Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

Graph rigidity and localization of multi-robot formations

ZHANG Fan (张 帆)

(*General Robotics Automation Sensing and Perception Laboratory, University of Pennsylvania, Philadelphia, PA 19104, USA*)

E-mail: zhangfan@grasp.cis.upenn.edu

Received Sept. 24, 2003; revision accepted Dec. 22, 2003

Abstract: This paper provides theoretical foundation for the problem of localization in multi-robot formations. Sufficient and necessary conditions for completely localizing a formation of mobile robots/vehicles in SE(2) based on distributed sensor networks and graph rigidity are proposed. A method for estimating the quality of localizations via a linearized weighted least-squares algorithm is presented, which considers incomplete and noisy sensory information. The approach in this paper had been implemented in a multi-robot system of five car-like robots equipped with omni-directional cameras and IEEE 802.11b wireless network.

Key words: Cooperative localization, Graph rigidity, Multi-robot formation
Document code: A **CLC number:** TP18

INTRODUCTION

In order for a team of mobile robots to navigate autonomously in some desired formations and further perform cooperative tasks, such as surveillance and target acquisition, they must be able to localize themselves in the formation as well as in a global reference frame (Belta and Kumar, 2001; Das *et al.*, 2002). Therefore, how to estimate the robots' positions and orientations (poses) in a precise and efficient way is of particular interest. Our interest was in localizing a team of heterogeneous robots in SE(2), and in localizing targets with information obtained from heterogeneous sensors. Specifically, we are interested in conditions under which all robots in the formation can be localized in the environment, and in minimizing the relative and absolute uncertainty in the estimates. We aimed to derive necessary and sufficient conditions for localizing a formation of three or more robots in SE(2) from distributed camera measurements, quantifying the quality of the resulting estimates, and adapting the team formation to improving these estimates.

Recent research addressed the problem of network localization in non-deterministic domains. Examples of fusing observations from heterogeneous sensors to estimate the state of a robot team include the distributed Kalman filter (Roumeliotis and Bekey, 2000) and maximum likelihood (Howard *et al.*, 2002) methods. These approaches consider communication and computational cost but do not address the impact of robot formation on the quality of the solution obtained. Other studies investigated generating optimal sensing trajectories for robots engaged in target tracking tasks given the robot state are known exactly (Spletzer and Taylor, 2003). The work presented in this paper addresses the combination of these two problems. Given the fact that the quality of estimates obtained from measurements depends on how well sensors can be localized, we extend these ideas to find an optimal formation control scheme which will facilitate not only maximal target localization but also consider the robot configuration estimate quality.

Also relevant to this work is the recent literature that uses graphs to model sensor networks and cooperative control schemes (Desai *et al*., 2001; Pappas *et al*., 2001). Results on graph rigidity theory (Laman, 1970; Roth, 1982; Whiteley and Tay, 1985) can be directly applied to multi-robot systems in R^2 (Olfati-Saber and Murray, 2002; Eren *et al*., 2002). However, relatively little attention has been paid on networks with bearing observations, which is particularly important for networks of cameras.

In Section II, the problem will be modeled topologically by using graph theory notations. Then in Section III, four types of sensing constraints are provided and a matrix form representation for those constraints is derived. In Section IV, analytical and computational criteria for localizability of multi-robot formations are presented. Then in Section V, we propose a method to compute the quality of coordinates estimations, and Section VI compares different estimation approaches based on this method. Experimental results will be shown in Section VII, and conclusive remarks will be given in Section VIII.

MODELING

Consider a planar world, $W=R^2$, occupied by a team $R=\{R_1, R_2, \ldots, R_n\}$ of n robots. Assume that all sensory information obtained by each robot can be transformed to a reference frame located at the center of that robot. Thus each robot in R can be regarded as a point robot. We define a global reference frame, F, by forming a virtual robot R_0 with fixed configuration $q_0=0$ in F (Fig.1). The configuration of the team R in the frame F can be denoted by $\hat{\boldsymbol{q}}=\text{column}\{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n\}\in R^{3n}$, where $\boldsymbol{q}_i=(p_i, \theta_i)\in R^3$ is a parameterization of SE(2) with $\boldsymbol{p}_i=(x_i, y_i)^{\mathrm{T}}$ and θ_i being the estimates of absolute position and orientation of the ith robot. Similarly, we also define a body reference frame, B_j, which is located at the jth robot. The configuration of R in the frame B_j is $\tilde{\boldsymbol{q}}=\text{column}\{q_1^j, q_2^j, \ldots, q_n^j\}$, where $\boldsymbol{q}_i^j=(\boldsymbol{p}_{ji}, \theta_{ji})^{\mathrm{T}}$ with $\boldsymbol{p}_{ji}=(x_{ji}, y_{ji})^{\mathrm{T}}$ and θ_{ji} being the estimates of relative position and orientation of the ith robot about the jth robot, and $\boldsymbol{q}_j^j=0$.

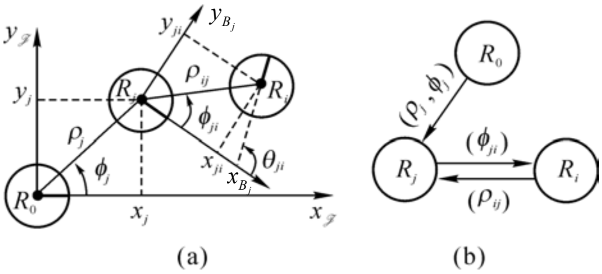

Fig.1 Modeling of sensory information
(a) A formation of 2 robots in SE(2): R_i has range measurements about R_j; R_j has absolute measurements about itself and bearing measurements about R_i. A body reference frame, B_j, has been attached to R_j; (b) sensing graph

The physical configurations of the robots coupled with the characteristics of the hardware and the requirements of the sensing and control algorithms dictate a physical network or a formation of n robots in SE(2). This formation can be represented by a directed graph called sensing graph, $G_s=(V_s, \varepsilon_s, Z, P)$ where $V_s=R\cup R_0$ is a finite set of vertices. The edge set $\varepsilon_s\subset V_s\times V_s$ consists of labeled edges that represent the presence of sensor measurements (observations) between robots. In this paper, we consider three types of exteroceptive sensors: range sensor, bearing sensor, and GPS sensor. Thus the measurement set Z will contain three types of sensory information: (i) range between two robots, ρ_{ij}, (ii) bearing of one robot in relation to another, ϕ_{ij}, and (iii) absolute position of a robot in the frame F, (x_j, y_j), which is physically obtained by GPS sensors. P is the set of variances that represent the quality of measurements.

In a sensing graph G_s, the jth vertex has an incoming edge from the ith vertex labeled by (ρ_{ij}, ϕ_{ij}) whenever robot R_i can sense robot R_j. Corresponding to the three types of sensory information shown above, we use (i) a shorthand relative range edge (ρ_{ij}) to denote $(\rho_{ij}$, null), (ii) a relative bearing edge (ϕ_{ij}) for (null, $\phi_{ij})$ and (iii) an absolute range-bearing edge (ρ_i, ϕ_i) pointed from R_0 to R_j respectively. Thus, the uncertainty set P consists of variances $\sigma_{\rho_{ij}}^2$, $\sigma_{\phi_{ij}}^2$ and covariance matrices $\text{diag}(\sigma_{x_j}^2, \sigma_{y_j}^2)$.

We also consider a communication network associated with the group R, which describes the information flow between the robots. The communication network can be denoted by an undirected graph, called communication graph, $G_c=(R, \varepsilon_c)$, where the edge set ε_c denotes the pairs of robots that can communicate with each other (Throughout this paper, assume that there are omni-directional transmiters and receivers on each robot).

Based on the physical sensing network and communication network, a computational network associated with the group R can be designed. This network is represented by a directed graph called formation control graph, $G_f=(V_f, \varepsilon_f)$, where $V_f = V_s \cap R=R$ and $\varepsilon_f = \varepsilon_s \cap \varepsilon_c$. In this paper, we assume that each robot can listen to every other robot in R. Thus G_c will be a complete graph where G_f is isomorphic to a subgraph of G_s, which contains all vertices and edges in G_s except for R_0 and the edges associated with it. In Section VI, we will show that under a well designed computational network, the configuration of a formation of n robots in SE(2) can be completely estimated with comparable quality.

CONSTRAINT MATRIX

Considering the physical sensor network associated with a formation of n robots in a body reference frame B_j, the sensory information available within the computational network is coupled with the configurations of robots in the formation by various types of sensing equations. Given a set of local (relative) measurements,

$$z = \{(\rho_{ij}, \phi_{ij}) | (i,j) \in \{1,2...,n\}\},$$

we can write sensing equations as follows:

$$\begin{aligned}
&\text{Type 1:} \quad \rho_{ik} = \sqrt{(\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})^{\mathrm{T}} (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})} \\
&\text{Type 2:} \quad \phi_{ji} = \tan^{-1}(y_{ji} / x_{ji}) \\
&\text{Type 3:} \quad \phi_{ik} - \phi_{ki} + \pi = \theta_{ik} = \theta_{jk} - \theta_{ji} \qquad (1) \\
&\text{Type 4:} \quad \phi_{ij} - \phi_{ik} = \cos^{-1} \frac{(\boldsymbol{p}_{ji} - \boldsymbol{p}_{jj})^{\mathrm{T}} (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})}{\|\boldsymbol{p}_{ji} - \boldsymbol{p}_{jj}\| \cdot \|\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk}\|}
\end{aligned}$$

Suppose that robots' configurations in SE(2) are simply maintained and consider the physical sensor network as in quasi-static situations. Then each sensing equation shown above can be regarded as a formation constraint on robots' configurations: Type 1 is a range constraint specifying the relationship between range-type sensory information and positions of two robots in the formation; Type 2 is a bearing constraint relating bearing information with position of each robot in relation to the body reference point, R_j; Type 3 and Type 4 constraints state relative orientation and relative bearing information by robots' configurations respectively. Particularly, Type 4 constraints can be regarded as substitution for Type 1 via cosine theory. Differentiating these constraint equations yields

$$[\boldsymbol{r}_{1,ik} \quad \boldsymbol{r}_{2,ik}] \begin{pmatrix} \dot{\boldsymbol{p}}_{ji} \\ \dot{\boldsymbol{p}}_{jk} \end{pmatrix} = \boldsymbol{0}$$

$$[-y_{ji} / \boldsymbol{p}_{ji}^{\mathrm{T}} \boldsymbol{p}_{ji} \quad x_{ji} / \boldsymbol{p}_{ji}^{\mathrm{T}} \boldsymbol{p}_{ji}] \begin{pmatrix} \dot{x}_{ji} \\ \dot{y}_{ji} \end{pmatrix} = 0$$

$$[-1 \quad 1] \begin{pmatrix} \dot{\theta}_{ji} \\ \dot{\theta}_{jk} \end{pmatrix} = 0$$

$$[\boldsymbol{b}_{1,jik} \quad \boldsymbol{b}_{2,jik} \quad \boldsymbol{b}_{3,jik}] \begin{pmatrix} \dot{\boldsymbol{p}}_{ji} \\ \dot{\boldsymbol{p}}_{jj} \\ \dot{\boldsymbol{p}}_{jk} \end{pmatrix} = \boldsymbol{0}$$

where

$$\boldsymbol{r}_{1,ik} = \frac{\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk}}{\sqrt{(\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})^{\mathrm{T}} (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})}}$$

$$\boldsymbol{r}_{2,ik} = \frac{\boldsymbol{p}_{jk} - \boldsymbol{p}_{ji}}{\sqrt{(\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})^{\mathrm{T}} (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})}}$$

$$\begin{aligned}
\boldsymbol{b}_{1,jik} = &\frac{2[(\boldsymbol{p}_{ji} - \boldsymbol{p}_{jj})^{\mathrm{T}} (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})]}{(\boldsymbol{p}_{ji} - \boldsymbol{p}_{jj})^2 (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})^2} (2\boldsymbol{p}_{ji} - \boldsymbol{p}_{jj} - \boldsymbol{p}_{jk}) \\
&- \frac{2[(\boldsymbol{p}_{ji} - \boldsymbol{p}_{jj})^{\mathrm{T}} (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})]^2}{(\boldsymbol{p}_{ji} - \boldsymbol{p}_{jj})^4 (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})^2} (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jj}) \\
&- \frac{2[(\boldsymbol{p}_{ji} - \boldsymbol{p}_{jj})^{\mathrm{T}} (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})]^2}{(\boldsymbol{p}_{ji} - \boldsymbol{p}_{jj})^2 (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})^4} (\boldsymbol{p}_{ji} - \boldsymbol{p}_{jk})
\end{aligned}$$

$$\boldsymbol{b}_{2,jik}=\frac{2[(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jj})^{\mathrm{T}}(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jk})]}{(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jj})^2(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jk})^2}(\boldsymbol{p}_{jk}-\boldsymbol{p}_{ji})+\frac{2[(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jj})^{\mathrm{T}}(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jk})]^2}{(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jj})^4(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jk})^2}(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jj})$$

$$\boldsymbol{b}_{3,jik}=\frac{2[(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jj})^{\mathrm{T}}(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jk})]}{(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jj})^2(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jk})^2}(\boldsymbol{p}_{jj}-\boldsymbol{p}_{ji})+\frac{2[(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jj})^{\mathrm{T}}(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jk})]^2}{(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jj})^2(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jk})^4}(\boldsymbol{p}_{ji}-\boldsymbol{p}_{jk})$$

Given sensory information on M measurements, we can form m formation constraints and correspondingly m differential equations. By rearranging the first order terms, a matrix form representation can be written as

$$\boldsymbol{K}_{\mathrm{f}}(\tilde{\boldsymbol{q}})\cdot\dot{\tilde{\boldsymbol{q}}}=0,\quad \tilde{\boldsymbol{q}}=\mathrm{column}\{\boldsymbol{q}_1^j,\boldsymbol{q}_2^j,\ldots,\boldsymbol{q}_n^j\}$$

$$\boldsymbol{K}_{\mathrm{f}}=\begin{bmatrix}\cdots & \boldsymbol{r}_{1,ik} & \cdots & \boldsymbol{r}_{2,ik} & \cdots & \cdots & \cdots\\ \cdots & \left(\dfrac{-y_{ji}}{\boldsymbol{p}_{ji}^{\mathrm{T}}\boldsymbol{p}_{ji}},\dfrac{x_{ji}}{\boldsymbol{p}_{ji}^{\mathrm{T}}\boldsymbol{p}_{ji}}\right) & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & -1 & \cdots & 1 & \cdots & \cdots\\ \cdots & \boldsymbol{b}_{1,jik} & \cdots & \boldsymbol{b}_{2,jik} & \cdots & \boldsymbol{b}_{3,jik} & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}$$

where all level dots denotes zeros. The matrix $\boldsymbol{K}_{\mathrm{f}}$ of dimension $m\times 3n$ is called a constraint matrix for a formation of n robots in SE(2) in a body reference frame.

COMPLETELY LOCALIZABLE FORMATION

Naturally, any directed graph with $n\geq 2$ vertices in 2D space can be viewed as a sensing graph for a formation of n robots in SE(2). By localizing the formation completely, all $3n$ coordinates in the configuration space of team R can be estimated algebraically.

Definition 1 A sensing graph of a formation of n robots in SE(2) is said to be rigid if the Euclidean distance (range) ρ_{ij} and the bearing ϕ_{ij} between each pair of robots are constant along any smooth trajectory the formation may undergo.

Definition 2 A formation of n robots in SE(2) is said to be completely localizable in a body reference frame if all $3n$ coordinates of those n robots can be estimated by relative sensory information and a corresponding set of constraint equations.

Theorem 1 A sensing graph of a formation of n robots in SE(2) is rigid only if

$$R = 3n - 2n_{\mathrm{g}} - n_{\mathrm{b}} - n_{\mathrm{r}} \leq 0,\ (n\geq 2),$$

where n_{g}, n_{b} and n_{r} are numbers of measurements obtained by GPS sensors, bearing sensors and range sensors respectively.

Proof For two robots in SE(2), R_i and R_j, it is easy to verify that sensory information of at least one absolute position measurement (x_j, y_j) obtained by GPS sensor, one absolute bearing measurement ϕ_{j0} between R_j and the virtual robot R_0, two relative bearing (ϕ_{ij} and ϕ_{ji}) and one relative range measurement (ρ_{ij} or ρ_{ji}) is necessarily needed to compute the remaining $3n-n_{\mathrm{g}}=4$ coordinates in a global reference frame by equations

$$\begin{aligned}\theta_j&=\phi_{j0}-\phi_j+\pi\\ x_i&=x_j+x_{ji}\cos(\theta_j)-y_{ji}\sin(\theta_j)\\ y_i&=y_j+x_{ji}\sin(\theta_j)+y_{ji}\cos(\theta_j)\\ \theta_i&=\theta_j+\theta_{ji}\end{aligned}\qquad(2)$$

where

$$\begin{aligned}\phi_j&=\tan^{-1}(y_j/x_j)\\ x_{ji}&=\rho_{ji}\cos(\phi_{ji})\\ y_{ji}&=\rho_{ji}\sin(\phi_{ji})\\ \theta_{ji}&=\phi_{ji}-\phi_{ij}+\pi\end{aligned}\qquad(3)$$

Hence the previous theorem holds for two robots in SE(2). Now adding a third robot R_k into the scenario, additional sensory information of three relative measurements (e.g. ρ_{kj}, ϕ_{kj}, ϕ_{jk}) or one absolute plus one relative measurement (e.g. (x_k, y_k), ϕ_{ki}) is necessary to localize R_k based on previous localization of R_i and R_j. The same condition can be used for localization of every robot added into the existing formation. Therefore, Theorem 1 holds for a formation of arbitrary n robots in SE(2).

Theorem 2 Consider a formation of n robots in SE(2), whose vertex set $\{\boldsymbol{q}_1^j,\boldsymbol{q}_2^j,\ldots,\boldsymbol{q}_n^j\}$ is not contained in any hyperplane within R^3. Suppose $\tilde{\boldsymbol{q}} \in R^{3n}$ is a regular point maximizing the rank of $\boldsymbol{K}_f$. The formation is completely localizable if and only if its sensing graph is rigid and

$$\text{rank}\{\boldsymbol{K}_f(\tilde{\boldsymbol{q}})\} = 3n-3, \quad (n \geq 2).$$

Proof Consider a maintained formation in SE(2) with vertex set $\tilde{\boldsymbol{q}} = \text{column}\{\boldsymbol{q}_1^j, \boldsymbol{q}_2^j, ..., \boldsymbol{q}_n^j\} \in R^{3n}$. The set of points, $\boldsymbol{M} \in R^{3n}$, which are congruent to the formation is a smooth manifold. Suppose the affine span of these vertices is R^3, which means that the vertices are not contained in any hyperplane in R^3. Then $\boldsymbol{M}$ arises from the 1-dimensional manifold of orthogonal transformations and the 2-dimensional manifold of translations in SE(2). The tangent space $\boldsymbol{T}$ to $\boldsymbol{M}$ at $\tilde{\boldsymbol{q}}$ with dimension dim($\boldsymbol{T}$) = dim($\boldsymbol{M}$) = 3 is a subset of kernel$\{\boldsymbol{K}_f\}$. Thus we have

$$\text{rank}\{\boldsymbol{K}_f\} = \dim(\tilde{\boldsymbol{q}}) - \text{nullity}(\boldsymbol{K}_f) \leq 3n - \dim(\boldsymbol{T}).$$

At a regular point $\tilde{\boldsymbol{q}}$ which maximize the rank$\{\boldsymbol{K}_f\}$, the tangent space $\boldsymbol{T}$ is equal to the nullspace of $\boldsymbol{M}$, which makes dim($\tilde{\boldsymbol{q}}$)–nullity($\boldsymbol{K}_f$)=3n–dim($\boldsymbol{T}$).

For any formation on n robots in SE(2), one can specify no more than $3n$–3 formation constraints independently to localize those robots in a body reference frame. In addition to the sufficient condition proposed by Theorem 2, we also present necessary conditions for rigid sensing graphs or completely localizable formations in SE(2):

Condition 1 A minimum number of one Type 1 constraint must be included into which a completely localizable formation in SE(2) is specified.

Condition 2 A minimum number of one Type 2 constraint must be included into which a completely localizable formation in SE(2) is specified.

Condition 3 A minimum number of n–1 Type 3 constraint must be included into which a completely localizable formation in SE(2) is specified.

Proof Consider the localization of robots' positions in a body reference frame. It has been shown in the proof of Theorem 1 that for a formation of two robots in SE(2), at least one relative range measurement (ρ_{ij} or ρ_{ji}), which specifies one Type 1 constraint, must be obtained. In order to use Eq.(2) and Eq.(3) to calculate the position of R_i in R_j's reference frame, a relative bearing measurement (ϕ_{ji}) and a Type 2 constraint, which relates it to the position of R_i, must be also specified.

It is well known from the Henneberg construction method that any rigid graph in d-dimensional space can be constructed from a simply maintained link by either adding a new vertex with d edges to the existing graph, or removing an existing maintenance edge while adding a new vertex with edges to the end vertices of the removed edge and to another d–1 vertices in the existing graph. Considering it as a sensing graph of a multi-robot formation, the Henneberg method can be used to construct any localizable (only for positions) formations in R^2 based on the simplest localizable (only for positions) formation built by two robots as above. Therefore, the necessity of one Type 1 constraint and one Type 2 constraint is hereditary for localizing any formation of $n \geq 2$ robot in SE(2).

For Condition 3, the localization of robots in SE(2) includes the estimations of robots' orientations, which call for Type 3 constraints naturally. Consider the configuration space for relative orientations of n robots in a body reference frame B_j; it is obvious that the dimension of this configuration space is n–1, given that the orientation of the jth robot, the origin of B_j, is fixed to be zero. Hence, only n–1 constraints of Type 3 are independent and necessary for estimating all relative orientations of n–1 robots in B_j. Note that the sensing graph representation for bearing measurements must be a connected graph (Fig.2).

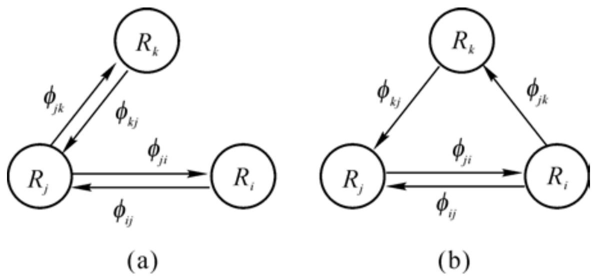

Fig.2 Bearing sensing graphs for three robots in SE(2). In a rigid bearing sensing graph, (b) can be always transformed trigonometrically into (a) based on available exteroceptive sensory information

For further matters, we know that the relative positions of n robots in R^2 can be estimated with local uniqueness in a body reference frame by $2n–3$ formation constraints of Type 1 (Olfati-Saber and Murray, 2002; Eren *et al*., 2002). Furthermore, a globally unique localization that results in R^2 can be also guaranteed if the sensing graph is 3-connected and redundantly rigid (Eren *et al*., 2004). However, the uniqueness of localizations cannot be preserved in SE(2) but will be up to rotation. As an example, let us consider a group of two robots in SE(2). The position of robot R_i in R_j's reference frame, (x_{ji}, y_{ji}), cannot be uniquely determined unless a constraint of Type 2, which provides the rotation angle by a relative bearing measurement, is specified.

Within any completely localizable formation, the configurations of n robots in SE(2) can be estimated in a body reference frame by a set of constraints which satisfies Theorem 2 and the necessary conditions shown above. Let $\boldsymbol{p}_j = (x_j, y_j)^{\mathrm{T}} \in R^2$ be the absolute location of the body reference point in relation to a fixed virtual robot in its global reference frame and let $\boldsymbol{T}$ be the rotation matrix formed by $\theta_j = \phi_{j0} - \phi_j + \pi$, where $\phi_j = \tan^{-1}(y_j / x_j)$. Then giving the transformation $(\boldsymbol{p}_j, \boldsymbol{T}) \in \mathrm{SE}(2)$, the absolute configuration of the formation can be also determined in the global reference frame F.

ESTIMATION ERRORS

Uncertainty or errors exist in the preceding localization process since there always exists noise within sensory information acquired through the physical network. Considering the uncertainty in the model, we can generalize m constraints as

$$\boldsymbol{H} \cdot \boldsymbol{z} = g(\tilde{\boldsymbol{q}}) + \boldsymbol{L} \cdot \boldsymbol{r}_0, \tag{4}$$

where $\boldsymbol{z}$ is the vector of measurements, $\boldsymbol{r}_0 \sim N(0, \boldsymbol{R}_0)$ is the vector of measurements' noise (assuming zero-mean independent noise and hence a diagonal covariance matrix $\boldsymbol{R}_0$), g is the set of functions of $\tilde{\boldsymbol{q}}$, $\boldsymbol{H}$ denotes linear combinations of measurements while $\boldsymbol{L}$ denotes linear combinations of measurements' noises. For Type 3 constraint in Eq.(1), for example, we have

$$\sigma_{\phi_{ij}}^2 + \sigma_{\phi_{ji}}^2 = \sigma_{\theta_{ij}}^2,$$

and thus

$$\boldsymbol{H} = [-1 \quad 1]; \quad \boldsymbol{L} = [\,1 \quad 1].$$

Let $\boldsymbol{r}=\boldsymbol{L}\cdot\boldsymbol{r}_0$ with $\boldsymbol{r} \sim N(0, \boldsymbol{R})$ denoting the vector of combined measurements' noise, where $\boldsymbol{R}$ is a covariance matrix. Since $\Delta \boldsymbol{r} = \boldsymbol{L}\cdot\Delta\boldsymbol{r}_0$, we can multiply both sides of this equation by their transposes and take the expectation values on both sides, which yields

$$\boldsymbol{R} = \boldsymbol{L}\, \boldsymbol{R}_0\, \boldsymbol{L}^{\mathrm{T}}.$$

Theorem 3 The total number of constraint equations, m, which can be used to estimate the configuration of a formation of n robots in SE(2), must satisfy

$$3n - 3 \le m \le M, \quad (n \ge 2),$$

where M = size ($\boldsymbol{z}$) is the number of measurements.

The left part of inequality can be satisfied by Theorem 2. The right part of inequality is obvious since the row rank of $\boldsymbol{L}$ must not exceed its column rank in order to make $\boldsymbol{R}$ full-ranked, which is a required condition for weighted least squares (WLS) algorithm we used to solve Eq.(4). Physically Theorem 3 implies that only $M – (3n–3)$ redundant constraints can be useful for enhancing the quality of estimation in one endeavor. Let $\boldsymbol{G}$ be the Jacobian of $\boldsymbol{g}$ evaluated at current value of $\tilde{\boldsymbol{q}}$. Given that $q_j^j = 0$ in B_j, $\boldsymbol{G}$ has dimension $m\times(3n–3)$ and can be obtained by deleting the 3 columns corresponding to the jth robot coordinates q_j^j in $\boldsymbol{K}_{\mathrm{f}}$. Then a linearized version of the least squares algorithm can be written as

$$\Delta\tilde{\boldsymbol{q}} = (\boldsymbol{G}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{G})^{-1}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{H}\Delta\boldsymbol{z}, \tag{5}$$

where $\boldsymbol{W} = \boldsymbol{R}^{-1}$ is the weight matrix. Now squaring Eq.(5) by multiplying both sides of the equation by their transposes respectively and then taking the expectations on both sides, we can get the covariance matrix $\boldsymbol{C}$ for coordinates estimations as

$$\boldsymbol{C}=(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{G})^{-1}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{R}\boldsymbol{W}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{G})^{-\mathrm{T}},$$

or simply

$$\boldsymbol{C}=(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{G})^{-1}. \tag{6}$$

DEPENDENCE ON COMPUTATIONAL NETWORK

Obviously, the quality of estimations, which is given by $\boldsymbol{C}$, is affected by the specific set of measurements and corresponding constraint equations we chose to construct the constraint matrix $\boldsymbol{G}$, linear matrix $\boldsymbol{L}$ and eventually compute $\boldsymbol{C}$. This provides us a natural strategy for comparing the quality of different approaches of coordinates estimations based on different sets of measurement information.

As an example, let us consider a team of $n=3$ robots in SE(2), $\boldsymbol{R}=\{R_i, R_j, R_k\}$. The configuration of the team in B_j is $\tilde{\boldsymbol{q}}=\text{column}\{\boldsymbol{q}_i^j,\boldsymbol{q}_j^j,\boldsymbol{q}_k^j\}$, where $\boldsymbol{q}_j^j=(0,0,0)^{\mathrm{T}}$ (Fig.3). The physical conditions are limited such that the total sensory information available in our computational network is

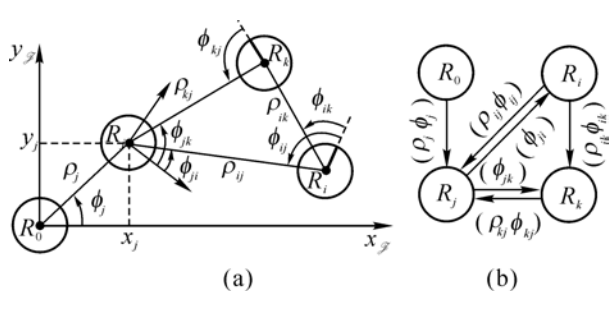

Fig.3 Modeling of sensory information
(a) physical network of 3 robots in SE(2): R_i has range and bearing measurements about R_j and R_k; R_j has absolute measurements of itself and bearing measurements of R_i and R_k; R_k has range and bearing measurements of R_j. A body reference frame, B_j, has been attached to R_j. (b) sensing graph

$$\boldsymbol{z}=(\rho_{ij},\rho_{ik},\phi_{ij},\phi_{ik},\phi_{ji},\phi_{jk},\rho_{kj},\phi_{kj})^{\mathrm{T}}$$
$$\boldsymbol{R}_0=\mathrm{diag}(\sigma_{\rho_{ij}}^2,\sigma_{\rho_{ik}}^2,\ldots,\sigma_{\phi_{kj}}^2)$$

Due to the costs of computation and communications within the network, we want to have the least possible amount of sensory information and minimum number of constraints to make the formation completely localizable. From Theorem 2 we see that only $3n–3$ independent constraints are needed to estimate all $3n$ coordinates successfully. Thus the sensing graph representation for a formation specified by $3n–3$ independent constraints is called a minimally rigid graph for a team of $n\geq2$ robots in SE(2). By this means, two subsets of measurements can be selected to construct a minimum number of 6 formation constraints: (1) ρ_{ij}, ρ_{ik}, (ϕ_{ij}, ϕ_{ji}), ϕ_{ji}, ρ_{kj}, (ϕ_{jk}, ϕ_{kj}) where parentheses mean a couple of measurements used to form one constraint; (2) ρ_{ij}, ρ_{ik}, (ϕ_{ij}, ϕ_{ik}), (ϕ_{ij}, ϕ_{ji}), ϕ_{jk}, (ϕ_{jk}, ϕ_{kj}). The constraint matrix $\boldsymbol{G}$ and numerical matrix $\boldsymbol{L}$ for those two approaches are

$$\boldsymbol{L}_1=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&1&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&1\end{bmatrix}$$

$$\boldsymbol{L}_2=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&1&0&1&0&0&0\\0&0&0&0&0&1&0&1\\0&0&0&0&0&1&0&0\end{bmatrix}$$

$$\boldsymbol{G}_1=\begin{bmatrix}\boldsymbol{r}_{1,ij}&0&0&0\\\boldsymbol{r}_{1,ik}&0&\boldsymbol{r}_{2,ki}&0\\0&1&0&0\\\left(\dfrac{-y_{ji}}{\boldsymbol{p}_{ji}^{\mathrm{T}}\boldsymbol{p}_{ji}}\;\dfrac{x_{ji}}{\boldsymbol{p}_{ji}^{\mathrm{T}}\boldsymbol{p}_{ji}}\right)&0&0&0\\0&0&\boldsymbol{r}_{2,jk}&0\\0&0&0&1\end{bmatrix}$$

$$
\boldsymbol{G}_2=\begin{bmatrix} \boldsymbol{r}_{1,ij} & 0 & 0 & 0 \\ \boldsymbol{r}_{1,ik} & 0 & \boldsymbol{r}_{2,ki} & 0 \\ \boldsymbol{b}_{1,jik} & 0 & \boldsymbol{b}_{3,jik} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \left(\dfrac{-y_{jk}}{\boldsymbol{p}_{jk}^{\mathrm{T}}\boldsymbol{p}_{jk}} \quad \dfrac{x_{jk}}{\boldsymbol{p}_{jk}^{\mathrm{T}}\boldsymbol{p}_{jk}}\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Together with $\boldsymbol{R}_0$, two covariance matrices, $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, can be established through Eq.(6). Computing and comparing the determinants or traces of $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, which are proportional to the volume and circumference of the rectangular regions enclosing the covariance ellipsoids, we can find better estimation approach with less uncertainty.

EXPERIMENTAL RESULTS

To demonstrate the previous strategy, we present some experimental results with a team of five car-like mobile robots equipped with omni-directional cameras as their only sensor (Fig.4). The communication among the robots relies on IEEE 802.11 networking. In order to facilitate the visual processing, each robot is marked with a different color providing unique sensor identification for each robot. A calibrated overhead camera with an external computer is used to localize the team in the environment. This external computer also collected sensor information broadcasted within the computational network.

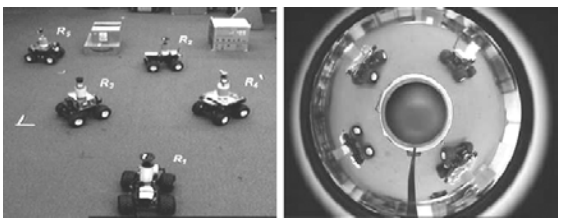

Fig.4 A formation of five car-like robots (left) and a sample image from omni-directional camera (right)

A limitation of the omni-directional cameras we used was that their resolution decreases with the distance of the objects. After 1.5 meters, errors within range measurements could not be represented by a normal distribution with zero offset, while the bearing measurements present normal distribution with zero-mean and approximately constant variance whenever a target could be detected. Thus we assume that range data were subject to normally distributed noise with variance proportional to the fourth power of the range for values smaller than 1.5 meters, while bearing readings suffered from normally distributed noise of constant variance. Range values grater than 1.5 meters were ignored.

Consider a formation of five mobile robots in a plane (Fig.4). As constrained by the physical conditions of our sensors and experimental environment, in a snapshot, all sensory information available in the computational network is represented in Fig.5a. Using our Matlab algorithm, a subset of measurements [size(z)=21 in Fig.5a] can be chosen such that the quality of coordinates estimation by those measurements is relatively the highest at the moment.

Table 1 shows how the quality of estimations is governed by different sets of measurement information we used in some typical cases. The first case in Table 1 assumes availability of all sensory information shown in Fig.5a, while Case 6 assumes limited information for only a minimally rigid sensing graph as shown in Fig.5b. Case 2 considers all sensory information in Fig.5a except for ϕ_{53}, which did not contribute to the improvement of esti-

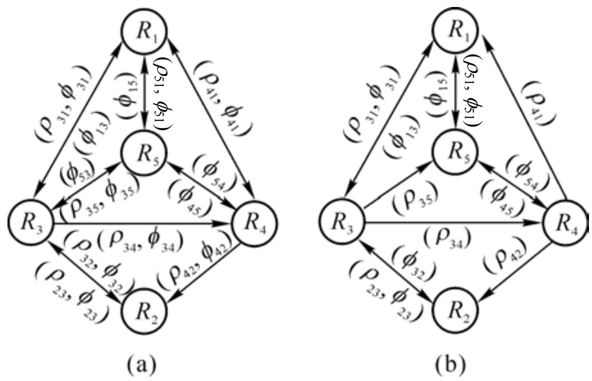

Fig.5 Graph of two computational networks

(a) Computational network containing redundant sensory information; (b) computational network containing only necessary sensory information for a minimally rigid formation

Table 1 Quality of coordinates estimations under different sets of relative measurements

	Trace (***C***)
Case 1	0.0142
Case 2	0.0142
Case 3	0.0171
Case 4	0.0228
Case 5	0.1231
Case 6	0.1505

mation quality. Case 3 and Case 4 are two transitional cases between Case 1 and Case 6, in which two different subsets of redundant information are considered. Case 5 represents another optional set of measurement information for getting a minimally sensing graph, in which ρ_{53} is replaced by ϕ_{42} from Fig.5b.

CONCLUSIONS

In this paper, we provided a strategy of modeling multi-robot teams in SE(2) by using the graphic models. Sufficient and necessary conditions for rigid sensing graphs and complete localizable formations of robot teams in SE(2) were derived. A method of computing and comparing the quality of different estimations approaches was also proposed based on limited sensory information available in the computational networks. Experimental data were used to show the effect of redundant sensory information on the coordinate estimation quality. The analytical and computational criteria for rigid sensing graphs and completely localizable formations we present in this paper can be generally applied to various kinds of mobile robots or autonomous vehicles in SE(2) and is scalable with the number of sensors or robots in a formation. Future work includes the extension of our strategy to dynamic situations, in which robots will move cooperatively to achieve localizations with minimized uncertainty.

References

Belta, C., Kumar, V., 2001. Motion Generation for Formation of Robots: A Geometric Approach. Proc. of Intl. Conf. on Robotics and Automation, Seoul, Korea.

Das, A., Spletzer, J., Kumar, V., Taylor, C., 2002. Ad Hoc Networks for Localization and Control. Proc. of the 41st IEEE Conf. on Decision and Control, Las Vegas, NV.

Desai, A., Kumar, V., Ostrowski, J., 2001. A Theoretical Framework for Modeling and Controlling Formation of Mobile Robot. IEEE Transactions on Robotics and Automations.

Eren, T., Belhumeur P.N., Anderson B.D.O., Morse, A.S., 2002. A Framework for Maintaining Formations based on Rigidity. Proceedings of the IFAC Congress, p.2752-2757.

Eren, T., Goldenberg, D., Whiteley, W., Yang, Y.R., Morse, A.S., Anderson, B.D.O., Belhumeur, P.N., 2004. Rigidity and Randomness in Network Localization. Proc. of the IEEE INFOCOM Conference, Hong Kong.

Howard, A., Matari, M., Sukhatme, C., 2002. Team Localization: A Maximum Likelihood Approach. Technical Report IRIS-01-415, Institute for Robotics and Intelligent Systems Technical Report, University of Southern California.

Laman, G., 1970. On graphs and rigidity of plane skeletal structures. *Journal of Engineering Mathematics*, **4**: 331-340.

Olfati-Saber, R., Murray, R.M., 2002. Graph Rigidity and Distributed Formation Stabilization of Multi-vehicle systems. Proc. of the 41st IEEE Conf. on Decision and Control, Las Vegas, NV.

Pappas, G.J., Tabuada P., Lima, P., 2001. Feasible Formations of Multi-agent Systems. Proceedings of the American Control Conference, Arlington, Virginia.

Roth, B., 1982. Rigidity and flexible frameworks. *The American Mathematical Monthly*, **88**:6-21.

Roumeliotis, S., Bekey, G., 2000. Collective Localization: A Distributed Kalman Filter Approach to Localization of Groups of Mobile Robots. Proc. of the IEEE Intl. Conf. on Robotics and Automation, San Francisco, p.2958-2965.

Spletzer, J., Taylor, C.J., 2003. Dynamic sensor planning and control for optimally tracking targets. *International Journal of Robotics Research*, **22**:7-20.

Whiteley, W., Tay, T., 1985. Generating isostatic frameworks. *Structural Topology*, **11**:21-69.