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Arc-length technique for nonlinear finite element analysis*

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Abstract: Nonlinear solution of reinforced concrete structures, particularly complete load-deflection response, requires tracing of the equilibrium path and proper treatment of the limit and bifurcation points. In this regard, ordinary solution techniques lead to instability near the limit points and also have problems in case of snap-through and snap-back. Thus they fail to predict the complete load-displacement response. The arc-length method serves the purpose well in principle, received wide acceptance in finite element analysis, and has been used extensively. However modifications to the basic idea are vital to meet the particular needs of the analysis. This paper reviews some of the recent developments of the method in the last two decades, with particular emphasis on nonlinear finite element analysis of reinforced concrete structures.

Key words: Arc-length method, Nonlinear analysis, Finite element method, Reinforced concrete, Load-deflection path
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INTRODUCTION

Complete investigation of the nonlinear behavior of structures must follow the equilibrium path; identify and compute the singular points like limit or bifurcation points, whose secondary branches in the equilibrium path must be examined and followed. Several techniques to achieve the solution pattern on the equilibrium path were presented in literature. Load controlled Newton-Raphson method was the earliest method in this regard; but it fails near the limit point. To overcome difficulties with limit points, displacement control techniques were introduced. However for structural systems exhibiting snap-through or snap-back behavior, these techniques lead to error. One way to overcome the problem is by adopting a technique to switch between load and displacement controls (Sabir and Lock, 1972), by using the artificial springs of Wright and Gaylord (1968), or by abandoning the equilibrium iterations in the close vicinity of limit point (Bergan and Soreide, 1978; Bergan *et al*., 1978). To obtain a more general technique, the arc-length method for structural analysis, originally developed by Riks (1972; 1979) and Wempner (1971) and later modified by several scholars, is used. Various forms of arc-length method followed the original work of Riks and Wempner. This paper reviews recent developments of the arc-length method, discusses its various key issues, particularly during the last two decades.

Unlike the load control method in which the load is kept constant during a load step or in the displacement control method in which displacement is kept constant during increment, in the arc-length method, the load-factor at each iteration is modified so that the solution follows some specified path until convergence is achieved. The basic

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idea of the method can be explained as follows.

Since the method treats the load-factor as a variable, it becomes an additional unknown in equilibrium equations resulting from finite element procedure, and yield (N+1) unknowns, where N is the number of elements in the displacement vector. The solution of (N+1) unknowns requires an additional constraint equation expressed in terms of current displacement, load-factor and arc-length. Two approaches, fixed arc-length and varying arc-length are generally used. In the former the arc-length is kept fixed for current increment, whereas in the latter case, new arc-length is evaluated at the beginning of each load step to ensure the achievement of the solution procedure. Simplification of the constraint equation leads to a quadratic equation, whose roots are used for determining the load-factor. Proper selection of root is one of the key issues of the method, whose details will be discussed in subsequent sections. Generally, for the first increment, the trial value of the load-factor is assumed as 1/5 or 1/10 of total load. For further increments the load-factor is computed according to the rate of convergence of the solution process. In case of divergence from the solution path, the arc-length is reduced and computations are done again. The computation time of the solution process is also of major concern in finite element analysis, so a maximum number of iterations are preset; and if solution does not converge in the specified number of iterations then the load step is reduced and the process is started over again.

ARC-LENGTH TECHNIQUE

The equilibrium equation of nonlinear system can be written as

$$\boldsymbol{g}_i(\lambda_i) = \boldsymbol{f}_i - \lambda_i \boldsymbol{q} \tag{1}$$

where $\boldsymbol{f}_i$ is vector of internal equivalent nodal forces, $\boldsymbol{q}$ is the external applied load vector, λ is the load-level parameter, and $\boldsymbol{g}_i$ is out-of-balance force vector. The arc-length method is aimed to find the intersection of Eq.(1) with constant s termed as the arc-length, and can be written in differential form as

$$s = \int \sqrt{\mathrm{d}\boldsymbol{p}^{\mathrm{T}}\mathrm{d}\boldsymbol{p} + \mathrm{d}\lambda^2 \psi^2 \boldsymbol{q}^{\mathrm{T}}\boldsymbol{q}} \tag{2}$$

or in increment form, written as

$$a = \Delta\boldsymbol{p}^{\mathrm{T}}\Delta\boldsymbol{p} + \Delta\lambda^2\psi^2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q} - \Delta l^2 = 0 \tag{3}$$

where $\Delta\boldsymbol{p}$ is vector of incremental displacement, $\Delta\lambda$ is incremental load-factor, Δl is fixed radius of desired intersection, and ψ is the scaling parameter for loading terms. With some simplification Eqs.(1) and (3) can be directly used to compute the iterative change in displacement vector and load-factor, and are written as

$$\begin{Bmatrix} \delta\boldsymbol{p} \\ \delta\lambda \end{Bmatrix} = -\begin{bmatrix} \boldsymbol{K}_{\mathrm{T}} & -\boldsymbol{q} \\ 2\Delta\boldsymbol{p}^{\mathrm{T}} & 2\Delta\lambda\psi^2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q} \end{bmatrix}^{-1} \begin{Bmatrix} \boldsymbol{g}_{\mathrm{old}} \\ a_{\mathrm{old}} \end{Bmatrix} \tag{4}$$

where $\delta\boldsymbol{p}$ is iterative change in displacement vector, $\delta\lambda$ is iterative change in load-factor, $\boldsymbol{K}_{\mathrm{T}}$ is the tangential stiffness matrix, and $\boldsymbol{g}_{\mathrm{old}}$ and a_{old} are the previous values of out-of-balance load vector and arc-length. After the iterative change $\delta\boldsymbol{p}$ and $\delta\lambda$ have been computed, the displacement vector and load-factor are updated.

Alternatively, instead of solving Eqs.(1) and (3) directly, constraint equation can be introduced by following the technique of Baltoz and Dhatt (1979) for displacement control at single point (Crisfield, 1981). According to the technique, the iterative change of displacement for the new unknown load level $\Delta\lambda_{i+1} = \Delta\lambda_i + \delta\lambda$ is written as

$$\delta\boldsymbol{p} = -\boldsymbol{K}_{\mathrm{T}}^{-1}\boldsymbol{g} + \delta\lambda\boldsymbol{K}_{\mathrm{T}}^{-1}\boldsymbol{q} = \delta\boldsymbol{g} + \delta\lambda\delta\boldsymbol{p}_{\mathrm{T}} \tag{5}$$

and the incremental displacement for the next increment can be written as

$$\Delta\boldsymbol{p}_{i+1} = \Delta\boldsymbol{p}_i + \delta\boldsymbol{p} \tag{6}$$

Substituting values from Eqs.(5) and (6) into the constraint equation yields the expression

$$c_1\delta\lambda^2 + c_2\delta\lambda + c_3 = 0 \tag{7}$$

where

$$\begin{aligned} c_1 &= \delta \boldsymbol{p}_{\mathrm{T}}^{\mathrm{T}}\delta \boldsymbol{p}_{\mathrm{T}} + \psi^2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q} \\ c_2 &= 2\delta \boldsymbol{p}_{\mathrm{T}}(\Delta\boldsymbol{p} + \delta\boldsymbol{g}) + 2\Delta\lambda\psi^2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q} \\ c_3 &= (\Delta\boldsymbol{p} + \delta\boldsymbol{g})^{\mathrm{T}}(\Delta\boldsymbol{p} + \delta\boldsymbol{g}) - \Delta l^2 + \Delta\lambda^2\psi^2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q} \end{aligned} \tag{8}$$

Eq.(7) is solved to get the value of $\delta\lambda$, and so to define the iterative change completely. This equation leads to two results of $\delta\lambda$. The method for selection of proper value will be discussed in subsequent sections. The method is graphically shown in Fig.1.

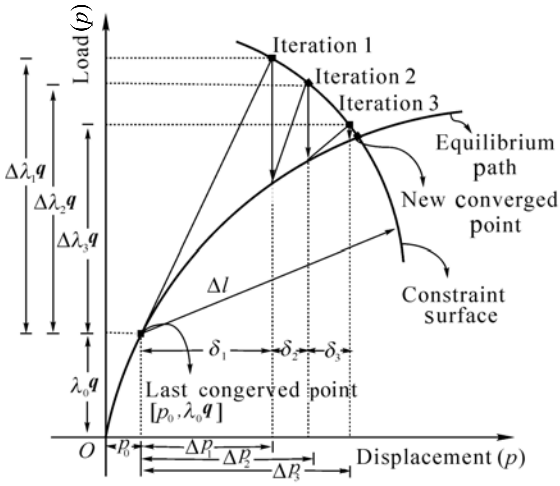

Fig.1 Arc-length procedure for specific iteration

Originally, the work of Riks and Wempner advocated that the iterative change should be made orthogonal to the predictor solution $(\Delta\boldsymbol{p}_{\mathrm{p}}, \Delta\lambda_{\mathrm{p}})$ (Fig.2). Ramm (1981) advocated the use of making iterative change orthogonal to secant change $(\Delta\boldsymbol{p}_{\mathrm{o}}, \Delta\lambda_{\mathrm{o}}\psi\boldsymbol{q})$ (Fig.3); the technique of Ramm is closely related to the work of Riks and Wempner. To avoid the dependency of solution process on either predictor or secant change, Fried (1984) suggested use of $(\delta\boldsymbol{p}_{\mathrm{T}}, (1/\psi^2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q}))$ instead of $(\Delta\boldsymbol{p}_{\mathrm{o}}, \Delta\lambda_{\mathrm{o}})$. All these techniques are termed as linearised versions of arc-length method, in which the constraint equation leads to only one solution, hence no issue of selection of roots. However it is possible that the method may sometime miss the equilibrium path, and lead to numerical difficulties.

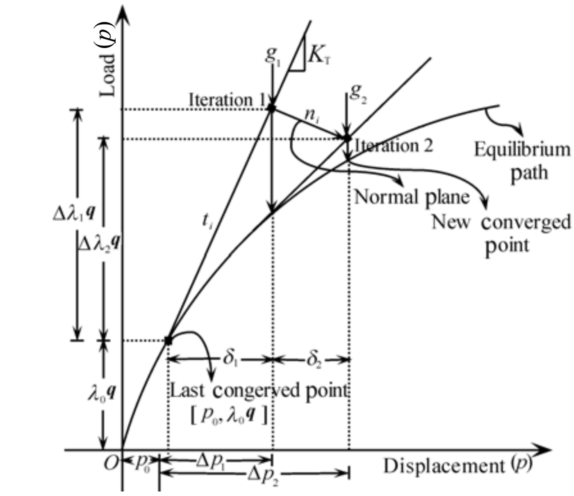

Fig.2 Arc-length method (Riks, 1972; 1979; Wempner, 1971)

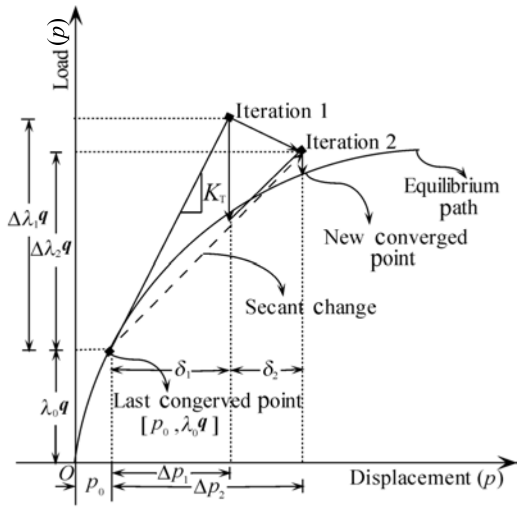

Fig.3 Arc-length method (Ramm, 1981)

ARC-LENGTH METHOD DURING THE 80'S

After the introduction of arc-length method by Riks (1972; 1979) and Wempner (1971), the method received wide acceptance in the field of finite element analysis. However as Crisfield (1981) stated, the Riks method was not suitable for standard finite element analysis even with modified Newton-Raphson (mN-R) procedure, because equations proposed by Riks destroy the banded nature of the stiffness matrix. For one-dimensional problem with N displacement variables, Crisfield (1981) gave the modification of the method and suggested the fixation of incremental length Δl during load increment. Therefore instead of applying the constraint Eq.(3), the following equation was used

$$\Delta \boldsymbol{p}_i^{\mathrm{T}} \Delta \boldsymbol{p}_i = \Delta l^2 \tag{9}$$

The proposed technique is termed as cylindrical arc-length method. Solution of Eq.(9) yields to a quadratic equation similar to Eq.(7) but with $\psi = 0$, and thus two roots. To avoid doubling back on original load/deflection path, Crisfield suggested that the angle between the incremental displacement vector before the current iteration and incremental load vector after the current iteration must be minimum. To achieve this, the appropriate root is the one that gives positive angle, and in case that both roots are positive, the appropriate root is the one that is closest to the linear solution. Crisfield applied this method for large-deflection elasto-plastic analysis of imperfectly stiffened plates and shells and got satisfactory results.

Crisfield (1983) pointed out that although the arc-length method works well and has been successfully incorporated into finite element programming, yet only partial success was achieved in solving the material nonlinearity of beams and slabs with significant strain softening. However the same problem when solved by the displacement control method with simple line searches yielded acceptable results. Thus the author suggests the use of line search scheme with arc-length method. The suggested scheme uses fixed load level and seeks a scalar η_i such that the energy Φ at $\boldsymbol{p}_{i+1}$ is stationary in the direction of η_i, i.e.

$$\begin{aligned}\left[\frac{\partial \Phi}{\partial \eta_i}\right]_{i+1}^{\mathrm{T}} &= \left[\frac{\partial \Phi}{\partial \boldsymbol{p}}\right]_{i+1}^{\mathrm{T}} \frac{\partial \Phi}{\partial \eta_i} = \boldsymbol{g}_{i+1}(\boldsymbol{p}_{i+1}(\eta_i))^{\mathrm{T}} \delta_i \\ &= s_j(\eta_{i,j}) = 0\end{aligned} \tag{10}$$

Eq.(10) is too stringent a condition to meet in practice and instead it is desirable to satisfy

$$\left|s_j(\eta_{i,j})\right| < \mu \left|s_0(\eta_{i,0} = 0)\right| \tag{11}$$

where

$$\left|s_0(\eta_{i,0} = 0)\right| = \delta_i^{\mathrm{T}} \boldsymbol{g}_i(\boldsymbol{p}_i) \tag{12}$$

where $\boldsymbol{g}_i$ is the out-of-balance force vector at the end of the previous iteration. The near optimum value of μ is suggested as 0.8. The procedure works well; but if Eq.(12) is violated, then a new arc-length $\eta_{i,2}$ must be tried as a second attempt to satisfy the equation. The simplest method to compute second attempt is to use linear interpolation or extrapolation using s_1 and s_0 values.

Forde and Stiemer (1987) introduced a general arc-length procedure based on orthogonality principles. In their work the authors suggested selection of an arbitrary direction $\boldsymbol{n}^{(i)}$ with reference to tangent $\boldsymbol{t}^{(i)}$ of the current incremental load-displacement configuration. The scalar product of these vectors yields a residual $\boldsymbol{g}^{(i)}$. The tangential and normal vectors ($\boldsymbol{t}$ and $\boldsymbol{n}$) consist of m dimensions from the displacement vector and one dimension from the load parameter. These components are combined using a scaling factor β to form vectors with $m+1$ dimension, which can be written as

$$\boldsymbol{t}^{(i)} = \boldsymbol{u}^{(i)} + \beta\lambda^{(i)} \tag{13}$$

$$\boldsymbol{n}^{(i)} = \Delta \boldsymbol{u} + \beta\Delta\lambda \tag{14}$$

The scalar product of $\boldsymbol{t}^{(i)}$ and $\boldsymbol{n}^{(i)}$ results in the general expression for $\Delta\lambda$, as follows

$$\Delta\lambda = \frac{\boldsymbol{g}^{(i)} - \boldsymbol{u}^{(i)^{\mathrm{T}}} \Delta \boldsymbol{u}^{\mathrm{II}}}{\beta^2 \lambda^{(i)} + \boldsymbol{u}^{(i)^{\mathrm{T}}} \Delta \boldsymbol{u}^{\mathrm{I}}} \tag{15}$$

The expression can be simplified for particular cases of orthogonality. The use of the method simplifies the solution process. The method reveals exactly similar results for $\Delta\lambda$ as obtained by Crisfield (1981) but without solving quadratic equation and selection of proper root.

ARC-LENGTH METHOD DURING THE 90'S

Al-Rasby (1991) gave modified arc-length method almost similar to that of Forde and Stiemer (1987) but introduced scaling matrices for calculation of the arc-length for the purpose of non-dimensionalizing the vectors that define the arc-length constraint equations. These matrices can also be used to gauge the relative contribution of load and displacement components to the arc- length constraint equation. Based on the idea, the equation for $\Delta\lambda$ takes the form

$$\Delta\lambda = \frac{\boldsymbol{g}^{(i)} - \boldsymbol{U}^{(i)^{\mathrm{T}}} \boldsymbol{V}_{\mathrm{s}} \delta^{(i)}}{\boldsymbol{U}^{(i)^{\mathrm{T}}} \boldsymbol{V}_{\mathrm{s}} \delta^{(i)} + \lambda^{(i)} \boldsymbol{R}_{\mathrm{ref}}^{\mathrm{T}} \boldsymbol{W}_{\mathrm{s}} \boldsymbol{R}_{\mathrm{ref}}} \tag{16}$$

where $\boldsymbol{V}_{\mathrm{s}}$ is the displacement scaling diagonal matrix, and $\boldsymbol{W}_{\mathrm{s}}$ is the load scaling diagonal matrix. By varying the residual $\boldsymbol{g}$, all of the known arc-length methods can be used. With proper variation of $\boldsymbol{V}_{\mathrm{s}}$ and $\boldsymbol{W}_{\mathrm{s}}$, load control and displacement control methods are the special cases of this technique. Since variation of the scaling diagonal matrices required to follow the solution process, the variation can be done in infinite variety. However the author suggests the following four types of variations:

i) $\boldsymbol{V}_{\mathrm{s}} \equiv \boldsymbol{W}_{\mathrm{s}} \equiv \boldsymbol{I}$. Setting both diagonal scaling matrices equal to identity means that all parameters are considered equally. This may not be the ideal case, since on the load-displacement path, at particular points some parameters are more significant than others. Also this procedure does not introduce any scaling on either load or displacement component.

ii) $\boldsymbol{V}_{\mathrm{s}} \equiv \boldsymbol{I}$ and $[\boldsymbol{W}_{\mathrm{s}}]_{i,i} = (\boldsymbol{R}_{\mathrm{ref}})_t^{-2}$. No sum on i. This is equivalent to specifying $\boldsymbol{R}_{\mathrm{ref}}^{\mathrm{T}} \boldsymbol{W}_{\mathrm{s}} \boldsymbol{R}_{\mathrm{ref}} \equiv \boldsymbol{I}$.

iii) $\boldsymbol{V}_{\mathrm{s}} \equiv \boldsymbol{I}$ and $\boldsymbol{W}_{\mathrm{s}} = c\boldsymbol{I}$,

Where $c = \left[\dfrac{\delta^{(1)\mathrm{T}} \delta^{(1)}}{\Delta\lambda_1^2 \boldsymbol{R}_{\mathrm{ref}}^{\mathrm{T}} \boldsymbol{R}_{\mathrm{ref}}} \right] CSP$

and $CSP = \left[\dfrac{\lambda^{(1)}}{\Delta\lambda_1} \right] \dfrac{\delta^{(1)} \boldsymbol{R}_{\mathrm{ref}}}{\boldsymbol{U}^{(i)} \boldsymbol{R}_{\mathrm{ref}}}$ where $\boldsymbol{\delta}^{(i)}$ is the initial displacement vector in the first step, $\Delta\lambda_1$ is the initial load parameter in the first step and *CSP* is the current stiffness parameter.

iv) $\boldsymbol{V}_{\mathrm{s}} = [\boldsymbol{K}]_{i,i}$ and $\boldsymbol{W}_{\mathrm{s}} = [\boldsymbol{K}]_{i,i}^{-1}$, no sum on i. In this scaling procedure the diagonal terms of the stiffness matrix are used as scaling parameters.

Fafard and Massicotte (1993) modified the method by making use of the advantages of arc-length method given by Crisfield (1981) and Ramm (1981). Fafard and Massicotte in their work used the constraint equation given by Ramm (1981), and used updated hyper-plane technique to evaluate arc-length as follows

$$\Delta\lambda = \frac{\{\Delta \boldsymbol{p}_i\}^{\mathrm{T}} \{\boldsymbol{K}^{-1} \boldsymbol{g}\}}{\{\Delta \boldsymbol{p}_i\}^{\mathrm{T}} \{\boldsymbol{K}^{-1} \boldsymbol{q}\}} \tag{17}$$

and the current displacement was computed as given in Eq.(6). This displacement does not fall on the defined hypersphere; so to bring it on the hypersphere, a reduction in the computed displacement is suggested, as

$$\|\Delta \boldsymbol{p}_{i+1}\| = \Delta l = \alpha \|\Delta \boldsymbol{p}_{i+1}\| \tag{18}$$

from which

$$\alpha = \frac{\Delta l}{\|\Delta \boldsymbol{p}_{i+1}\|} \tag{19}$$

Hence the desired incremental displacement can be expressed as

$$\{\Delta \boldsymbol{p}_{i+1}\} = \alpha \{\Delta \boldsymbol{p}_i\} + \alpha \{\delta_i\} \tag{20}$$

Since the technique combines the advantages of two methods, it can be effectively used for load-deflection tracing in nonlinear analysis and it can also overcome both the individual drawbacks of the other methods and the convergence problem; however near failure point convergence can cause problem, which can be avoided by reducing the Δl value. In addition Fafard and Massicotte (1993) gave the geometrical representation of arc-length methods introduced by Crisfield and Ramm, which helps to visualize the method geometrically.

Carrera (1994) gave a modification for selection of proper root of the nonlinear constraint equation. The author argues that although for some problems arc-length method works well; yet it fails in some cases, and mostly it is due to the selection of the proper root of the governing nonlinear constraint equation. Thus Carrera proposed selection of root that is closest to constraint linear solution. The method requires computation of the linear solution, which is obtained from available data. This simplifies the matter, which otherwise, as in the case of Crisfield's method, requires additional computation of angle before selection of the proper root.

Fan (1994) researched variable step-length incremental/iterative methods; reviewed a few existing methods; and proposed three different ways to compute the arc-length, i.e., zero incremental displacement norm, zero residual force norm and zero incremental work norm, to be used with cylindrical arc-length and load/displacement control methods. The validity of the proposed strategy was verified by solving two cylindrical shell problems with snap-back. Observation of the results obtained revealed the validity of the derived methods; but there was no indication of their validity in three dimensional space problems.

Convergence to predefined deformation state is important in some situations. For example, in the assessment of dynamic characteristics of a structure, convergence at first yield is important. The arc-length method does not include the formulation to achieve this type of solution. Teng and Luo (1998) introduced modification to the existing arc-length method based on the concept of accumulated arc-length. Accumulated arc-length is defined as the sum of all arc-lengths up to and including the current load step. According to the procedure, first, conventional arc-length method is used to trace the load-displacement path. When the path approaches predefined state, the accumulated arc-length process is started. To achieve the desired level of convergence, the arc-length must be modified. This modification is done by introducing a new parameter γ

$$\gamma(L_i) = \lambda_i - \lambda_{\mathrm{d}} \tag{21}$$

where λ_i is the converged load-factor, λ_{d} is the desired load level and L_i is the sum of arc-lengths up to the current increment. Then the desired arc-length increment for the next loading step l_{d} is computed by making the accumulated arc-length L_{d} satisfy the following equation.

$$\gamma(L_{\mathrm{d}}) = 0 \tag{22}$$

hence

$$L_{i+1} = L_i + l_{\mathrm{d}} \tag{23}$$

To solve the above equation, Taylor expansion is used, and two possibilities are suggested, i.e. quadratic and linear expansion. If the new arc-length obtained is greater than the one obtained from normal procedure then the arc-length from normal procedure is used and subsequent evaluation of arc-length according to accumulated arc-length procedure is done in the next step. In addition, alternative quadratic approach is also suggested which even eliminates the issue of selection of appropriate root (here accumulated arc-length is defined in terms of load-factor); then Taylor expansion is used to solve the resulting equation for desired arc-length. Although the procedure given is used to solve numerical problems and reported results are satisfactory, the issue of imaginary roots is not touched in this research work.

In arc-length method the issue of complex roots arising from solution of quadratic equation is a factor that leads to divergence of the solution. Several attempts had been made to modify the method to overcome this problem; however the issue is still alive in certain situations, and efforts to overcome the problem continue. In an attempt to resolve this issue, Zhou and Murray (1994) introduced a relaxation factor β to reflect the contribution of residual force vector to displacement. The range of β is from 0 to 1. The method includes traditional arc-length method as a subset when β=1. The concept of the technique is to apply fraction of residual force in iterative process, which increases the computational cost of the process, but the probability of solution failure can be diminished. Proper value of factor β always ensures positive discriminant of quadratic equation, hence the real roots. Acceptable range of value for β can be determined by setting the discriminant of the quadratic equation greater than or equal to zero, which leads to quadratic equation in terms of β. Solution of the new equation leads to β values, from which proper value is selected as

$$0 < \beta \le \beta_{\max} \quad (24)$$

and

$$\beta_{\max} = \min(1, \beta_2) \quad (25)$$

Although rare, but near zero, $\beta_{\max}$ values can reduce the efficiency considerably. To overcome this situation research work suggested the use of full Newton-Raphson method when $\beta_{\max}$ is less than 0.001. This modified technique has been successfully applied to the post buckling analysis of buried pipe segments.

Lam and Morley (1992) proposed a new strategy to deal with the complex roots arising from the solution of the quadratic equation. Authors proposed the resolution of out-of-balance load vector into parallel and orthogonal components with respect to fixed nodal load vector in case of divergence. To achieve this they introduced a scaling parameter η in the usual constraint equation. The procedure becomes general arc-length procedure when η equals 1. The value of parameter η is initially set to 1; with the consequent divergence of solution process due to complex roots being computed by using simple line-search technique. The line-search leads to a quadratic equation in terms of η. Suitable value of this scaling parameter are chosen close to 1. This newly computed value of η is then fed back into original constraint equation which is solved again. This way the problem of complex roots can be avoided successfully. The authors further report that generally the quadratic equation solved for η gives real values, but in case it lead to complex values, the following procedure is adopted:

i) Compute the displacement due to the orthogonal component of out-of-balance load vector (orthogonal to applied load vector) and update the displacement.

ii) Compute associated nodal forces.

iii) Compute incremental loading parameter and the arc-length from orthogonal component of out-of-balance force vector.

iv) Reduce the loading parameter by multiplying the loading parameter by the ratio of original arc-length to arc-length associated with complex roots, and iterate the solution process.

Although this solution procedure imposes extra computational cost, it can counteract the problem of complex roots successfully.

To improve the workability of the arc-length procedure, particularly to avoid the track back of solution scheme due to incorrect loading parameter, Feng *et al.*(1996) suggested a formulation for computation of initial load increment, based on the technique of sign of determinant of current stiffness matrix. The technique and its use are reported in literature on finite element analysis. However, when used with iterative techniques for solution of equations in combination with incremental-iterative process, it does not work since the determinant of the stiffness matrix cannot be obtained as a by-product of the solver. Thus modification for use with iterative solvers is needed. The modification is done by using the derivative of the load parameter at current iteration, i.e.

$$\text{sign}(\Delta\lambda_k^0) = \text{sign}(\dot{\lambda}_k) \quad (26)$$

Where $\dot{\lambda}$ is the derivative of the load-factor. The formulation for both λ values in Eq.(26) is given in a relevant paper and is not repeated here. The numerical results reported present the validity of the technique. The validity of the technique was also reported by Bellini and Chulya (1987).

May and Duan (1997) worked with arc-length method for structures with strain softening, and argued that since failure of materials with strain softening is highly localized, global constraint equations are unable to produce convergence. They argued that inclusion of displacement of dominant nodes (nodes which are associated with failure zone) into the constraint equation leads to betterment. However these nodes can cause problem as any node might change its position with development of nonlinearity that leads to troublesome constraint equation. Thus they modified the existing arc-length method to address these problems and introduced local arc-length procedure. In its constraint equation, displacement associated with the regions of nonlinearity was used instead of total displacement vector. To automate the selection of dominant nodes in analysis research work proposes the use of dominant element in which some or all of the material is either in damaged zone or in failure zone. Thus the relative displacement vector for a plane element with n degrees of freedom is given as

$$\Delta\boldsymbol{\delta}=[\delta_1-\delta_n,\delta_2-\delta_1,\delta_3-\delta_2,...,\delta_n-\delta_{n-1}]^{\mathrm{T}} \quad (27)$$

where δ_1 to δ_n are the displacements at nodes within the element. In adjacent dominant elements relative displacement of common nodes is computed twice, since each element has its own relative displacement vector. The equations for arc-length and load-factor are then computed as

$$\sum_{e=1}^{m}\left(\Delta\delta_1\right)_e^{\mathrm{T}}\left(\Delta\delta_i\right)_e=\Delta l^2 \quad (28)$$

$$\Delta\lambda_i=\Delta\lambda_1-\frac{\sum_{e=1}^{m}\left(\Delta\boldsymbol{U}_{\mathrm{P}}\right)_e^{\mathrm{T}}\left(\Delta\boldsymbol{\delta}_{i-1}+\Delta\boldsymbol{U}_{\mathrm{F}}\right)_e}{\sqrt{\sum_{e=1}^{m}\left(\Delta\boldsymbol{U}_{\mathrm{P}}\right)_e^{\mathrm{T}}\left(\Delta\boldsymbol{U}_{\mathrm{P}}\right)_e}} \quad (29)$$

in which

$$\Delta\lambda_1=\frac{\Delta l}{\sqrt{\sum_{e=1}^{m}\left(\Delta\boldsymbol{U}_{\mathrm{P}}\right)_e^{\mathrm{T}}\left(\Delta\boldsymbol{U}_{\mathrm{P}}\right)_e}} \quad (30)$$

where $\Delta\boldsymbol{U}_{\mathrm{P}}$ is the relative reference deformation in the element and $\Delta\boldsymbol{U}_{\mathrm{F}}$ is the relative incremental deformation due to unbalanced force. The technique mentioned successfully predicts the load-displacement behavior of materials with strain softening as mentioned in reported results. However, the reported method was applied to plain concrete element and there was no mention of the use of the method for compression softening, which is frequent and dominant in concrete structures. Hellweg and Crisfield (1998) also attempted to solve a double cantilever beam with softening type damage model and reported divergence of the solution even if proper root for cylindrical arc-length method was selected. The authors argue about the situation when divergence is due to the sharp snap-back, in which neither root can lead the solution on the proper path. Therefore they suggested that the selection of proper root be based on the minimum residual norm. However to preserve computer resources the idea is applied only when sharp snap-back is observed, otherwise the usual method is continued.

Regarding the issue of selection of root when divergence occurs, Kweon and Hong (1994) discarded every existing possibility of selection of roots and suggested restarting of the corresponding load step with arc-length reduced to half. To prevent the number of iterations being too large the authors also suggested that the maximum number of iterations be fixed. If the number of iterations exceeds the maximum number, then the load step is restarted with arc-length reduced to half. Generally the direction of load step is evaluated according to the sign of the determinant of stiffness matrix, but Kweon and Hong (1994) suggested that the load direction must be independent of the sign of the determinant of the stiffness matrix; and imposed the condition that the first incremental displacement vector in the present load step should make an acute angle with the total incremental displacement vector at the last step. They argued that the criterion leads to better results, and that numerical problems for post buckling analysis do not occur. Although the reported results of post buckling analysis of isotropic shell under compression showed the validity of the reported procedure, the method will probably lead to slower convergence as the arc-length is reduced to half of previous or no convergence at all for problems with critical points.

While performing geometrical nonlinear post-buckling analysis of truss members Kuo and Yang (1995) found that most existing arc-length control methods failed to predict load-displacement response, and argued that it is because iterations were not performed in proper direction for problems with multi-winding loops. To overcome this problem, Kuo and Yang introduced two control parameters for detecting change in direction and for guiding the direction of iteration. It was accomplished by obtaining the dot product of two adjacent tangent vectors; and once negative value is obtained, the direction of loading should be reversed. The technique has advantage in that negative value will only be obtained when the control passes a limit point;

however for large variation in curvature it is possible that the above parameter may not yield negative value. In such cases a vector directed along the secant line of N-dimensional curve is defined. Then the dot product of two adjacent tangent vectors along this newly defined secant vector will give the correct sign. The validation of the proposed technique is given by solving numerical examples.

APPLICATION IN NONLINEAR ANALYSIS OF REINFORCED CONCRETE STRUCTURES

Most nonlinear analyses of reinforced concrete structures use Newton-Raphson (NR) or modified Newton-Raphson (m-NR) method to trace the equilibrium path. Modified Newton-Raphson method leads to some simplification, but convergence of the solution process is slow. Both of these methods work well with linear or bilinear material relationship, but become inefficient with higher degree of nonlinearity, i.e., cracking, bond-slip, material nonlinear behavior, time dependent effects, etc., are introduced. Furthermore a disadvantage of these methods is that without some special technique the descending branch of the load-displacement path cannot be traced. An early method of dealing with limit points was to use the displacement-control of Baltoz and Dhatt (1979). The method has been applied successfully in many situations; however the method is not suitable for problems with many degrees of freedom, which is quite common in case of nonlinear analysis of reinforced concrete structures. A further problem when trying to apply the method to highly material nonlinear finite elements, such as concrete in general and reinforced concrete in particular, is that the initial solution which makes the basis of further iterations, may be well away from the final equilibrium state, and thus can lead to numerical difficulties and/or divergence of solution process. In addition strain-softening is one of the major effects in reinforced concrete analysis, which must be included to produce a realistic response. The effect can lead to localized failure of material and often exhibit snap-back behavior in load-deformation space. In such cases, traditional methods fail to trace the complete load-deflection response of reinforced concrete structures. Therefore the arc-length method is used in combination with Newton-Raphson method to trace the complete response in load-deformation space. The method has been successfully applied by several researchers for the analysis of reinforced concrete structures, who reported good agreement between experimental and analyzed results. In the following table, a few examples are summarized to show the validity of the method. Details of problems included are given in relevant references.

Table 1 summarizes five examples, among which examples 1 to 3 were analyzed by Foster (1992) and examples 4 and 5 were analyzed by Crisfield (1983). All the examples included were compared with the modified Newton-Raphson method to demonstrate the validity and effectiveness of the arc-length method. For all five examples mentioned, the modified Newton-Raphson method almost failed to converge at the early stages. In addition Table 1 also shows the faster convergence rate of the solution process when the arc-length method is combined with line searches.

Lam and Morley (1992) used their modified arc-

Table 1 Comparison of arc-length method and modified Newton-Raphson method

No.	Application	Total load steps	Total iterations		
			Modified Newton-Raphson method	Arc-length method	Arc-length method with line search
1	Single span RC deep beam	13	682	341	150
2	Two span RC deep beam	13	420	120	111
3	Shear panel	17	-	117	108
4	Jain Kennedy slab	14	-	246	86
5	McNeice slab	16	-	244	98

length method to analyze doubly reinforced concrete beam with fixed and sliding support and a cantilever beam with nearly uniform bending moment. For both of the examples, strain-softening constitutive relation was incorporated in the analysis which produced singular points on the load-deflection path and strain localization. Due to the use of the arc-length method, it was possible to trace the complete load-deflection path of the examples which otherwise was not possible with traditional methods.

CONCLUSION

The arc-length method has become a powerful tool to use with finite element formulation for complete analysis of the load-deflection path. After introduction, the method got high praise of researchers and many research papers on it have been published. This paper reviews some of the modifications and provides in-depth insight of the method. In addition to papers on the modification of the existing method, a good number of papers have also been published for use of the method in finite element analysis. It can be seen from the review that the method is performing quite well, but to the best knowledge of the authors, the validity of the method has not reported for three-dimensional solid modeling of reinforced concrete structures in general or for strain softening in compression in particular.

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