Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

# A general version of the Morse-Sard theorem*

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Received Sept. 30, 2003; revision accepted Jan. 8, 2004

**Abstract:** Let $k$, $m$, $n$ be positive integers, and $k\geq 2$, $\alpha\in(0,1]$, $0<r<\min\{m,n\}$ an integer, $d=r+(m-r)/(k+\alpha)$, and if $f\in C^{k,\alpha}(\mathbb{R}^m, \mathbb{R}^n)$, $A=C_r(f)=\{x\in\mathbb{R}^m \mid \mathrm{rank}(Df(x))\leq r\}$, then $f(A)$ is $d$-null. Thus the statement posed by Arthur Sard in 1965 can be completely solved when $k\geq 2$.

**Key words:** Hausdorff measure, Rectifiable, Morse decomposition

**Document code:** A **CLC number:** O174.12

INTRODUCTION

The Morse-Sard theorem is a fundamental theorem in analysis, especially in the basis of transversality theory and differential topology. The classical Morse-Sard theorem states that the image of the set of critical points of a function $f$: $\mathbb{R}^m\rightarrow\mathbb{R}^n$ of class $C^{m-n+1}$ has zero Lebesgue measure in $\mathbb{R}^n$. It was proved by Morse (1939) in the case $n=1$ and by Sard (1942) in general case. So it is called.

Due to its theoretical importance, the Morse-Sard theorem was generalized in many directions. Many of these generalizations are related to Hausdorff measures and Hausdorff dimensions.

Let $A$ be a non-empty bounded subset of $\mathbb{R}^m$, and $0\leq d\leq m$, for each $\delta>0$ let

$$H_\delta^d(A)=\inf\{\sum_i(\mathrm{diam}(U_i))^d : A \text{ is covered by sets } U_i \text{ with } 0<\mathrm{diam}(U_i)\leq\delta\},$$

where the infimum is over all coverings $\{U_i\}$ of $A$ by a (finite or countable) collection of sets with diameters at most $\delta$. We may defined

$$H^d(A)=\lim_{\delta\rightarrow 0}H_\delta^d(A).$$

We call $H^d(A)$ the $d$-dimensional Hausdorff measure of $A$.

It is easy to see that there is a number $d$ at which $H^d(A)$ jumps from $\infty$ to 0; we call this number $d$ the Hausdorff (or Hausdorff-Besicovitch) dimension of $A$ which we denote by $\dim_H(A)$. Thus

$$\dim_H(A)=\sup\{d:H^d(A)=\infty\}=\inf\{d:H^d(A)=0\}.$$

Let $t>0$ be a real number, we recall that a subset $A\subset\mathbb{R}^m$ is $t$-finite if $H^t(A)<\infty$, that $A\subset\mathbb{R}^m$ is $t$-null if $H^t(A)=0$, and that $A\subset\mathbb{R}^m$ is $t$-sigma-finite if $A$ is the countable union of $t$-finite sets. Obviously, every subset of $\mathbb{R}^m$ is $m$-sigma-finite.

Sard himself proved that if

$$C_r(f)=\{x\in\mathbb{R}^m \mid \mathrm{rank}(Df(x))\leq r\},$$

*Project supported by the National Natural Science Foundation of China (No. 10171090) and the Scientific Research Fund of Zhejiang Provincial Education Department (No. 20030341), China

then for any $\varepsilon>0$ there is $k\in N$ such that if $f$ is $C^k$ then $f(C_r(f))$ has zero Hausdorff measure of dimension $r+\varepsilon$ (Sard, 1965). This result was made more precise by Federer (Federer, 1969), who proved that if $k\in N$ then the Hausdorff measure of dimension $r+(m-r)/k$ of $f(C_r(f))$ is zero. Later, Yomdin (1983) proved that the Hausdorff dimension of $f(C_r(f))$ is at most $r+(m-r)/(k+\alpha)$, provided that $f\in C^{k,\alpha}$, where $k\in N$ and $\alpha\in[0,1]$. Bates (1993) improved the hypothesis of the classical Morse-Sard theorem from $f\in C^{m-n+1}$ to $f\in C^{m-n,1}$. More recently, Norton (1994) improved the hypothesis of the classical Morse-Sard theorem from $f\in C^{m-n,1}$ to $f\in C^{m-n,z}$.

For $m$, $n$, $k\in N$, and $\alpha\in[0,1]$, $f\in C^{k,\alpha}$ and $A$ has rank $r$ for $f$ can be defined as follows:

**Definition 1** A function $f$: $\mathbb{R}^m\to\mathbb{R}^n$ is of class $C^{k,\alpha}$ if $f$ is $C^k$ on $\mathbb{R}^m$ and $k$th derivative $D^kf$ satisfies the $\alpha$-Hölder condition: for every compact subset $U$ of $\mathbb{R}^m$ there is an $M>0$ such that

$$\left\|D^k f(x)-D^k f(y)\right\|\le M\left\|x-y\right\|^{\alpha},\ \text{ for all } x,y\in U.$$

**Definition 2** $A$ is a set of rank $r$ for $f$: $\mathbb{R}^m\to\mathbb{R}^n$ if the rank of $Df(x)$ is at most $r$ for every $x\in A$, i.e., $A\subset C_r(f)$, where $r$ is a non-negative integer and

$$C_r(f):=\{x\in\mathbb{R}^m\ |\text{rank}(Df(x))\le r\}.$$

For $\alpha\in(0,1)$, we denote $\Lambda_\alpha$ the Lipschitz space of those continuous functions $f$: $\mathbb{R}^m\to\mathbb{R}^n$ for which

$$\|f\|_{\alpha}=\sup_{x\in\mathbb{R}^m}|f(x)|+\sup_{x,h\in\mathbb{R}^m,|h|>0}\frac{|f(x+h)-f(x)|}{|h|^{\alpha}}<\infty.$$

A function $f$: $\mathbb{R}^m\to\mathbb{R}^n$ belongs to $\Lambda_1$ if $f$ is continuous and

$$\|f\|_{1}=\sup_{x\in\mathbb{R}^m}|f(x)|+\sup_{x,h\in\mathbb{R}^m,|h|>0}\frac{|f(x+h)-f(x)|}{|h|^{1}}<\infty.$$

For $s=k+\alpha>1$, the class $\Lambda_s$ is then defined inductively as the space of $C^k$ functions $f$: $\mathbb{R}^m\to\mathbb{R}^n$ for which

$$\|f\|_{s}=\|f\|_{s-1}+\|Df\|_{s-1}<\infty.$$

When $s>1$ is not an integer, each element of the Hölder class $C^s$ coincides locally with some member of $\Lambda_s$. For $k\in N$, the Lipschitz space $\Lambda_k$ strictly contains $C^k$. A function $f\in C^k$ coincides locally with some member of $\Lambda_k$ provided that the $(k-1)$-derivative of $f$ satisfies the local Zygmund condition. For details, see Stein (1970).

For $t\in N$, a set $A\subset\mathbb{R}^m$ is called $(H^t,t)$-rectifiable if $H^t(A)<\infty$ and $H^t$-almost all of $A$ is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^t$ to $\mathbb{R}^m$. These sets are the generalized surfaces of geometric measure theory. They include countable unions of immersed manifolds (as long as the total area stays finite) and arbitrary subset of $\mathbb{R}^t$. For details, see Federer (1969).

Without confusion, for $A\subset\mathbb{R}^m$, we denote diam($A$) by $|A|$.

The aim of this note is to prove the statement posed by Sard (1965). It is a corollary of the following conjecture.

**Conjecture 1** Let $k$, $m$, $n$ be positive integers, and $k\ge2$, $\alpha\in(0,1]$, $0<r<\min\{m,n\}$ an integer, $d=r+(m-r)/(k+\alpha)$, and if $f\in C^{k,\alpha}(\mathbb{R}^m,\mathbb{R}^n)$, $A=C_r(f)=\{x|\text{rank}(Df(x))\le r\}$ then $f(A)$ is $d$-null.

This conjecture is a corollary of our main result. And our main result is the following theorem.

**Theorem 1** Fix $t$, $k$, $m$, $n\in N$ and $r\in N\cup\{0\}$ with $t>r$, $\alpha\in(0,1]$, $k\ge2$, set $d=r+(t-r)/(k+\alpha)$. Let $A\subset\mathbb{R}^m$ be a $t$-sigma-finite set of rank $r$ for a $C^1$ differentiable map $f$: $\mathbb{R}^m\to\mathbb{R}^n$. If $f\in\Lambda_{k+\alpha}$ and $A$ is $t$-rectifiable, then $f(A)$ is $d$-null.

Note that if $k+\alpha>1$ and $t>r$, then $t<r+(t-r)/(k+\alpha)$, and the conclusion of the theorem is trivial. If $t=r$, then the statement of the theorem is false, for example, consider the projection $\pi:\mathbb{R}^m\to\mathbb{R}^n$ defined by

$$\pi(x_1,\ldots,x_r,\ldots,x_m)=(x_1,\ldots,x_r,0,\ldots,0).$$

It is to see $A=\mathbb{R}^m$ is a set of rank $r$ for $\pi$, and $\pi$ is $C^\infty$, but $f(A)=\mathbb{R}^r$ is not $r$-null.

DECOMPOSITIONS OF CRITICAL SETS

**Lemma 1** Fix $\eta>0$. Let $A\subset\mathbb{R}^m$ be a set of rank $r$ for a $C^1$ differentiable map $f:\mathbb{R}^m\to\mathbb{R}^n$. If $s>2$, $f\in\Lambda_s$, and $A$ is $t$-rectifiable for some $t\in N$, then there exists a decomposition $\{A_2\}$ of $A$ such that $A_0$ is $t$-null, $\sum_i |A_i|^t \le H^t(A)+1$, and for each $i\in N$, there is a constant $M_i>0$ and a splitting of $\mathbb{R}^m$ such that for any set $S\subset\mathbb{R}^m$,

$$|f(S\cap A_i)|\le(\|f\|_1+\eta)|S'|+M_i|S'||A_i|+\eta|A_i|^s .$$

where the splitting of $E=\mathbb{R}^m$ is $E=E'\oplus E''$ with $\dim(E')\le r$, and $S'$ is the orthogonal projection of $S$ in the space $E'$.

To prove this lemma, we will need the following generalization of the Morse decomposition lemma.

**Lemma 2** Let $h:\mathbb{R}^m\to\mathbb{R}^n$ be a $C^1$ and $\Lambda_s$ map with $s>1$. If $h\equiv0$ on $A\subset\mathbb{R}^m$, then for each $\eta>0$ there exists a decomposition $\{A_i\}$ of $A$ such that $A_0$ is countable, and for each $i\in N$, there is a closed ball $B_i\subset\mathbb{R}^{ni}$ and a $C^1$ map $\psi_i:B_i\to\mathbb{R}^m$ such that

1. $A_i\subset\psi_i(B_i)$ and $|x-y|\le|\psi_i(x)-\psi_i(y)|\le(1+\eta)|x-y|$ for all $x, y\in B_i$;

2. For all $x, x'\in\psi_i^{-1}(A_i)$ and $y$ in the line segment $xx'$, $|h(\psi_i(y))-h(\psi_i(x))|\le M_i|x-y|^s$, for some constant $M_i>0$.

To know the proof of this lemma, see Norton (1994) or Jiang and Xi (2000).

**Proof of Lemma 1** Let $E=\mathbb{R}^m$, we first prove the lemma assuming that with respect to some splitting $E=E'\oplus E''$, the map $f$ is of the form

$$f(z,y)=(h(z), g(z,y))$$

for some $h$, $g$ such that $D_2g\equiv0$ on $A$. Since $A$ is $t$-rectifiable, combining Lemma 2 with the definition of rectifiability, we obtain a countable decomposition $\{A_i\}$ of $A$ such that $A_0$ is $t$-null, and for each $i\in N$, there exists a ball $B_i\subset\mathbb{R}^t$ and a $C^1$ map $\psi_i:B_i\to E$ such that

1. $A_i\subset\psi_i(B_i)$ and each point of $\psi_i^{-1}(A_i)$ is a Lebesgue density point;

2. $|x-y|\le|\psi_i(x)-\psi_i(y)|\le2|x-y|$ for all $x, y\in B_i$;

3. For all $x, x'\in\psi_i^{-1}(A_i)$ and $y\in xx'$, $|D_2g(\psi_i(y))|\le M_i|x-y|^{s-1}$.

To obtain the desired decomposition of $A$, we will further decompose each $A_i$.

Fix $i\in N$, suppose that $(z,y)$, $(z',y')\in S\cap A_i$. Then

$$|f(z,y)-f(z',y')|\le|f(z,y)-f(z,y')|+|f(z',y)-f(z',y')|$$
$$\le\|f\|_1|S'|+|g(z',y)-g(z',y')|.$$

Given $\eta>0$, since $s>2$, we can fix a positive integer $P$ such that $2^sM_iP^{2-s}<\eta$. Since each $x\in\psi_i^{-1}(A_i)$ is a density point, there exists $\varepsilon_x>0$ such that if $Q(x,\varepsilon)$ is a cube of edge $\varepsilon<\varepsilon_x$ centered at $x\in\psi_i^{-1}(A_i)$, then

$$\frac{H^t(Q\cap\psi_i^{-1}(A_i))}{H^t(Q)}\ge1-P^{-t}.\cdots\cdots(*)$$

The collection $\Im=\{Q(x,\varepsilon):x\in\psi_i^{-1}(A_i),\varepsilon<\varepsilon_x\}$ is a Vitali family for $\psi_i^{-1}(A_i)$, and so there exists a sequence of pairwise disjoint cubes $\{Q_j\}\subset\Im$ such that $\psi_i^{-1}(A_i)\backslash\cup_j(x_j, \varepsilon_j)$ is $t$-null and

$$\sum_j\varepsilon_j^t<H^t(\psi_i^{-1}(A_i))+1 .$$

Fix $j$, consider $(z,y)$, $(z',y')\in Q_j\cap\psi_i^{-1}(A_i)$. By Condition (*), it follows that there are $x_0,\ldots,x_p$ in $Q_j\cap\psi_i^{-1}(A_i)$ such that $x=x_0$, $y=x_p$, and $|x_l-x_{l+1}|<2|Q_j|\cdot P^{-1}$ for all $l$. Let $\gamma:[0, 1]\to E$ be the composition of $\psi_i$ with the piecewise linear path connecting the points $x_l$. If $\tilde{\gamma}:[0,1]\to E$ is the composition of $\gamma$ with the projection onto $\{z'\}\times E''$, then

$$|g(z,y)-g(z',y')|=|\int_0^1 D_2g(\tilde{\gamma}(t))\tilde{\gamma}'(t)\mathrm{d}t|$$
$$\le\int_0^1|D_2g(\tilde{\gamma}(t))-D_2g(\gamma(t))||\tilde{\gamma}'(t)|\mathrm{d}t$$
$$+\int_0^1|D_2g(\gamma(t))||\tilde{\gamma}'(t)|\mathrm{d}t.$$

Since $|\tilde{\gamma}'(t)|\le2|Q_j|$ and $|\tilde{\gamma}(t)|\le2|Q_j|$ for all $t$, we have

$$|D_2g(\tilde{\gamma}(t))-D_2g(\gamma(t))||\tilde{\gamma}'(t)|\le 2\|f\|_2|S'||Q_j|.$$

Finally, property 1 of the map $\psi_i$ in Proof of Lemma 1 implies that for each $l$, we have $|x_l-x_{l+1}|\le 2|Q_j|/P$, and so from property 3, it follows that

$$\int_0^1|D_2g(\gamma(t))||\tilde{\gamma}'(t)|\,\mathrm{d}t\le 2|Q_j|M_i\sum_{l=0}^{P-1}|x_l-x_{l+1}|^{s-1}$$
$$\le 2^sM_iP^{2-s}|Q_j|^s.$$

Thus, proceeding as above, and applying the inequality $2^sM_iP^{2-s}<\eta$, we obtain

$$|f(S\cap\psi_i(Q_j))|\le\|f\|_1|S'|$$
$$+M_i|S'||Q_j|+\eta|Q_j|^s.$$

For each $i\in N$, we now choose a collection of cubes $Q_j$ associated to $A_i$ as above. The decomposition $\{A_i\}$ is given by a denumeration of the sets $\psi_i(Q_j)\cap A$, and $A_0=A\backslash A_i$ then satisfies the lemma.

For the general case, we apply the implicit function theorem locally to find a diffeomorphism $\varphi$: $E=E'\oplus E''\to E$ such that $f\circ\varphi$ is of the split form above.

Note that if we compose the map $f$ in Lemma 2 with a $C^1$ diffeomorphism $\varphi$: $E\to E$, then we obtain the estimate

$$|(f\circ\varphi)(x)-(f\circ\varphi)(y)|\le(\|f\|_s+\eta)\|\varphi\|_1^s|x-y|^s.$$

In other words, we can compose Morse decompositions with a $C^1$ diffeomorphism $\varphi$ and only increase the constants by the $C^1$norm of $\varphi$.

## PROOF OF THE MAIN THEOREM

Fix $t$, $s>1$, $r\in N$. We consider a $t$-finite set $A\subset\mathbb{R}^m$ of rank-$r$ points for a $C^1$ map $f$:$\mathbb{R}^m\to\mathbb{R}^n$. Since $Df$ is continuous, the function $x\to\mathrm{rank}(Df(x))$ is lower semi-continuous on $\mathbb{R}^m$, and so its level sets are Borel sets. Consequently, there is a Borel set $A'$ of rank-$r$ points of $f$ which contains $A$ and satisfies $H^t(A^t)=H^t(A)$.

For $\mu$, $\eta>0$, we suppose for the moment that there is a decomposition $\{A_i\}$ of $A'$ such that for each $i\in N$ and $\delta\in(0, M_i^{-1})$, choose a covering $\{S_j\}$of $A_i$ such that $|S_j|=\varepsilon_j<\delta$ for each $j$ and

$$\sum_j\varepsilon_j^t\le H^t(A_i)+\delta.$$

For each $j$, let $S_j'$, $S_j''$ denote the orthogonal projections of $S_j$ in the subspaces $E_i'$ and $E_i''$, respectively. Obviously, $|S_j'|\le|S_j|$, and there exists an $r$-cube $Q_j'\subset E_i'$ of diameter $\le(1+r)^{1/2}\varepsilon_j$ which contains $S_j'$. The product $Q_j'\times S_j''$ then contains $S_j$ and has diameter $\le(1+r)^{1/2}\varepsilon_j$. Given $\mu>0$, we partition $Q_j'$ into $([\mu^{-1}\varepsilon_j^{1-s}]+1)^r$ sub-cubes of equal edge$\le\mu\varepsilon_j^s$. For every such sub-cube $\tilde{Q}$, we have $|\tilde{Q}|\le r^{1/2}\mu\varepsilon_j^s$, so that

$$|f(A_i\cap(\tilde{Q}\times S_j''))|\le(P(\eta,\mu)+M_i\delta)\varepsilon_j^s$$

for some $p(\eta,\mu)$.

For $d=r+(t-r)/s$ and $P=P(\eta, \mu)$, we then have

$$H^d(f(A_i))\le\sum_j([\mu^{-1}\varepsilon_j^{1-s}]+1)^r((P+M_i\delta)\varepsilon_j^s)^d$$
$$\le 2^r(P+M_i\delta)^d\mu^{-r}(H^t(A_i)+\delta),$$

so that letting $\delta$ tending to 0 gives

$$H^d(f(A_i))\le 2^rP^d\mu^{-r}H^t(A_i).$$

Since the $A_i$'s are disjoint Borel sets, we have

$$H^d(f(A))\le\sum_{i=1}^{\infty}H^d(f(A_i))$$
$$\le 2^rP^d\mu^{-r}H^t(A).\cdots\cdots(**)$$

**Proof of Theorem 1** If $s=k+\alpha>2$, Lemma 1 asserts if $A$ is $t$-rectifiable, then $A\backslash A_0$ admits a decomposition of the form above, the inequality (**) above holds for all $\eta$, $\mu>0$ with

$$P = P(\eta, \mu) = (\|f\|_1 + \eta + 1)\mu.$$

Since $A_0$ is $t$-null, so $f(A_0)$ is $d$-null by the result in Jiang and Xi (2000). Then the inequality (**) is valid for all $\eta$, $\mu$>0. Since $d>r$, tending $\eta$, $\mu$ to 0 in the inequality (**) gives $H^d(f(A))=0$.

ACKNOWLEDGEMENTS

The author thanks Prof. Wen Zhiying for his helpful comments. The work was done during the author's post-doctor period in Zhejiang University. He also thanks Prof. Yin Yongcheng for some corrections and language comments.

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