http://www.zju.edu.cn/jzus; E-mail: jzus@zju.edu.cn
ISSN 1009 – 3095 *Journal of Zhejiang University SCIENCE* V.5, No.1, P.68 – 74, Jan., 2004

A SISO mixed H_2/l_1 optimal control problem and its solution*

WU Jun (吴俊)†, HU Xie-he (胡协和), CHU Jian (褚健)
(*National Key Laboratory of Industrial Control Technology, Institute of Advanced Process Control, Zhejiang University, Hangzhou 310027, China*)
†E-mail: jwu@iipc.zju.edu.cn

Received Dec.21, 2002; revision accepted Mar.26, 2003

Abstract: Study of the SISO mixed H_2/l_1 problem for discrete time systems showed that there exists a unique optimal solution which can be approximated within any prescribed missing error bound in l_2 norm with solvable suboptimal solutions and solvable superoptimal solutions.

Key words: Mixed H_2/l_1 problem, Existence, Uniqueness, Approximation
Document code: A **CLC number:** O224

INTRODUCTION

One of the methods for solving optimal regulation and tracking problems in linear systems is the H_2 optimal control scheme, where the controller minimizes the expected value of an H_2 norm criterion. It is well known, however, that the performance of the H_2 optimal controllers may be highly sensitive to parameter uncertainty (Doyle, 1982). This is why alternative methods that have been tried may probably be suboptimal with respect to the H_2 norm criterion of the nominal system on one hand, but will guarantee some kind of robustness with respect to the large uncertainty on the other hand.

The mixed H_2/l_1 control for discrete time systems presented by Voulgaris (1994) can be viewed as one of the discrete time robust H_2 control methods. The mixed H_2/l_1 control problem is to minimize the H_2 norm of the closed loop map between some exogenous white noise inputs and some regulated outputs, while the l_1 norm of the transfer function between inputs and outputs affected by model uncertainty is constrained to be below some level γ. Such a design will ensure good nominal performance to white noise inputs and, at the same time, will guarantee stability robustness against causal perturbations of induced l_∞ norm less than $1/\gamma$.

In Voulgaris' (1994; 1995) works, a SISO mixed H_2/l_1 problem of minimizing the H_2 norm of the closed loop map while maintaining its l_1 norm at a prescribed level was studied. It was shown that there exists a unique optimal solution which is the finite impulse response for this problem which can be reduced to a finite dimensional quadratic programming problem. In Wu and Chu's (1996; 1999) works, another SISO mixed H_2/l_1 problem of minimizing the H_2 norm of the closed loop map while maintaining the l_1 norm of the other closed loop map at a prescribed level was studied. It was shown that the optimal cost of this infinite dimensional quadratic programming problem can be approximated by solving finite dimensional quadratic programming problems. An upper approximate method and a lower approximate method were given. In Salapaka *et al*. (1995b), an approximate problem solved by finite dimensional quadratic programming technique was developed for the MIMO mixed H_2/l_1 problem whose optimal solution was approximated in weak-* topology. In Elia and Dahleh's work, the MIMO mixed H_2/l_1 problem was treated with specialized results on duality

* Project supported by the National Natural Science Foundation of China (No.60174026) and the Scientific Research Foundation for Returned Overseas Chinese Scholars of Zhejiang Province (No.J20020546)

theory and some special cases were solved successfully. In addition, a sequential quadratic programming algorithm was used by Jacques and Ridgely (1995) and Jacques *et al*. (1996) to solve the fixed order mixed H_2/l_1 problem.

This paper considers the SISO mixed H_2/l_1 problem presented in Wu and Chu's (1996; 1999) works. We will give the results on the existence and the uniqueness of the optimal solution. Moreover the optimal solution is shown to be approximated within any prescribed missing error bound in l_2 norm with solvable suboptimal solutions and solvable superoptimal solutions. A simple numerical example is given to demonstrate the theory results developed.

NOTATIONS

R denotes the field of real numbers, R^m denotes the m-dimensional real vectors, C denotes the field of complex numbers, ∂D denotes the unit circle in C, and Z_+ denotes the nonnegative integers.

A SISO causal LTI (Linear Time Invariant) transfer function $\hat{G}$ can be described as

$$\hat{G} = G(0) + G(1)\lambda + G(2)\lambda^2 + \cdots \quad (G(k) \in R, \forall k \in Z_+). \tag{1}$$

As $\hat{G}$ can be represented uniquely by its impulse response sequence $[G(0), G(1), G(2), \cdots]^{\mathrm{T}}$, $\hat{G}$ and its impulse response sequence are not differentiated in notation in this paper. Define

$$l_e = \{\hat{G} \mid \hat{G} = G(0) + G(1)\lambda + G(2)\lambda^2 + \cdots, G(k) \in R, \forall k \in Z_+\}, \tag{2}$$

$$l_1 = \left\{\hat{G} \,\middle|\, \hat{G} \in l_e, \sum_{k=0}^{\infty} \left| G(k) \right| < \infty \right\}, \tag{3}$$

$$l_2 = \left\{\hat{G} \,\middle|\, \hat{G} \in l_e, \sum_{k=0}^{\infty} (G(k))^2 < \infty \right\}, \tag{4}$$

$$Rl_1 = \{\hat{G} \mid \hat{G} \in l_1, \hat{G} \text{ is a rational function of } \lambda\}. \tag{5}$$

$\forall \hat{G} \in l_1$, the l_1 norm of $\hat{G}$ is given by $\|\hat{G}\|_1 = \sum_{k=0}^{\infty} \left|G(k)\right|$. It is easy to verify that $\forall \hat{G}_1, \hat{G}_2 \in l_1$, $\|\hat{G}_1 \hat{G}_2\|_1 \leqslant \|\hat{G}_1\|_1 \|\hat{G}_2\|_1$. $\forall \hat{G}_1, \hat{G}_2 \in l_2$, the inner product of $\hat{G}_1$ and $\hat{G}_2$ is given by $\langle \hat{G}_1, \hat{G}_2 \rangle = \sum_{k=0}^{\infty} G_1(k) G_2(k)$. Then l_2 is a Hilbert space, and $\forall \hat{G} \in l_2$, the l_2 norm (or H_2 norm) of $\hat{G}$ is

$$\|\hat{G}\|_2 = (\langle \hat{G}, \hat{G} \rangle)^{1/2} = \left(\sum_{k=0}^{\infty} (G(k))^2\right)^{1/2}. \tag{6}$$

Notice that $l_1 \subset l_2$ and $\|\hat{G}\|_2 \leqslant \|\hat{G}\|_1$.

$\forall N \in Z_+$, the N-th truncation operator is defined as:

$$F_N: l_e \to R^{N+1}: F_N(G(0) + G(1)\lambda + \cdots) = G(0) + G(1)\lambda + \cdots + G(N)\lambda^N \tag{7}$$

PROBLEM FORMULATION

Given an admissible discrete time plant $\hat{P}$ as

$$\begin{cases} y = \hat{P}_{11} u + \hat{P}_{12} w_1 + \hat{P}_{13} w_2 \\ z_1 = \hat{P}_{21} u + \hat{P}_{22} w_1 + \hat{P}_{23} w_2 \\ z_2 = \hat{P}_{31} u + \hat{P}_{32} w_1 + \hat{P}_{33} w_2 \end{cases} \tag{8}$$

where $\hat{P}_{ij}(i = 1, 2, 3; j = 1, 2, 3)$ are rational causal LTI transfer functions, w_1 and w_2 are single exogenous inputs, z_1 and z_2 are single regulated outputs, u is the single control input, and y is the single measured output. $\hat{P}$ is assumed to be stabilizable. $\hat{P}_{11}$ and $\hat{P}_{22}$ are assumed to be strictly causal. The discrete time controller is

$$u = \hat{C} y \tag{9}$$

Let $\hat{\Psi}$ denote the closed loop transfer function between w_1 and z_1, $\hat{\Phi}$ denote the closed loop transfer function between w_2 and z_2. Given a constant γ, the objective is to find a causal LTI controller $\hat{C}$ which stabilizes $\hat{P}$ and minimizes $\|\hat{\Phi}\|_2$ subject to $\|\hat{\Psi}\|_1 \leqslant \gamma$.

By incorporating the Youla parametrization of stabilizing controllers (Francis, 1987), it is obtained that

$$\hat{\Psi} = \hat{P}_{22} + \hat{P}_{21} \hat{C} (1 - \hat{P}_{11} \hat{C})^{-1} \hat{P}_{12} = \hat{T}_1 - \hat{Q} \hat{V}_1 \tag{10}$$

$$\hat{\Phi} = \hat{P}_{33} + \hat{P}_{31} \hat{C} (1 - \hat{P}_{11} \hat{C})^{-1} \hat{P}_{13} = \hat{T}_2 - \hat{Q} \hat{V}_2 \tag{11}$$

where $\hat{T}_1, \hat{T}_2, \hat{V}_1, \hat{V}_2 \in Rl_1$ are fixed SISO maps that depend on plant $\hat{P}$, and $\hat{Q} \in l_1$ is a free SISO parameter. Hence the mixed H_2/l_1 control problem can be stated as: Given $\hat{T}_1$, $\hat{T}_2$, $\hat{V}_1$, $\hat{V}_2 \in Rl_1$ and a constant γ, find $\hat{Q} \in l_1$ such that $\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leqslant \gamma$ and $\|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2$ is minimized. Without loss of generality (McDonald and Pearson, 1991), assume that

$$\hat{V}_1 = [V_1(0), \cdots, V_1(m-1), 1]^{\mathrm{T}} \in R^{m+1} \quad (12)$$

$$\hat{V}_2 = [V_2(0), \cdots, V_2(n-1), 1]^{\mathrm{T}} \in R^{n+1} \quad (13)$$

and

$$V_1(0) \neq 0 \quad (14)$$

$$V_2(0) \neq 0. \quad (15)$$

From Eq.(12), $\hat{V}_1$ can be writen as

$$\hat{V}_1 = \lambda^m + V_1(m-1)\lambda^{m-1} + \cdots + V_1(0) = \prod_{i=1}^{m}(\lambda - \lambda_i). \quad (16)$$

where $\lambda_i \in C$ are the zeros of $\hat{V}_1$. Let $\Lambda = \{\lambda_1, \cdots, \lambda_m\}$. Then for the mixed H_2/l_1 problem in this paper, an assumption similar to the usual rank assumptions made in H_2 optimization and l_1 optimization is described as

$$\Lambda \cap \partial D = \varphi \quad (17)$$

where φ represents the empty set.

Define

$$\xi = \left\{\hat{\Phi} \mid \hat{\Phi} = \hat{T}_2 - \hat{Q}\hat{V}_2, \ \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leqslant \gamma, \hat{Q} \in l_1\right\}, \quad (18)$$

$$\zeta = \{\hat{Q} \mid \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leqslant \gamma, \hat{Q} \in l_1\}, \quad (19)$$

$$\gamma_0 = \inf_{\hat{Q} \in l_1} \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1, \quad (20)$$

$$L = \frac{\|\hat{T}_1\|_1 + \gamma}{\prod_{i=1}^{m} |1 - |\lambda_i||}. \quad (21)$$

The mixed H_2/l_1 problem is described as

$$\mu = \inf_{\hat{\Phi} \in \xi} \|\hat{\Phi}\|_2 = \inf_{\hat{Q} \in \zeta} \|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2 \quad (22)$$

Throughout this paper, γ is assumed to belong to (γ_0, ∞). The following lemma shows that ζ is bounded.

Lemma 1 $\forall \hat{Q} \in \zeta, \|\hat{Q}\|_1 \leqslant L$.

Proof Space C_1 is defined as

$$\left\{\hat{G} \mid \hat{G} = G(0) + G(1)\lambda + \cdots, \sum_{k=0}^{\infty} |G(k)| < \infty, G(k) \in C, \forall k \in Z_+\right\}.$$

For any $\hat{G} \in C_1$, the C_1 norm of $\hat{G}$ can be given by $\|\hat{G}\|_1 = \sum_{k=0}^{\infty} |G(k)|$. Obviously, l_1 is a subset of C_1 and the C_1 norm in l_1 space is exactly the l_1 norm. With C_1 norm and $\|\hat{Q}\|_1 < \infty$, it follows that

$$\|\hat{T}_1\|_1 + \gamma \geqslant \|\hat{T}_1\|_1 + \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \geqslant \|\hat{Q}\hat{V}_1\|_1 \geqslant \left\|\prod_{i=1}^{m}(\lambda - \lambda_i)\hat{Q}\right\|_1 \geqslant \left|\left\|\lambda\prod_{i=2}^{m}(\lambda-\lambda_i)\hat{Q}\right\|_1 - \left\|-\lambda_1\prod_{i=2}^{m}(\lambda-\lambda_i)\hat{Q}\right\|_1\right| \geqslant |1 - |\lambda_1|| \left\|\prod_{i=2}^{m}(\lambda - \lambda_i)\hat{Q}\right\|_1 \geqslant \prod_{i=1}^{m}|1 - |\lambda_i|| \|\hat{Q}\|_1$$

The above means that $\|\hat{Q}\|_1 \leqslant L$.

More regirously, it can be seen that $\|\hat{Q}\hat{V}_1\|_1 = \left\|\prod_{i=1}^{m}(\lambda - \lambda_i)\hat{Q}\right\|_1$. From the logic point of view, the statement $\|\hat{Q}\hat{V}_1\|_1 \geqslant \left\|\prod_{i=1}^{m}(\lambda - \lambda_i)\hat{Q}\right\|_1$ is true because it implies that $\|\hat{Q}\hat{V}_1\|_1$ is larger than or equal to $\left\|\prod_{i=1}^{m}(\lambda - \lambda_i)\hat{Q}\right\|_1$. As a result, we write $\|\hat{Q}\hat{V}_1\|_1 \geqslant \left\|\prod_{i=1}^{m}(\lambda - \lambda_i)\hat{Q}\right\|_1$ in the above proof.

EXISTENCE AND UNIQUENESS OF OPTIMAL SOLUTION

It is seen easily that ξ is a nonempty convex subset in l_2 space. Moreover it is shown that

Theorem 1 ξ is a closed subset in l_2 space.

Proof Suppose $(\hat{\Phi}_1, \hat{\Phi}_2, \hat{\Phi}_3, \cdots)$ is a sequence in ξ such that $(\hat{\Phi}_1, \hat{\Phi}_2, \hat{\Phi}_3, \cdots)$ converges in l_2 norm. Since l_2 space is complete, there exists $\hat{\Phi}_0 = \lim_{i\to\infty}\hat{\Phi}_i \in l_2$. Noticing $V_2(0) \neq 0$ we have

$$\hat{Q}_0 = \frac{\hat{T}_2 - \hat{\Phi}_0}{\hat{V}_2} = Q_0(0) + Q_0(1)\lambda + \cdots \in l_e$$

This theorem will follow if $\hat{Q}_0 \in l_1$ and $\|\hat{T}_1 - \hat{Q}_0\hat{V}_1\|_1 \leqslant \gamma$. Next we will show that is the case:

If $\hat{Q}_0 \notin l_1$, i.e. $\|\hat{Q}_0\|_1 = \infty$.

Choose $N_0 \in Z_+$ such that $\|F_{N_0}\hat{Q}_0\|_1 > L$.

Since $\hat{\Phi}_0 = \hat{T}_2 - \hat{Q}_0\hat{V}_2$,

$$F_{N_0}\hat{\Phi}_0 = \begin{bmatrix} \Phi_0(0) \\ \vdots \\ \Phi_0(N_0) \end{bmatrix} = \begin{bmatrix} T_2(0) \\ \vdots \\ T_2(N_0) \end{bmatrix} - \begin{bmatrix} V_2(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ V_2(N_0) & \cdots & V_2(0) \end{bmatrix} \begin{bmatrix} Q_0(0) \\ \vdots \\ Q_0(N_0) \end{bmatrix} = F_{N_0}\hat{T}_2 - \mathscr{V}_2 F_{N_0}\hat{Q}_0.$$

Notice that $\mathscr{V}_2$ is invertible, $F_{N_0}\hat{Q}_0 = (\mathscr{V}_2)^{-1}(F_{N_0}\hat{T}_2 - F_{N_0}\hat{\Phi}_0) = \mathscr{U}_2(F_{N_0}\hat{T}_2 - F_{N_0}\hat{\Phi}_0)$, where

$$\mathscr{U}_2 = (\mathscr{V}_2)^{-1} = \begin{bmatrix} U_2(0,0) & \cdots & U_2(0,N_0) \\ \vdots & \vdots & \vdots \\ U_2(N_0,0) & \cdots & U_2(N_0,N_0) \end{bmatrix}$$

Define $\|\mathscr{U}_2\| = \max_{j\in\{0,\cdots,N_0\}} \sum_{i=0}^{N_0} |U_2(i,j)|$. It is easily seen that $\|F_{N_0}\hat{Q}_0\|_1 \leqslant \|\mathscr{U}_2\| \|F_{N_0}\hat{T}_2 - F_{N_0}\hat{\Phi}_0\|_1$. Choose ε_0 such that $0 < \varepsilon_0 < \dfrac{\|F_{N_0}\hat{Q}_0\|_1 - L}{\sqrt{N_0+1}\|\mathscr{U}_2\|}$. Since $\lim_{i\to\infty}\hat{\Phi}_i = \hat{\Phi}_0$, there are $\hat{Q}_i \in \zeta$ such that

$$\|\hat{T}_2 - \hat{Q}_i\hat{V}_2 - \hat{\Phi}_0\|_2 = \|\hat{\Phi}_i - \hat{\Phi}_0\|_2 < \varepsilon_0.$$

Thus

$$\|\hat{Q}_i\|_1 \geqslant \|F_{N_0}\hat{Q}_i\|_1 \geqslant \|F_{N_0}\hat{Q}_0\|_1 - \|F_{N_0}(\hat{Q}_0 - \hat{Q}_i)\|_1 \geqslant \|F_{N_0}\hat{Q}_0\|_1 - \|\mathscr{U}_2\| \|F_{N_0}\hat{\Phi}_i - F_{N_0}\hat{\Phi}_0\|_1 \geqslant \|F_{N_0}\hat{Q}_0\|_1 - \sqrt{N_0+1}\|\mathscr{U}_2\| \|\hat{\Phi}_i - \hat{\Phi}_0\|_2 > \|F_{N_0}\hat{Q}_0\|_1 - \sqrt{N_0+1}\|\mathscr{U}_2\|\varepsilon_0 \geqslant L,$$

which means $\hat{Q}_i \notin \zeta$ from Lemma 1. However $\hat{Q}_i \notin \zeta$ is a contradiction. Hence $\hat{Q}_0 \in l_1$.

If $\|\hat{T}_1 - \hat{Q}_0\hat{V}_1\|_1 > \gamma$. Choose $N_1 \in Z_+$ such that $\|F_{N_1}(\hat{T}_1 - \hat{Q}_0\hat{V}_1)\|_1 > \gamma$. It is easy to show that $F_{N_1}(\hat{T}_1 - \hat{Q}_0\hat{V}_1) = F_{N_1}\hat{T}_1 - F_{N_1}(\hat{Q}_0\hat{V}_1) = F_{N_1}\hat{T}_1 - \mathscr{V}_1 F_{N_1}\hat{Q}_0$, where

$$\mathscr{V}_1 = \begin{bmatrix} V_1(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ V_1(N_1) & \cdots & V_1(0) \end{bmatrix}.$$

Choose ε_1 such that

$$0 < \varepsilon_1 < \frac{\|F_{N_1}(\hat{T}_1 - \hat{Q}_0\hat{V}_1)\|_1 - \gamma}{\sqrt{N_1+1}\|\mathscr{V}_1\|\|\mathscr{U}_2\|}.$$

Since $\lim_{i\to\infty}\hat{\Phi}_i = \hat{\Phi}_0$, there are $\hat{Q}_i \in \zeta$ such that $\|\hat{T}_2 - \hat{Q}_i\hat{V}_2 - \hat{\Phi}_0\|_2 = \|\hat{\Phi}_i - \hat{\Phi}_0\|_2 < \varepsilon_1$. Thus

$$\|\hat{T}_1 - \hat{Q}_i\hat{V}_1\|_1 \geqslant \|F_{N_1}(\hat{T}_1 - \hat{Q}_i\hat{V}_1)\|_1 \geqslant \|F_{N_1}(\hat{T}_1 - \hat{Q}_0\hat{V}_1)\|_1 - \|F_{N_1}(\hat{T}_1 - \hat{Q}_i\hat{V}_1) - F_{N_1}(\hat{T}_1 - \hat{Q}_0\hat{V}_1)\|_1 \geqslant \|F_{N_1}(\hat{T}_1 - \hat{Q}_0\hat{V}_1)\|_1 - \|\mathscr{V}_1\| \|F_{N_1}(\hat{Q}_0 - \hat{Q}_i)\|_1 > \|F_{N_1}(\hat{T}_1 - \hat{Q}_0\hat{V}_1)\|_1 - \sqrt{N_1+1}\|\mathscr{V}_1\| \|\mathscr{U}_2\|\varepsilon_1 \geqslant \gamma.$$

The above means $\hat{Q}_i \notin \zeta$ which is a contradiction. Hence $\|\hat{T}_1 - \hat{Q}_0\hat{V}_1\|_1 \leqslant \gamma$.

From the following well known results in functional analysis (Conway, 1990).

Lemma 2 If E is a Hilbert space with norm $\|\cdot\|$, A is a closed convex nonempty subset of E, then there is a unique point $x^* \in A$ such that $\|x^*\| = \inf_{x\in A}\|x\|$.

We have

Corollary 1 For the mixed H_2/l_1 problem (22), there is a unique optimal solution $\hat{\Phi}^* \in \xi \subset l_2$ such that $\|\hat{\Phi}^*\|_2 = \mu = \inf_{\Phi\in\xi}\|\hat{\Phi}\|_2$

From the above Corollary, we know that $\hat{Q}^* = \dfrac{\hat{T}_2 - \hat{\Phi}^*}{\hat{V}_2} \in \zeta \subset l_1$ and $\hat{\Psi}^* = \hat{T}_1 - \hat{Q}^*\hat{V}_1 \in l_1$ are unique. The uniqueness of $\hat{Q}^*$ means that the optimal mixed H_2/l_1 controller $\hat{C}^*$ is unique.

APPROXIMATION TO THE OPTIMAL SOLUTION

The mixed H_2/l_1 Eq.(22) is in fact an infinite dimensional quadratic programming problem. Although we have proven in the last section that the unique optimal solution of this problem exists, presently we lack the method to get the optimal solution exactly except for some special cases (Elia and Dahleh, 1997). An upper approximate method and a lower approximate method of the optimal cost μ were developed in Wu and Chu's (1999) work. In this section we will show that using these upper and lower approximate methods, the optimal solution $\hat{\Phi}^*$ can be approximated too.

At first we will introduce briefly the approximate methods presented in Wu and Chu's (1999) work: $\forall N \in Z^+$, define the N-th truncted variable version of ξ

$$\xi_N = \left\{ \hat{\Phi} \middle| \hat{\Phi} = \hat{T}_2 - \hat{Q}\hat{V}_2, \ \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leqslant \gamma, \hat{Q} \in R^{N+1} \right\}. \quad (23)$$

and the N-th truncted constraint version of ξ

$$\xi_{+N} = \left\{ \hat{\Phi} \middle| \hat{\Phi} \in l_2, F_N(\hat{T}_2 - \hat{Q}\hat{V}_2 - \hat{\Phi}) = 0, \|F_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 \leqslant \gamma, \hat{Q} \in l_1, \|\hat{Q}\|_1 \leqslant L \right\} \quad (24)$$

For those nonempty ξ_Ns and ξ_{+N}s, the N-th truncated variable problem and the N-th truncated constraint problem of mixed H_2/l_1 Eq.(22) can be constructed as

$$\mu_N = \inf_{\hat{\Phi} \in \xi_N} \|\hat{\Phi}\|_2 \quad (25)$$

$$\mu_{+N} = \inf_{\hat{\Phi} \in \xi_{+N}} \|\hat{\Phi}\|_2 \quad (26)$$

Eqs.(25) and (26) are finite dimensional optimization problems which can be successfully solved with many numerical optimization techniques (Wu and Chu, 1999). The following lemmas proven in Wu and Chu's (1999) work show that the optimal cost μ of Eq.(22) is approximated by μ_N and μ_{+N}.

Lemma 3 $\mu_N \geqslant \mu_{N+1}$ and $\lim_{N\to\infty} \mu_N = \mu$.

Lemma 4 $\mu_{+N} \leqslant \mu_{N+1}$ and $\lim_{N\to\infty} \mu_{+N} = \mu$.

Next we will investigate whether the optimal solution $\hat{\Phi}^*$ of Eq.(22) can be approximated using the above approximate methods. Similar to Theorem 1, we can prove that ξ_N and ξ_{+N} are closed subsets in l_2 space. Hence Eq.(25) and Eq.(26) have unique optimal solutions from Lemma 2. Let $\hat{\Phi}_N^*$ denote the optimal solution of Eq.(25), $\hat{\Phi}_{+N}^*$ denote the optimal solution of Eq.(26). In this paper, $\hat{\Phi}_N^*$ is called the suboptimal solution, which is feasible, of the mixed H_2/l_1 Eq.(22); while $\hat{\Phi}_{+N}^*$ is called the superoptimal solution, which is infeasible, of the mixed H_2/l_1 Eq.(22). The following theorems give an upper bound of $\|\hat{\Phi}_N^* - \hat{\Phi}^*\|_2$ and $\|\hat{\Phi}_{+N}^* - \hat{\Phi}^*\|_2$:

Theorem 2 $\|\hat{\Phi}_N^* - \hat{\Phi}^*\|_2 \leqslant \sqrt{2\mu_N^2 - 2\mu^2} \leqslant \sqrt{2\mu_N^2 - 2\mu_{+N}^2}$

Proof For any integer $N \geqslant 0$, by Parallelogram Law, it gives

$$\|\hat{\Phi}^* + \hat{\Phi}_N^*\|_2^2 + \|\hat{\Phi}^* - \hat{\Phi}_N^*\|_2^2 = 2(\|\hat{\Phi}^*\|_2^2 + \|\hat{\Phi}_N^*\|_2^2).$$

Hence

$$\|(\hat{\Phi}^* + \hat{\Phi}_N^*)/2\|_2^2 = (\|\hat{\Phi}^*\|_2^2 + \|\hat{\Phi}_N^*\|_2^2)/2 - (\|\hat{\Phi}^* - \hat{\Phi}_N^*\|_2^2/4).$$

However, ξ is convex, which means $(\hat{\Phi}^* + \hat{\Phi}_N^*)/2 \in \xi$, and hence
$\|(\hat{\Phi}^* + \hat{\Phi}_N^*)/2\|_2^2 \geqslant \|\hat{\Phi}^*\|_2^2$. Therefore

$$\|\hat{\Phi}^* - \hat{\Phi}_N^*\|_2 \leqslant \sqrt{2\|\hat{\Phi}_N^*\|_2^2 - 2\|\hat{\Phi}^*\|_2^2} \leqslant \sqrt{2\mu_N^2 - 2\mu^2} \leqslant \sqrt{2\mu_N^2 - 2\mu_{+N}^2}.$$

Theorem 3 $\|\hat{\Phi}_{+N}^* - \hat{\Phi}^*\|_2 \leqslant \sqrt{2\mu^2 - 2\mu_{+N}^2} \leqslant \sqrt{2\mu_N^2 - 2\mu_{+N}^2}$

Proof Similar to the proof of Theorem 2.

Noticing μ_N and μ_{+N} converge to μ, we have the following corollary which shows both $\hat{\Phi}_N^*$ and $\hat{\Phi}_{+N}^*$ are convergent to $\hat{\Phi}^*$ in l_2 norm.

Corollary 2 $\lim_{N\to\infty} \|\hat{\Phi}_N^* - \hat{\Phi}^*\|_2 = 0$, $\lim_{N\to\infty} \|\hat{\Phi}_{+N}^* - \hat{\Phi}^*\|_2 = 0$

Corresponding to the feasible suboptimal solution $\hat{\Phi}_N^*$, the suboptimal element $\hat{Q}_N^* = \dfrac{\hat{T}_2 - \hat{\Phi}_N^*}{\hat{V}_2}$ can be obtained, and then the suboptimal mixed H_2/l_1 controller $\hat{C}_N^*$ can be obtained using Youla parametrization.

NUMERICAL EXAMPLE

To show the approximate approachs can be used efficiently for the mixed H_2/l_1 problem, we study a filtering problem which was studied with the dual theory in Elia and Dahleh's (1997) work. Considering the closed loop system

$$\begin{cases} w_2 = (\hat G + \hat\Delta) z \\ z = \hat K w_2 + w_1 \end{cases}$$

We want to design a controller $\hat K$ for the plant $\hat G = \frac{0.2}{1+0.5\lambda}$ such that:

- the variance of the nominal error signal z ($\hat\Delta = 0$) due to a white gaussian zero meaninput signal w_1 is minimized;
- the closed loop system is guaranteed to remain stable in the presence of model uncertainty, represented as an additive perturbation $\hat\Delta$ with $\|\hat\Delta\|_{l_\infty\text{-}ind} < 1$.

In other words, we want to find the $\hat K$ such that $\|(1-\hat G\hat K)^{-1}\|_2$ is minimized and $\|\hat K(1-\hat G\hat K)^{-1}\|_1 \leqslant 1$. By incorporating the Youla parametrization. We have that

$\hat N_G = 0.2$, $\hat M_G = 1 + 0.5\lambda$, $\hat X_G = -2.5\lambda$, $\hat Y_G = 1$

$$\hat G = \frac{\hat N_G}{\hat M_G},\ \hat K = -\frac{\hat X_G + \hat Q\hat M_G}{\hat Y_G - \hat Q\hat N_G},\ \hat Q \in l_1$$

Let

$$\hat\Phi = 10(1-\hat G\hat K)^{-1} = 10\hat M_G\hat Y_G - 10\hat Q\hat M_G\hat N_G = (10+5\lambda) - \hat Q(2+\lambda) = \hat T_2 - \hat Q\hat V_2$$

$$\hat\Psi = 4\hat K(1-\hat G\hat K)^{-1} = -4\hat M_G\hat X_G - 4\hat Q\hat M_G^2 = (10\lambda + 5\lambda^2) - \hat Q(4+4\lambda+\lambda^2) = \hat T_1 - \hat Q\hat V_1$$

Then the filtering problem is exactly the mixed H_2/l_1 problem

$$\mu = \inf_{\substack{\hat Q\in l_1 \\ \|\hat T_1 - \hat Q\hat V_1\|_1 \leqslant 4}} \|\hat T_2 - \hat Q\hat V_2\|_2$$

Since

$$\hat V_1 = 4 + 4\lambda + \lambda^2 = (2+\lambda)^2$$

has no zeros on ∂D, we know $\hat\Phi^*$, $\hat Q^*$, $\hat\Psi^*$ and $\hat K^*$ exist and are unique, and use the upper and lower approximate methods to solve this problem. Some approximate results are shown in Table 1, where $\varepsilon_\mu = \mu_N - \mu_{+N}$, $\varepsilon_\Phi = \sqrt{2\mu_N^2 - 2\mu_{+N}^2}$. When $N \geqslant 26$, ε_μ and ε_Φ are less than our prescribed missing error bound $\varepsilon = 0.0020$.

This filtering problem was solved using the dual theory in Elia and Dahleh's (1997) work and get

$$\mu = 8.0829,\ \hat\Phi^* = \frac{8+5\lambda}{1+0.5\lambda},$$

$$\hat Q^* = \frac{1+2.5\lambda+1.25\lambda^2}{1+\lambda+0.25\lambda^2},\ \hat\Psi^* = -4$$

Table 1 Some approximate results

N	0	1	2	3	4	5	6	…
μ_N	infeasible	infeasible	9.0830	8.1726	8.1338	8.1338	8.1258	…
μ_{+N}	8.0000	8.0623	8.0777	8.0816	8.0826	8.0828	8.0829	…
ε_μ	infeasible	infeasible	1.0052	0.0910	0.0512	0.0510	0.0429	…
$\hat\Phi_N^*$	infeasible	infeasible	$\begin{bmatrix} 9.0000 \\ 0.5000 \\ -1.0000 \\ 0.5000 \end{bmatrix}$	$\begin{bmatrix} 8.0976 \\ 0.9512 \\ -0.4756 \\ 0.2683 \\ -0.1341 \end{bmatrix}$	$\begin{bmatrix} 8.0563 \\ 0.9718 \\ -0.4859 \\ 0.2430 \\ -0.1215 \\ 0.0088 \end{bmatrix}$	$\begin{bmatrix} 8.0563 \\ 0.9718 \\ -0.4859 \\ 0.2430 \\ -0.1215 \\ 0.0088 \\ 0.0000 \end{bmatrix}$	$\begin{bmatrix} 8.0473 \\ 0.9764 \\ -0.4882 \\ 0.2441 \\ -0.1220 \\ 0.0610 \\ 0.0134 \\ -0.0067 \end{bmatrix}$	…
$\hat\Phi_{+N}^*$	8	$\begin{bmatrix} 8 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 8.0000 \\ 1.0000 \\ -0.5000 \end{bmatrix}$	$\begin{bmatrix} 8.0000 \\ 1.0000 \\ -0.5000 \\ 0.2500 \end{bmatrix}$	$\begin{bmatrix} 8.0000 \\ 1.0000 \\ -0.5000 \\ 0.2500 \\ -0.1250 \end{bmatrix}$	$\begin{bmatrix} 8.0000 \\ 1.0000 \\ -0.5000 \\ 0.2500 \\ -0.1250 \\ 0.0625 \end{bmatrix}$	$\begin{bmatrix} 8.0000 \\ 1.0000 \\ -0.5000 \\ 0.2500 \\ -0.1250 \\ 0.0625 \\ -0.0312 \end{bmatrix}$	…
ε_Φ	infeasible	infeasible	5.8737	1.7199	1.2891	1.2861	1.1795	…

and the optimal controller $\hat{K}^* = \frac{-1-0.5\lambda}{0.8+0.5\lambda}$. Comparing the approximate results and the optimal results, we get Table 2. It turned out that, as N increases, the suboptimal and superoptimal costs converge to the optimal cost; the suboptimal and superoptimal solutions converge to the unique optimal solution in l_2 norm. Moreover, we can see from Tables 1 and 2 that $|\mu_N - \mu| \leqslant \varepsilon_\mu$, $|\mu_{+N} - \mu| \leqslant \varepsilon_\mu$, $\|\hat{\Phi}_N^* - \hat{\Phi}^*\|_2 \leqslant \varepsilon_\Phi$ and that $\|\hat{\Phi}_{+N}^* - \hat{\Phi}^*\|_2 \leqslant \varepsilon_\Phi$.

Table 2 Distances from suboptimal results and superoptimal results to optimal results

N	suboptimal results		superoptimal results	
	$\mu_N - \mu$	$\|\hat{\Phi}_N^* - \hat{\Phi}^*\|_2$	$\mu_{+N} - \mu$	$\|\hat{\Phi}_{+N}^* - \hat{\Phi}^*\|_2$
0	infeasible	infeasible	−0.0829	1.1547
1	infeasible	infeasible	−0.0206	0.5774
2	1.0001	1.2583	−0.0052	0.2887
3	0.0897	0.1346	−0.0013	0.1443
4	0.0509	0.0917	−0.0003	0.0722
5	0.0509	0.0917	−0.0001	0.0361
6	0.0429	0.0745	0.0000	0.0180
…	…	…	…	…

CONCLUSIONS

In this paper we have shown that there exists a unique optimal solution of the SISO mixed H_2/l_1 problem. The optimal cost and optimal solution can be approximated within any prescribed missing error bound using upper and lower approximate methods which are similar to the FMV (Finitely Many Variables) and FME (Finitely Many Equations) methods in l_1 theory (Dahleh, 1992; Staffans, 1993). Although the upper and lower approximate methods based on Q design (Boyd *et al*., 1988) do not necessarily capture the structure of the mixed H_2/l_1 problem (Dahleh and Khammash, 1993; Dahleh and Diaz-Bobillo, 1995), its desired property of convergence to the optimal point is important and useful in practical system design.

References

Boyd, S.P., Balakrishnan, V. and Barratt, C.H., 1988. A new CAD method and associate architecture for linear controllers. *IEEE Trans. Automat. Contr.*, **33**(3): 268 − 283.

Cohen, A. and Shaked, U., 1997. Robust discrete-time H_2 control. *International Journal of Control*, **67**(2): 213 − 232.

Conway, J.B., 1990. A Course in Functional Analysis. Springer-Verlag, New York.

Dahleh, M.A., 1992. BIBO stability robustness for coprime factor perturbations. *IEEE Trans. Automat. Contr.*, **37**(3): 352 − 355.

Dahleh, M.A. and Khammash, M., 1993. Controller design for plants with structured uncertainty. *Automatica*, **29**(1): 37 − 56.

Dahleh, M.A. and Diaz-Bobillo, I.J., 1995. Control of Uncertain Systems: A Linear Programming Approach. Prentice Hall, New Jersey.

Doyle, J.C., 1982. Analysis of feedback systems with structured uncertainty. *Proceedings of the Institution of Electrical Engineers*, *Pt D*, **129**(6): 242 − 250.

Elia, N. and Dahleh, M.A., 1997. Controller design with multiple objectives. *IEEE Trans. Automat. Contr.*, **42**(5): 596 − 613.

Francis, B.A., 1987. A Course in H_∞ Control Theory. Springer-Verlag, Berlin.

Jacques, D.R. and Ridgely, D.B., 1995. A Fixed-Order, Mixed-Norm Control Synthesis Method for Discrete Linear Systems. Proc. Amer. Contr. Conf., Seattle, p. 1936 − 1940.

Jacques, D.R., Canfield, R.A. and Ridgely, D.B., 1996. A MATLAB toolbox for fixed-order, mixed-norm control synthesis. *IEEE Control Systems Magazine*, **16**(5): 36 − 43.

Kaminer, I., Khargonekar, P.P. and Rotea, M.A., 1993. Mixed H_2/H_∞ control for discrete time systems via convex optimization. *Automatica*, **29**(1): 57 − 70.

McDonald, J.S. and Pearson, J.B., 1991. l^1 optimal control of multivariable systems with output norm constraints. *Automatica*, **27**(2): 317 − 329.

Salapaka, M.V., Dahleh, M. and Voulgaris, P., 1995a. Mixed Objective Control Synthesis: Optimal l_1/H_2 Control. Proc. Amer. Contr. Conf., Seattle, p.1438 − 1442.

Salapaka, M.V., Dahleh, M. and Voulgaris, P., 1995b. MIMO Optimal Control Design: The Interplay of the H_2 and the l_1 norms. Proc. IEEE Conf. Decision and Control, New Orleans, p.3682 − 3687.

Staffans, O.J., 1993. The four-block model matching problem in l^1 and infinite-dimensional linear programming. *SIAM J. Contr. Optim.*, **31**(3): 747 − 779.

Voulgaris, P., 1994. Optimal H_2/l_1 Control: the SISO Case. Proc. IEEE Conf. Decision and Control, Orlando, **4**: 3181 − 3186.

Voulgaris, P., 1995. Optimal H_2/l_1 control via duality theory. *IEEE Trans. Automat. Contr*, **40**(11): 1881 − 1888.

Wu, J. and Chu, J., 1996. Mixed H_2/l_1 Control for Discrete Time Systems. Proc. 13th IFAC Congress, San Francisco, **G**: 453 − 457.

Wu, J. and Chu, J., 1999. Approximation methods of scalar mixed H_2/l_1 problem for discrete time systems. *IEEE Trans. Automat. Contr.*, **44**(10): 1869 − 1874.