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Multiplicative perturbations of *C*-regularized resolvent families

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Abstract: This paper presents two new theorems for multiplicative perturbations of *C*-regularized resolvent families, which generalize the previous related ones for the resolvent families.

Key words: Multiplicative perturbation, Resolvent family, *C*-regularized resolvent family, Generator
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INTRODUCTION

Da Prato and Iannelli (1980) first introduced the notion of resolvent family on a Banach space X. From then on, many researchers have studied the theories of resolvent families in the past 20 years due to their wide and interesting applications to problems in physics, engineering, biology, etc. (Chang and Shaw, 1997; Lizama, 1993; 1995; Prűss, 1993; and the references therein)

The concept of C-regularized resolvent family is introduced by Zheng and Sun (2000) and Li (2002), where C is an injective bounded operator on a Banach space X. The definition is stated below.

Definition 1 Let X be a Banach space and A be a closed linear operator in X. $L(X)$ is the space of all bounded linear operators. $C\in L(X)$ is injective and $a\in L^1_{\text{loc}}(\mathbb{R}^+)$. A linear operator family $\{T(t)_{t\geq 0}\}\subset L(X)$ is called a C-regularized resolvent family with the generator A if the following conditions are satisfied:

(a) $T(t)$ is strongly continuous on $\mathbb{R}^+$ and $T(0)=C$;

(b) $T(t)A\subset AT(t)$ for $t\geq 0$ and $A=C^{-1}AC$;

(c) $T(t)x=Cx+\int_0^t a(t-s)T(s)Ax\mathrm{d}s,\ \forall x\in D(A)$, $t\geq 0$.

In addition, if there are some constants $M,\omega\geq 0$ such that $\|T(t)\|\leq M\mathrm{e}^{\omega t}$ for $t\geq 0$, then $\{T(t)\}_{t\geq 0}$ is called an exponentially bounded C-regularized resolvent family.

Obviously, the notions of C-regularized resolvent families are natural extensions of resolvent families, C-semigroups and C-regularized cosine operator functions, which are the special cases $C=I$, $a(t)=1$ and $a(t)=t$, respectively.

This paper deals with multiplicative perturbations of C-regularized resolvent families. Some new results regarding multiplicative perturbations of C-regularized resolvent families are presented in the second part.

MULTIPLICATIVE PERTURBATION THEOREMS

Throughout this paper, X is a Banach space. We write $\rho(A)$ and $\rho_C(A)=\{\lambda\in\mathbb{C},\ \lambda-A$ is injective and $R(C)\subset R(\lambda-A)\}$ for the resolvent set and the C-regularized resolvent set of A.

Theorem 1 Let A be a closed densely defined linear operator in X and $B \in L(X)$. $C \in L(X)$ is injective. $BC=CB$ and $A=C^{-1}AC$. The following statements hold.

(a) If BA generates a C-regularized resolvent family on X, then so does AB.

(b) If AB generates a C-regularized resolvent family on X, then so does BA provided $\rho(BA) \neq \varnothing$.

Proof (a) Let $\{T(t)\}_{t\geq 0}$ be the C-regularized resolvent family on X generated by BA. Then by Definition 1, noting the denseness and closedness of A, we have

$$T(t)BA \subset BAT(t), \int_0^t a(t-s)T(s)x\mathrm{d}s \in D(BA) = D(A), \quad t \geq 0, x \in X \tag{1}$$

and

$$T(t)x = Cx + BA\int_0^t a(t-s)T(s)x\mathrm{d}s, t \geq 0, x \in X. \tag{2}$$

So $x \mapsto A\int_0^t a(t-s)T(s)Bx\mathrm{d}s$ define a bounded linear operator family on X. Set

$$R(t)x = Cx + A\int_0^t a(t-s)T(s)Bx\mathrm{d}s, \ t \geq 0, x \in X. \tag{3}$$

We see easily, by Eq.(2) and the fact that the graph norms of A and BA are equivalent, that $\{R(t)\}_{t\geq 0}$ is a strongly continuous family of bounded linear operators on X. Furthermore, by Eqs.(1), (2) and (3), we obtain that

$$\begin{aligned} R(t)ABx &= CABx + A\int_0^t a(t-s)T(s)BABx\mathrm{d}s \\ &= ABCx + ABA\int_0^t a(t-s)T(s)Bx\mathrm{d}s \\ &= ABR(t)x, \ t \geq 0, x \in D(AB), \end{aligned} \tag{4}$$

$$BR(t)x = BCx + BA\int_0^t a(t-s)T(s)Bx\mathrm{d}s = T(t)Bx, \quad t \geq 0, x \in X.$$

It follows that for $t \geq 0$, $x \in X$,

$$B\int_0^t a(t-s)R(s)x\mathrm{d}s = \int_0^t a(t-s)T(s)Bx\mathrm{d}s \in D(A)$$

and

$$AB\int_0^t a(t-s)R(s)x\mathrm{d}s = A\int_0^t a(t-s)T(s)Bx\mathrm{d}s = R(t)x - Cx.$$

By Eq.(4) and the closedness of AB, this indicates

$$R(t)x - Cx = \int_0^t a(t-s)R(s)ABx\mathrm{d}s, x \in D(AB). \tag{5}$$

Combined with $C^{-1}ABC = C^{-1}ACB = AB$, Eqs.(4) and (5) show that AB generates a C-regularized resolvent family $\{R(t)\}_{t\geq 0}$.

(b) Suppose AB generates a C-regularized resolvent family. Take $\mu \in \rho(BA)$ and put

$$A_0 = (\mu - BA)B, \ B_0 = A(\mu - BA)^{-1}.$$

Then A_0 is a closed linear operator and $B_0 \in L(X)$. By the hypothesis we know that $B_0A_0=AB$ generates a C-regularized resolvent family and $B_0C=CB_0$. Since $\rho(BA) \neq \varnothing$, we have $C^{-1}BAC = BA$. Then it follows that

$$\begin{aligned} C^{-1}A_0C &= \mu B - C^{-1}BABC = \mu B - C^{-1}BACB \\ &= (\mu - BA)B = A_0. \end{aligned}$$

Thus A_0, B_0 satisfy the conditions of (a) and $A_0B_0= BA$ generate a C-regularized resolvent family on X.

Before the next perturbation theorem, we must give the following two results.

Lemma 1 If $R_C(\lambda, A) = R_C(\lambda, \overline{A})$ for some $\lambda \in \rho_C(A) \cap \rho_C(\overline{A})$, then $C^{-1}AC = C^{-1}\overline{A}C$, where $R_C(\lambda, A) = (\lambda - A)^{-1}C$ is the C-resolvent.

Proof It is easy to verify. We omit its proof.

In the following, we assume that A is dense, $a \neq 0$ a.e. and $T(t)$, $a(t)$ are exponentially bounded, i.e., there exist M, $\omega \geq 0$ such that $\|T(t)\| \leq Me^{\omega t}$ and $|a(t)| \leq Me^{\omega t}$. Then we can denote by $\hat{a}(\lambda)$ and $\hat{T}(t)$ the Laplace transforms of $a(t)$ and $T(t)$.

Theorem 2 Suppose that $\{T(t)\}_{t\geq 0} \subset L(X)$ is a strongly continuous bounded linear operator family. Then $\{T(t)\}_{t\geq 0}$ is an exponentially bounded C-regularized resolvent family with the generator A. If and only if there exists $\omega \geq 0$ such that

(a) $A = C^{-1}AC$;

(b) $\dfrac{1}{\hat{a}(\lambda)} \in \rho_C(A)$ for $\lambda > \omega$ with $\hat{a}(\lambda) \neq 0$;

(c) $(\lambda - \lambda\hat{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}T(t)x\mathrm{d}t$ for $\forall x \in X$, $\lambda > \omega$.

If (a) is replaced by $A \subset C^{-1}AC$, then $\{T(t)\}_{t\geq 0}$ is an exponentially bounded C-regularized resolvent family with the generator $C^{-1}AC$.

Proof The first conclusion has been proved by Zheng and Sun (2000) and Li (2002). Let $C^{-1}AC = \overline{A}$. To prove the second conclusion, we only need to show $\overline{A} = C^{-1}\overline{A}C$. We see easily that $\lambda - \lambda\hat{a}(\lambda)\overline{A}$ is injective for $\lambda > \omega$.

Let $y = (\lambda - \lambda a(\lambda)A)^{-1}Cx$, $x \in X$, $\lambda > \omega$. We have $y \in D(A) \subset D(\overline{A})$ and $(\lambda - \lambda\hat{a}(\lambda)\overline{A})y = (\lambda - \lambda\hat{a}(\lambda)\overline{A})(\lambda - \lambda\hat{a}(\lambda)A)^{-1}Cx = Cx$, which imply $y = (\lambda - \lambda\hat{a}(\lambda)\overline{A})^{-1}Cx$, i.e.,

$$(\lambda - \lambda\hat{a}(\lambda)\overline{A})^{-1}Cx = (\lambda - \lambda\hat{a}(\lambda)A)^{-1}Cx. \quad (6)$$

By the assumption of $a(t)$ and the uniqueness theorem for Laplace transform, it follows that there exists $\lambda > \omega$ such that $\hat{a}(\lambda) \neq 0$. Then by Eq.(6) and Lemma 1, we have $\overline{A} = C^{-1}AC = C^{-1}\overline{A}C$.

The next perturbation theorem will be based on the following assumption (H).

(H) $a(t)$ is of bounded variation on each compact interval $[0,T]$, $T \geq 0$ and $a(0)=0$.

In this case we define the operator $S(t)$ by $S(t) = \int_0^t T(t-s)x\mathrm{d}a(s)$ for $x \in X$ and $t \in [0,T]$.

Theorem 3 Let A be a generator of an exponentially bounded C-regularized resolvent family $\{T(t)\}_{t\geq 0}$ on X. $a(t)$ satisfies the assumption (H). Let $B \in L(X)$, $R(B) \subset R(C)$. If B satisfies the following conditions:

(M1) For $f \in C([0,\infty], X)$,

$\int_0^t S(t-s)C^{-1}Bf(s)\mathrm{d}s \in D(A)$ for $t \geq 0$ and

$\left\| A\int_0^t S(t-s)C^{-1}Bf(s)\mathrm{d}s \right\| \leq M\int_0^t e^{\omega(t-s)}\|f(s)\|\mathrm{d}s$ for $t \geq 0$;

(M2) There exists a bounded injective operator C_1 on X such that $R(C_1) \subset R(C)$, $C_1A(I+B) \subset A(I+B)C_1$, then $C_1^{-1}A(I+B)C_1$ generates an exponentially bounded C_1-regularized resolvent family on X.

Proof We first define a family of operator $T_n(t)$ $(t \geq 0, n \in N_0)$ on X as follows: $T_0(t)x = T(t)x$ and

$$T_n(t)x = A\int_0^t S(t-s)C^{-1}BT_{n-1}(s)x\mathrm{d}s, x \in X, n \in N. \quad (7)$$

Next we will show by induction that

(i) for $x \in X$, $T_n(t)x \in C([0,\infty], X)$,

(ii) for $t \geq 0$, $\|T_k(t)\| \leq M^{k+1}\dfrac{t^k}{k!}e^{\omega t}$.

For n=0, these follow immediately from the hypothesis. If (i), (ii) are true for n=k, then, by Eq.(7) and (M1) we can easily obtain that $T_{k+1}(t)x \in C([0,\infty], X)$ and

$$\begin{aligned}\|T_{k+1}(t)x\| &= \left\| A\int_0^t S(t-s)C^{-1}BT_k(s)x\mathrm{d}s \right\| \\ &\leq M\int_0^t e^{\omega(t-s)}\|T_k(s)x\|\mathrm{d}s \\ &\leq M^{k+2}\frac{t^{k+1}}{(k+1)!}e^{\omega t}\|x\|.\end{aligned}$$

So (i), (ii) hold. Set

$$R(t)x = \sum_{n=0}^{\infty} T_n(t)C^{-1}C_1x, t \geq 0, x \in X. \quad (8)$$

By (ii) and $C^{-1}C_1 \in L(X)$ it is easy to see that the series Eq.(8) uniformly converges on any compact interval of $\mathbb{R}^+$, which implies that the operator $R(t)$ is well defined. Obviously we have $\|R(t)\| \leq M\|C^{-1}C_1\|e^{(\omega+M)t}$ and $R(t)$ is strongly continuous on $\mathbb{R}^+$. By Eqs.(7) and (8), we know that the following equation holds:

$$R(t)x = T(t)C^{-1}C_1x + A\int_0^t S(t-s)C^{-1}BR(s)x\mathrm{d}s, \quad x \in X, t \geq 0. \quad (9)$$

For $x \in X$ and large enough λ, we can define the Laplace transform of $R(t)$ by

$$\widehat{R}(\lambda)x=\int_0^{\infty}\mathrm{e}^{-\lambda t}R(t)x\mathrm{d}t.$$

Taking Laplace transform with Eq.(9) for large enough λ, we have

$$\widehat{R}(\lambda)x=(\lambda-\lambda\widehat{a}(\lambda)A)^{-1}C_1x+\lambda\widehat{a}(\lambda)A(\lambda-\lambda\widehat{a}(\lambda)A)^{-1}B\widehat{R}(\lambda)x,\qquad x\in X.$$

Then we have

$$(I+B)\widehat{R}(\lambda)x=(\lambda-\lambda\widehat{a}(\lambda)A)^{-1}C_1x+\lambda(\lambda-\lambda\widehat{a}(\lambda)A)^{-1}B\widehat{R}(\lambda)x,\quad x\in X,$$

which implies $R((I+B)\widehat{R}(\lambda))\subset D(A)$ and $(\lambda-\lambda\widehat{a}\cdot(\lambda)A)(I+B)\widehat{R}(\lambda)x=C_1x+\lambda B\widehat{R}(\lambda)x, x\in X$, i.e.,

$$(\lambda-\lambda\widehat{a}(\lambda)A(I+B))\widehat{R}(\lambda)x=C_1x,x\in X.\qquad(10)$$

In the following we will prove that when λ is large enough the operator $\lambda-\lambda\widehat{a}(\lambda)A(I+B)$ is injective. Since for $\forall x\in D(\lambda-\lambda\widehat{a}(\lambda)A(I+B))$

$$(\lambda-\lambda\widehat{a}(\lambda)A(I+B))x=(\lambda-\lambda\widehat{a}(\lambda)A)(I-\lambda\widehat{a}(\lambda)A(\lambda-\lambda\widehat{a}(\lambda)A)^{-1}B)x,\qquad(11)$$

we only need prove $I-\lambda\widehat{a}(\lambda)A(\lambda-\lambda\widehat{a}(\lambda)A)^{-1}B$ is injective. Set

$$W(t)x=A\int_0^t S(t-s)C^{-1}Bx\mathrm{d}s, x\in X, t\geq 0.\qquad(12)$$

Hence by (M1), we have

$$\|W(t)\|\leq\frac{M}{\omega}(\mathrm{e}^{\omega t}-1)\quad(t\geq 0)\quad\text{if}\quad\omega>0$$

or

$$\|W(t)\|\leq Mt\quad(t\geq 0)\quad\text{if}\quad\omega=0\qquad(13)$$

We denote by $\widehat{W}(\lambda)$ the Laplace transform of $W(t)$ for large enough λ. Then by Eqs.(12) and (13), we have $\|\lambda\widehat{a}(\lambda)A(\lambda-\lambda\widehat{a}(\lambda)A)^{-1}B\|=\|\lambda\widehat{W}(\lambda)\|\to 0$, as $\lambda\to\infty$, which implies that $\|\lambda\widehat{a}(\lambda)A(\lambda-\lambda\widehat{a}(\lambda)A)^{-1}B\|<1$ for large enough λ. Therefore $I-\lambda\widehat{a}(\lambda)A(\lambda-\lambda\widehat{a}(\lambda)A)^{-1}B$ is invertible, i.e., $\lambda-\lambda\widehat{a}(\lambda)A(I+B)$ is injective. When λ is large enough, by Eq.(10) we obtain that

$$(\lambda-\lambda\widehat{a}(\lambda)A(I+B))^{-1}C_1x=\int_0^{\infty}\mathrm{e}^{-\lambda t}R(t)x\mathrm{d}t, x\in X.$$

Thus by Theorem 2, $C_1^{-1}A(I+B)C_1$ generates an exponentially bounded C_1-regularized resolvent.

Corollary 1 Let A be a generator of a C-regularized resolvent family $\{T(t)\}_{t\geq 0}$ on X. The conditions in Theorem 3 hold. If we also suppose $(\omega,\infty)\in\rho(A)$, then $A(I+B)$ generates an exponentially bounded C_1-regularized resolvent family.

Proof By Theorem 3 and its proof we know $C_1^{-1}A\cdot(I+B)C_1$ generates an exponentially bounded C_1-regularized resolvent family and for large enough λ the operator $I-[\lambda\widehat{a}(\lambda)A(\lambda-\lambda\widehat{a}(\lambda)A)^{-1}B]$ is invertible. Because of the uniqueness property of Laplace transform and the assumption on $a(t)$, we see easily that there exists large enough $\lambda>0$ such that $\widehat{a}(\lambda)\neq 0$. So by Eq.(11) and $(\omega,\infty)\in\rho(A)$, it follows that $\rho(A(I+B))\neq\varnothing$, which implies $C_1^{-1}A(I+B)C_1=A(I+B)$.

Definition 2 A Banach space $(Z,\|\cdot\|_Z)$ is said to satisfy condition (Z) with respect to $\{T(t)\}_{t\geq 0}$ which is a C-regularized resolvent family with the generator A if

(Z1) Z is continuously embedded in $[R(C)]$ and $[R(C)]$ is continuously embedded in X;

(Z2) for all continuous functions $f\in C([0,\infty],Z)$,

$$\int_0^t S(t-s)C^{-1}f(s)\mathrm{d}s\in D(A), t\geq 0;$$

(Z3)

$$\left\|A\int_0^t S(t-s)C^{-1}f(s)\mathrm{d}s\right\|\leq M\int_0^t \mathrm{e}^{\omega(t-s)}\|f(s)\|_Z\,\mathrm{d}s.$$

Corollary 2 Suppose Banach space Z satisfies condition (Z) with respect to a C-regularized resolvent family $\{T(t)\}_{t\geq 0}$ with the generator A.

$B \in L(X,Z)$. The condition (M2) in Theorem 3 holds. Then $C_1^{-1}A(I+B)C_1$ generates an exponentially bounded C_1-regularized resolvent family on X.

Proof For $\forall f \in C([0,\infty], X)$, we have $Bf \in C([0,\infty], Z)$. Because Z satisfies condition (Z), it is easy to know that

$$\int_0^t S(t-s)C^{-1}Bf(s)\mathrm{d}s \in D(A) \text{ for } t \geq 0$$

and

$$\left\| A\int_0^t S(t-s)C^{-1}Bf(s)\mathrm{d}s \right\| \leq M\int_0^t \mathrm{e}^{\omega(t-s)} \|Bf(s)\|_Z \mathrm{d}s$$
$$\leq M\|B\|_{L(X,Z)} \int_0^t \mathrm{e}^{\omega(t-s)} \|f(s)\|\mathrm{d}s,$$

which imply that the conditions in Theorem 3 are satisfied. We finish the proof.

References

Chang, J.C., Shaw, S.Y., 1997. Perturbation theory of abstract Cauchy problems and Volterra equations. *Nonlinear Anal., Theory, Methods & Appl.*, **30**(6): 3521-3528.

Da Prato, G., Iannelli, M., 1980. Linear integro-differential equation in Banach spaces. *Rend. Sem. Math. Padova*, **62**:207-219.

Li, F., 2002. Multiplicative Perturbations of *C*-Regularized Resolvent Families and Degenerate Cauchy Problem. Ph.D Dissertation, University of Science and Technology of China, Hefei (in Chinese).

Li, M., 2002. Report on Fractional Times Resolvent Families with Applications to Partial Differential Equations. Postdoctoral Dissertation, Huazhong University of Science and Technology, Wuhan (in Chinese).

Lizama, C., 1993. On Volterra equations associated with a linear operator. *Proc. Amer. Math. Soc.*, **118**:1159-1166.

Lizama, C., 1995. Uniform continuity and compactness for resolvent families of operators. *Acta Appl. Math.*, **38**:131-138.

Piskarev, S., Shaw, S.Y., 1995. Multiplicative perturbations of C_0-Semigroups and some applications to step responses and cumulative outputs. *J. Funct. Anal.*, **128**: 315-340.

Prűss, J., 1993. Evolutionary Integral Equations and Applications. *In*: Monographs in Mathematics. Birkhaűser Verlag.

Rhandi, A., 1993. Multiplicative perturbations of linear Volterra equations. *Proc. Amer. Math. Soc.*, **119**(2): 493-501.

Xiao, T.J., Liang, J., 2001. Multiplicative perturbations of *C*-segularized semigroups. *Computers & Math. with Appl.*, **41**:1215-1221.

Zheng, Q., 1992. Controllability of a class linear systems in banach space. *Proc. Amer. Math. soc.*, **123**:1241-1251.

Zheng, Q., Sun, Y., 2000. A generation of regularized resolvent families of operators. *Acta Math. Sci.*, **20**: 723-726.