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Random quadralinear forms and schur product on tensors

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Abstract: In this work, we made progress on the problem that $l_r \breve{\otimes} l_p \breve{\otimes} l_q$ is a Banach algebra under schur product. Our results extend Tonge's results. We also obtained estimates for the norm of the random quadralinear form $\boldsymbol{A}$: $l_r^M \times l_p^N \times l_q^K \times l_s^H \to C$, defined by: $\boldsymbol{A}(\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k, \boldsymbol{e}_s)= a_{ijks}$, where the (a_{ijks})'s are uniformly bounded, independent, mean zero random variables. We proved that under some conditions $l_r \breve{\otimes} l_p \breve{\otimes} l_q \breve{\otimes} l_s$ is not a Banach algebra under schur product.

Key words: Random tensors, Schur product, Banach algebra
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INTRODUCTION

We adopt the standard notation $l_p^N (1 \le p \le \infty)$ for the complex vector space $\boldsymbol{C}^N$ equipped with the norm $\| X \|_p := (\sum_{n=1}^{N} | x_n |^p)^{\frac{1}{p}}$.

The usual modifications are made to define l_∞^N and the infinite dimensional sequence space $l_p (1 \le p \le \infty)$. All of these are Banach spaces.

Let $\boldsymbol{A}$: $l_p^N \to l_q^M (1 \le p, q \le \infty)$ be a linear map and define the operator norm by

$$\| \boldsymbol{A} \|_{pq} := \sup\{\| \boldsymbol{A}x \|_q : \| x \|_p = 1\}.$$

The map $\boldsymbol{A}$ can be represented as an $M \times N$ matrix (a_{ij}) with respect to the standard bases. Motivated by problems on absolutely summing operators, Bennett *et al.*(1975) and Bennett (1977) obtained estimates for the probability distribution of $\| \boldsymbol{A} \|_{pq}$. They showed that, for all $1 \le p$, $q \le \infty$, the expectation $E(\| \boldsymbol{A} \|_{pq})$ is of the same order as that of the smallest possible value of $\| \boldsymbol{A} \|_{pq}$ when all the matrix entries a_{ij} are ± 1.

Notice that these results can also be interpreted as estimates for the norms of random bilinear forms. Problems involving the Von Neumann inequality led Varopoulos (1974) to work with norms of random trilinear forms on l_2^N. His results were extended and refined by Mantero and Tonge (1980). Let $\boldsymbol{A}: l_{p1}^N \times ... \, l_{pn}^N \to C$ be an n linear form with $\boldsymbol{A}(\boldsymbol{e}_{k_1}, ... \, \boldsymbol{e}_{k_n}) = a_{k_1, ... k_n}$, where the $\boldsymbol{e}_k$'s are the Standard unit basis vectors. There is a natural norm

$$\| \boldsymbol{A} \|_{p_1, ... p_n} := \sup\{| \boldsymbol{A}(x_1, ... x_n) | : \| x_i \|_{p_i} \le 1 (1 \le i \le n)\}.$$

In Mentero and Tonge (1980), Tonge get results which are useful in the study of Banach algebra structures on the tensor products $l_{p_1}\breve{\otimes}...\breve{\otimes}l_{p_n}$.

However, open problems were left, even in the case n=3.

In this paper, we report our further work based on Almasri *et al*.(2000)'s results. We proved that in a larger range, $l_r\otimes l_p\otimes l_q$ is not a Banach algebra.

Lemma 1 Let $1\le p\le 2$ and $2\le r,\ q<\infty$. Then there is an $\boldsymbol{A}=(a_{ijk})\in l_r^M\breve{\otimes}l_p^N\breve{\otimes}l_q^K$, with each $a_{ijk}=\pm1$, such that $\|\boldsymbol{A}\|_{rpq}^2<C\max(M^{\frac{2}{r}}N^{\frac{2}{p}-1},\ N^{\frac{2}{p}}K^{\frac{2}{q}(1-\frac{2}{r})},\ N^{\frac{2}{p}-\frac{2}{r}}K^{\frac{2}{r}+\frac{2}{q}(1-\frac{2}{r})})$. Where C is a positive constant independent of M, N, K.

Lemma 2 Let $2\le p,q,r<\infty$. Then there is an $\boldsymbol{A}=(a_{ijk})\in l_r^M\breve{\otimes}l_p^N\breve{\otimes}l_q^K$, with each $a_{ijk}=\pm1$, such that $\|\boldsymbol{A}\|_{rpq}^2<C\max\ (\ M^{\frac{2}{r}},\ \ N^{\frac{2}{r}+\frac{2}{p}(1-\frac{2}{r})},\ \ K^{\frac{2}{q}(1-\frac{2}{r})},\ N^{\frac{2}{p}(1-\frac{2}{r})}K^{\frac{2}{r}+\frac{2}{q}(1-\frac{2}{q})})$. Where C is a positive constant independent of M, N, K.

MAIN RESULT

Note If $\boldsymbol{A}=(a_{ijk})\in l_r^M\breve{\otimes}l_p^N\breve{\otimes}l_q^K$, has each $a_{ijk}=\pm1$, then $\|\boldsymbol{A}*\boldsymbol{A}\|_{rpq}=M^{\frac{1}{r}}N^{\frac{1}{p}}K^{\frac{1}{q}}$.

Theorem 1 Let $1<\min(p,q)\le 2\le\max(p,q)<\infty$ and $2\le r<\infty$. Then $l_r\breve{\otimes}l_p\breve{\otimes}l_q$ is not a Banach algebra under schur multiplication when one of the following two conditions holds:

(1) $\dfrac{1}{\min(p,q)}<\dfrac{2}{r}\times\dfrac{1}{\max(p,q)}+\dfrac{1}{2}$;

(2) $\dfrac{4}{qr}>(\dfrac{2}{p}-1)$.

Proof We consider the case where $1<p\le2\le q<\infty$. The other case is similar. If $\boldsymbol{A}=(a_{ijk})\in l_r^M\breve{\otimes}l_p^N\breve{\otimes}l_q^K$ has each $a_{ijk}=\pm1$, then $\|\boldsymbol{A}*\boldsymbol{A}\|_{rpq}=M^{\frac{1}{r}}N^{\frac{1}{p}}K^{\frac{1}{q}}$. Fix B>0. By Lemma 1, it is enough to show that we can find positive integers M, N, K with $M^{\frac{1}{r}}N^{\frac{1}{p}}K^{\frac{1}{q}}>BC\cdot\max(M^{\frac{2}{r}}N^{\frac{2}{p}-1},N^{\frac{2}{p}}K^{\frac{2}{q}(1-\frac{2}{r})},N^{\frac{2}{r}+\frac{2}{q}\times(1-\frac{2}{r})})$. Where C is a fixed positive number, independent of B, M, N, K. We can achieve this through three ways:

(1) Let $N=K=M^t$, t>0, we will show that $\dfrac{1}{p}<\dfrac{2}{r}\times\dfrac{1}{q}+\dfrac{1}{2}$. The proof follows below.

We have that

$$M^{\frac{1}{q}}>BCM^{\frac{1}{r}+t(\frac{1}{p}-1)},\ M^{\frac{1}{r}}>BCM^{t(\frac{1}{p}+\frac{1}{q}-\frac{4}{qr})}\tag{1}$$

Eq.(1) holds simultaneously for large M if there is a t>0 that satisfies

$$t(\frac{1}{p}+\frac{1}{q}-\frac{1}{r})<\frac{1}{r}<t(\frac{1}{q}-\ \frac{1}{p}+1).$$

Such a positive t exists if and only if

$$\frac{1}{p}<\frac{2}{r}\ \times\frac{1}{q}+\frac{1}{2}.$$

(2) Let $N=M^t$, $K=M^u$, t>u>0. Using a similar way to solve the inequality, we get

$$\frac{4u}{qr}>(\frac{2}{p}-1)t,\ \ (t>u>0).$$

Such positive t, u exists if and only if

$$\frac{4}{qr}>(\frac{2}{p}-1).$$

(3). Let $N=M^t$, $K=M^u$, u>t>0. We get

$$(\frac{2}{r}-\frac{2}{p}+1)t>(\frac{2}{r}-\frac{4}{qr})u,\ \ (0<t<u).$$

Such positive t, u exists if and only if the condition (2) holds.

So we finished the proof.

Using the same method and Lemma 2, we get the following result:

Theorem 2 Let $2 \le p,q,r < \infty$. Then $l_r \breve{\otimes} l_p \breve{\otimes} l_q$ is not a Banach algebra under schur multiplication when one of the following two conditions holds:

(1) $\frac{1}{q}+\frac{1}{p}>\frac{1}{2}$, $\frac{1}{r}+\frac{1}{q}>\frac{1}{2}$ or $\frac{1}{r}+\frac{1}{p}>\frac{1}{2}$;

(2) $1>\frac{p}{2}-\frac{p}{q}$.

Using an approach similar to that of Tonge, we can get an estimate on $\|A\|_{rpqs}^{r}$, and have the following theorem.

Theorem 3 Let $1 \le p \le 2, 2 \le q,r < \infty, s \ge 1$. Then there is an $A=(a_{ijk}) \in l_r^M \breve{\otimes} l_p^N \breve{\otimes} l_q^K$ with each $a_{ijk}=\pm 1$, such that

$$\|A\|_{rpqs}{}^2 < C\max(M^{\frac{2}{r}}N^{\frac{2}{p}-1}, N^{\frac{2}{p}}K^{\frac{2}{q}-\frac{4}{qr}}H^{\frac{2}{s}-\frac{4}{sr}},$$
$$N^{\frac{2}{P}-\frac{2}{r}}K^{\frac{2}{r}+\frac{2}{q}-\frac{4}{qr}}H^{\frac{2}{s}-\frac{4}{sr}}, N^{\frac{2}{p}-\frac{2}{r}}K^{\frac{2}{q}-\frac{4}{qr}}H^{\frac{2}{s}-\frac{4}{sr}+\frac{2}{r}}).$$

where C is a positive constant independent of M, N, and K.

Use of Theorem 3 yields the following:

Theorem 4 Let $1 \le p \le 2, 2 \le r,q < \infty, s \ge 1$. Then $l_r \breve{\otimes} l_p \breve{\otimes} l_q \breve{\otimes} l_s$ is not a Banach algebra under schur multiplication when all of the following conditions hold.

(1) $\frac{1}{r}+\frac{4}{qr}+\frac{4}{sr}>\frac{1}{p}+\frac{1}{q}+\frac{1}{s}$

(2) $1+\frac{1}{q}+\frac{1}{s}>\frac{1}{p}+\frac{1}{r}$.

Proof The proof is similar to that for Theorem 1.

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