study the asymptotic behavior of the solution for initial boundary value problems.

Key Words: Nonlinear, Predator-prey, Reaction diffusion, Singular perturbation
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INTRODUCTION

The nonlinear singularly perturbed problem is a very attractive object of study in international academic circles (De Jager *et al*., 1996). In the recent 10 years, many approximation methods, including the method of averaging, boundary layer method, matched asymptotic expansion method, and multiple scales method, have been developed and refined. Recently, many scholars (Bohé, 1999; Butuzov *et al*., 1999; O'Malley, Jr., 1999; Butuzov *et al*., 2001; Kelley, 2001) have done a great deal of work on the above problems. Using the method of differential inequality and other methods, Mo considered also a class of singularly perturbed nonlinear boundary value problems for the ordinary differential equation in (Mo, 1993; 1999), the reaction diffusion equations in (Mo, 1989; 2001a; Mo and Feng, 2001), the boundary value problems of elliptic equation in (Mo and Shao, 2001; Mo and Ouyang, 2001), the initial boundary value problems of hyperbolic equation in (Mo, 2001b) and the biomathematics equation in (Mo and Wang, 2002). In this work, with the use of a special and simple method, we studied a class of nonlinear singularly perturbed predator-prey reaction diffusion system in biomathematics.

Now we consider the following Robin Problem for the nonlinear reaction diffusion system:

$$\varepsilon\frac{\partial u_1}{\partial t} - Lu_1 = u_1 f_1(\lambda_1 - r_{11}u_1 - r_{12}u_2),\ (t, x) \in (0, T] \times \Omega \tag{1}$$

$$\varepsilon\frac{\partial u_2}{\partial t} - Lu_2 = u_2 f_2(-\lambda_2 + r_{21}u_1),\ (t, x) \in (0, T] \times \Omega \tag{2}$$

$$Bu_i \equiv u_i + a_i(x)\frac{\partial u_i}{\partial n} = g_i(x),\ a_i(x) \geqslant a_0 > 0,\ x \in \partial\Omega, i = 1, 2, \tag{3}$$

$$u_i = h_i(x),\ t = 0,\ i = 1, 2 \tag{4}$$

where ε is a positive small parameter which expresses larger diffusion coefficient, u_1 and u_2 stand for the numbers of prey and predator respectively, λ_i, r_{ij} are positive constants, λ_1 represents the intrinsic growth rate of the prey population, λ_2 refers to the death rate of the predator, r_{ij} are density coefficients of the prey and predator populations respectively, and

$$L = \sum_{i,j=1}^{n}\alpha_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\beta_i(x)\frac{\partial}{\partial x_i}$$

$$\sum_{i,j=1}^{n}\alpha_{ij}(x)\xi_{ij}\xi_{jk} \geqslant \lambda\sum_{i=1}^{n}\xi_i^2,\ \forall\,\xi_i \in R,\ \lambda > 0,$$

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$x \equiv (x_1, x_2, \cdots, x_n) \in \Omega$, Ω denotes a bounded region in R^n, $\partial\Omega$ signifies a boundary of Ω for class $C^{1+\alpha}$, ($\alpha \in (0,1)$ is a Hölder exponent), $\frac{\partial}{\partial n}$ denotes the outward derivative on $\partial\Omega$, and L is a uniformly elliptic operator. We will construct the asymptotic expansions of the solution and discuss their asymptotic behavior.

We need the following hypotheses:

[$\boldsymbol{H}_1$] α_{jk}, β_j, a_i, $g_i > 0$ and $h_i > 0$ with respect to variables are Hölder continuous in correspondence ranges and $g_i(x) = h_i(x)$, $i = 1, 2$ as $x \in \partial\Omega$.

[$\boldsymbol{H}_2$] $f_i(y)$ is continuous differentiable and

$f_1(\lambda_1) \leqslant 0$, $f_{1y}(y) \geqslant 0$, $f_2(-\lambda_2) \leqslant 0$, $f_{2y}(y) \leqslant 0$.

We now construct the formal asymptotic expansions for the solution of the problem Eqs.(1) – (4).

The reduced problem is

$$-Lu_1 = u_1 f_1(\lambda_1 - r_{11}u_1 - r_{12}u_2), \quad x \in \Omega, \tag{5}$$

$$-Lu_2 = u_2 f_2(-\lambda_2 + r_{21}u_1), \quad x \in \Omega, \tag{6}$$

$$Bu_i = g_i(x), \quad x \in \partial\Omega, \quad i = 1, 2. \tag{7}$$

We also assume that

[$\boldsymbol{H}_3$] there is a positive solution (U_1, U_2) of the problem Eqs.(5) – (7).

But (U_1, U_2) may not satisfy the boundary condition Eq.(4), so that we need to construct the initial layer correction (V_1, V_2).

We introduce a stretched variable (de Jager *et al.*, 1996):

$$\tau = \frac{t}{\varepsilon}.$$

And consider the initial boundary value problem for the parabolic system:

$$\frac{\partial V_1}{\partial \tau} - LV_1 = 0, \tag{9}$$

$$\frac{\partial V_2}{\partial \tau} - LV_2 = 0, \tag{10}$$

$$BV_i(\tau, x) = 0, \quad x \in \partial\Omega, \quad i = 1, 2, \tag{11}$$

$$V_i(0, x) = h_i(x) - U_i(x), \quad \tau = 0, \quad i = 1, 2. \tag{12}$$

It is easy to see that there is a positive solution (V_1, V_2) for the linear problem Eqs.(9) – (12). It is not difficult to see that (V_1, V_2) possesses initial layer behavior.

Now we prove that

$$u_i(t, x) = U_i(x) + V_i(t/\varepsilon, x) + O(\varepsilon), \quad (t, x) \in [0, T] \times (\Omega + \partial\Omega), \quad 0 < \varepsilon \ll 1, \quad i = 1, 2. \tag{13}$$

We have the following theorem:

Theorem. Under the hypotheses [H_1] – [H_3], there exists a solution (u_1, u_2) of the singularly perturbed Robin Problem Eqs.(1) – (4) for the non-linear predator-prey reaction diffusion system and holds the uniformly valid asymptotic expansions Eq.(13).

Proof. We first construct the auxiliary functions α_i and β_i:

$$\alpha_i = U_i + V_i - \delta_i \varepsilon, \quad i = 1, 2, \tag{14}$$

$$\beta_i = U_i + V_i + \delta_i \varepsilon, \quad i = 1, 2, \tag{15}$$

where δ_i are large enough positive constants, which will be decided below.

Obviously, we have

$$\alpha_i \leqslant \beta_i, \quad (t, x) \in [0, T] \times (\Omega + \partial\Omega), \quad i = 1, 2. \tag{16}$$

And for $x \in \partial\Omega$, there are positive constants M_{i1}, $i = 1, 2$, such that

$$B\alpha_i|_{x\in\partial\Omega} = B[U_i + V_i - \delta_i\varepsilon]_{x\in\partial\Omega} = g_i(x)|_{x\in\partial\Omega} + M_{i1}\varepsilon - \delta_i\varepsilon, \quad i = 1, 2,$$

thus selecting $\delta_i \geqslant M_{i1}$, we have

$$B\alpha_i|_{x\in\partial\Omega} \leqslant g_i(x)|_{x\in\partial\Omega}, \quad i = 1, 2, \tag{17}$$

Analogously, we obtain too that

$$B\beta_i|_{x\in\partial\Omega} \geqslant g_i(x)|_{x\in\partial\Omega}, \quad i = 1, 2, \tag{18}$$

We also have that

$$\alpha_i(0, x) \leqslant h_i(x) \leqslant \beta_i(0, x), \quad x \in \Omega, \quad i = 1, 2, \tag{19}$$

Now we prove that

$$\varepsilon\frac{\partial \alpha_1}{\partial t} - L\alpha_1 - \alpha_1 f_1(\lambda_1 - r_{11}\alpha_1 - r_{12}\alpha_2) \leqslant 0, \quad (t, x) \in (0, T) \times \Omega, \quad i = 1, 2, \tag{20}$$

$$\varepsilon\frac{\partial \alpha_2}{\partial t} - L\alpha_2 - \alpha_2 f_2(-\lambda_2 + r_{21}\alpha_1) \leqslant 0, \quad (t, x) \in (0, T) \times \Omega, \quad i = 1, 2, \tag{21}$$

and

$$\varepsilon \frac{\partial \beta_1}{\partial t} - L\beta_1 - \beta_1 f_1(\lambda_1 - r_{11}\beta_1 - r_{12}\beta_2) \geqslant 0,$$
$$(t,x) \in (0,T) \times \Omega,\ i = 1,2, \quad (22)$$

$$\varepsilon \frac{\partial \beta_2}{\partial t} - L\beta_2 - \beta_2 f_2(-\lambda_2 + r_{21}\beta_1) \geqslant 0,$$
$$(t,x) \in (0,T) \times \Omega,\ i = 1,2, \quad (23)$$

From the mean value theorem, for ε small enough $0 < \varepsilon \leqslant \varepsilon_1$, there exist positive constants M_{i2}, $i = 1,2$, such that

$$\varepsilon \frac{\partial \alpha_1}{\partial t} - L\alpha_1 - \alpha_1 f_1(\lambda_1 - r_{11}\alpha_1 - r_{12}\alpha_2) = [-LU_1 - U_1 f_1(\lambda_1 - r_{11}U_1 - r_{12}U_2)] + \left[\frac{\partial V_1}{\partial t} - L(V_1)\right] + [U_1 f_1(\lambda_1 - r_{11}U_1 - r_{12}U_2) - (U_1 + V_1 - \delta_1\varepsilon) f_1(\lambda_1 - r_{11}(U_1 + V_1 - \delta_1\varepsilon) - r_{12}(U_2 + V_2 - \delta_2\varepsilon))] = [U_1 f_1(\lambda_1 - r_{11}U_1 - r_{12}U_2) - (U_1 + V_1 - \delta_1\varepsilon) f_1(\lambda_1 - r_{11}(U_1 + V_1 - \delta_1\varepsilon) - r_{12}(U_2 + V_2 - \delta_2\varepsilon))] \leqslant -U_1[f_{1y}(*)](r_{11}\delta_1 + r_{12}\delta_2)\varepsilon + f_1(\lambda_1 - r_{11}(U_1 + V_1 - \delta_1\varepsilon) - r_{12}(U_2 + V_2 - \delta_2\varepsilon))\delta_1\varepsilon + M_{12}\varepsilon,$$

and

$$\varepsilon \frac{\partial \alpha_2}{\partial t} - L\alpha_2 - \alpha_2 f_2(-\lambda_2 + r_{21}\alpha_1) = [-LU_2 - U_2 f_2(-\lambda_2 + r_{21}U_1)] + \left[\frac{\partial V_2}{\partial t} - L(V_2)\right] + [U_2 f_2(-\lambda_2 + r_{21}U_1) - (U_2 + V_2 - \delta_2\varepsilon) f_2(-\lambda_2 + r_{21}(U_1 + V_1 - \delta_1\varepsilon))] = [U_2 f_2(-\lambda_2 + r_{21}U_1) - (U_2 + V_2 - \delta_2\varepsilon) f_2(-\lambda_2 + r_{21}(U_1 + V_1 - \delta_1\varepsilon))] \leqslant U_2[f_{2y}(**)]\varepsilon + f_2(-\lambda_2 + r_{21}(U_1 + V_1 - \delta_1\varepsilon))\delta_2\varepsilon + M_{22}\varepsilon,$$

where "$*$" denotes certain value between $\lambda_1 - r_{11}(U_1 + V_1 - \delta_1\varepsilon) - r_{12}(U_2 + V_2 - \delta_2\varepsilon)$ and $\lambda_1 - r_{11}U_1 - r_{12}U_2$, "$**$" denotes certain value between $-\lambda_2 + r_{21}(U_2 + V_2 - \delta_2\varepsilon)$ and $-\lambda_2 + r_{21}U_2$.

From the hypotheses and selecting large enough δ_i, $i = 1,2$, then we proved the inequalities Eq.(20) and Eq.(21) respectively.

Analogously, we can prove the inequalities Eq.(22) and Eq.(23).

Thus from Eqs.(16)–(23), we have that there exists a solution (u_1, u_2) of the problem Eqs.(1)–(4) and that

$$\alpha_i(t,x,\varepsilon) \leqslant u_i(t,x,\varepsilon) \leqslant \beta_i(t,x,\varepsilon),\ i = 1,2,$$
$$(t,x,\varepsilon) \in [0,T] \times (\Omega + \partial\Omega) \times [0,\varepsilon_1].$$

From Eqs.(14) and (15), we then obtain Eq.(13). The proof of the theorem is completed.

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