Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

Extreme points of norm closed nest algebra modules*

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Received Sept. 10, 2003; revision accepted Dec. 31, 2003

Abstract: Suppose that $\mathcal{U}$ is a norm closed nest algebra module. Using the characterization of rank one operators in $\mathcal{U}_{\perp}$, a complete description of the extreme points of the unit ball $\mathcal{U}_1$ is given.

Key words: Extreme point, Nest algebra module, Rank one operator
Document code: A **CLC number:** O177

INTRODUCTION

During the last decade there has been an interest in the Banach space geometry of non-self adjoint operator algebras. Subjects of investigation there include the extreme point structure of the unit ball (Hudson *et al.*, 1997; Moore and Trent, 1987) and the isometries between such algebras (Arazy and Solel, 1990; Kadison, 1951; Moore and Trent, 1989). One possible application of a characterization of extreme points is in the study of isometries. Kadison (1951) used the characterization of extreme points in his study of isometries on C^*-algebras. For the same end, Moore and Trent (1987) investigated the extreme points for nest algebras. Anoussis and Katsoulis (1996a) proved the surprising fact that for any nest with no finite atoms, the convex hull of the unitary elements which are, of course, extreme points, would suffice to recover the whole ball of the corresponding nest algebra. This result was refined to reach its final form by Davidson (1998). But there is little study on the extreme points of the unit ball of ideals, modules, etc. In our earlier work (Dong and Lu, 2002), we obtained a complete description of the extreme point structure of weakly closed nest algebra modules. In that paper, we made full use of the concrete description of weakly closed nest algebra modules given by Erdos and Power (1982), that is, each weakly closed nest algebra module is determined by a left order continuous order homomorphism ϕ from the nest $\mathcal{N}$ into itself and is of the following form:

$$\mathcal{U}_{\phi}=\{X\in\mathcal{B}(\mathcal{H}):XN\subseteq\phi(N),\ \forall N\in\mathcal{N}\}.$$

However, many nest algebra modules are not closed in the weak operator topology, such as Jacobson ideals, Larson ideals, etc. In this work, our focus is on norm closed nest algebra modules. Since we have no concrete description of norm closed nest algebra modules, the method in our earlier paper (Dong and Lu, 2002) could not work directly. As nest algebra modules are not algebras in general, we also could not apply the techniques in Theorem 6, the main result of Moore and Trent (1987). The techniques of annihilator will be used to prove the

*Project (No. M103044) supported by the Natural Science Foundation of Zhejiang Province, China

main result in this paper.

Let us introduce some notation and terminology. Throughout this paper, $\mathcal{H}$ represents a complex separable infinite-dimensional Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ the ideal of all compact operators on $\mathcal{H}$, and $\mathcal{F}(\mathcal{H})$ the set of finite rank operators on $\mathcal{H}$. A nest $\mathcal{N}$ is a chain of closed subspaces of Hilbert space $\mathcal{H}$ containing (0) and $\mathcal{H}$ which is complete in the sense that it is closed under arbitrary intersections and arbitrary closed spans. If N is an element of a nest $\mathcal{N}$, then $N\_=\vee\{L\in\mathcal{N}:L<N\}$. Clearly $N\_\in\mathcal{N}$ and $N\_\leq N$ (If $N\_\neq N$, $N\_$ is the immediate predecessor of N). Similarly we define N_+ (the immediate successor of N if $N\neq N_+$). If $\mathcal{N}$ is a nest, then the nest algebra $Alg(\mathcal{N})$ is the set of all operators T such that $TN\subseteq N$ for every element N in $\mathcal{N}$.

Suppose that $\phi:N\rightarrow\phi(N)$ is an order homomorphism of $\mathcal{N}$ into itself (that is, $N\leq N'$ implies $\phi(N)\leq\phi(N')$). Then the set

$$\mathcal{U}_\phi=\{X\in\mathcal{B}(\mathcal{H}):XN\subseteq\phi(N),\ \forall N\in\mathcal{N}\}$$

is clearly a weakly closed subset of $\mathcal{B}(\mathcal{H})$ and is easily seen to be a two sided $Alg(\mathcal{N})$-module. For each $N\in\mathcal{N}$, define

$$\phi_*(N)=\wedge\{\phi(N'):N'>N\};$$

and

$$\phi_\smile(N)=\vee\{N':\phi(N')<N\}.$$

Since $\mathcal{N}$ is complete, $\phi_*(N)$ and $\phi_\smile(N)$ are in $\mathcal{N}$.

If x and y are nonzero vectors in $\mathcal{H}$, we define the rank one operator $x\otimes y$ by

$$(x\otimes y)(z)=(z,y)x,\ \forall z\in\mathcal{H}.$$

The following lemma is the Lemma 1.2 in Erdos and Power (1982).

Lemma 1 Let $\mathcal{U}_\phi=\{X\in\mathcal{B}(\mathcal{H}):XN\subseteq\phi(N),\ \forall N\in\mathcal{N}\}$, where ϕ is an order homomorphism of $\mathcal{N}$ into $\mathcal{N}$, then the following are equivalent:

1) The rank-one operator $e\otimes f\in\mathcal{U}_\phi$;

2) There exists $N\in\mathcal{N}$, such that $e\in N$ and $f\in\phi_\smile(N)^\perp$;

3) There exists $M\in\mathcal{N}$, such that $e\in\phi_*(M)$ and $f\in M^\perp$.

Suppose that $\mathcal{X}$ is a Banach space, and $\mathcal{X}^*$ the dual space of $\mathcal{X}$. For $\mathcal{S}\subseteq\mathcal{X}$ and $\mathcal{W}\subseteq\mathcal{X}^*$, define

$$\mathcal{S}^\perp=\{x^*\in\mathcal{X}^*:x^*(x)=0,\ \forall x\in\mathcal{S}\};$$

$$\mathcal{W}_\perp=\{x\in\mathcal{X}:x^*(x)=0,\ \forall x^*\in\mathcal{W}\};$$

we will call $\mathcal{S}^\perp$ the annihilator of $\mathcal{S}$, and $\mathcal{W}_\perp$ the pre-annihilator of $\mathcal{W}$.

This paper deals mainly with study on the extreme point structure of norm closed nest algebra modules $\mathcal{U}$. The whole paper consists of three sections. Besides this introductory section, some auxiliary lemmas are given in Section 2. In Section 3, we characterize the extreme points of the unit ball of $\mathcal{U}$.

In the following, $\mathcal{U}$ always means a norm closed bi-module over the nest algebra $Alg(\mathcal{N})$. A closed subspace N in the Hilbert space $\mathcal{H}$ will always be identified with the projection $P(N)$ on it.

SOME LEMMAS

Definition Let $(\mathcal{X},\ \|\cdot\|)$ be a complex normed space, and let $\mathcal{X}_r$ denote the closed ball with center 0 and radius r. For $A\in\mathcal{X}_1$, if $B\in\mathcal{X}$ satisfies

$$\|A\pm B\|\leq 1,$$

then we will say that B is a contractive perturbation of A.

Lemma 2 Let $(\mathcal{X},\ \|\cdot\|)$ be a complex normed space, and let $A\in\mathcal{X}_1$. Then A is not an extreme point of $\mathcal{X}_1$ if and only if A has a nonzero contractive perturbation in $\mathcal{X}$.

Proof Suppose that B is a nonzero contractive perturbation of A, then

$$\|A\pm B\|\leq 1.$$

Thus, $A\pm B\in\mathcal{X}_1$ and $A=\dfrac{(A+B)+(A-B)}{2}$, $A\neq A+B$,

$A \neq A-B$. Hence A is not an extreme point of $\mathcal{X}_1$.

Suppose, on the contrary, that A is not an extreme point of $\mathcal{X}_1$. Thus, there exist $C, D \in \mathcal{X}_1$ such that $A = \frac{C+D}{2}$, $A \neq C$, $A \neq D$. Setting $B = \frac{C-D}{2} \neq 0$, we have

$$\| A \pm B \| \leq 1.$$

So B is a nonzero contractive perturbation of A.

Lemma 3 Suppose that $A, B \in \mathcal{B}(\mathcal{H})$, $\|A\| \leq 1$ and that $\|A \pm B\| \leq 1$. Then there exist bounded operators S and T such that $B = S(I - A^*A)^{\frac{1}{2}}$ and $B = (I - AA^*)^{\frac{1}{2}} T$.

Proof By the proof of Lemma 1 in Moore and Trent (1987).

The following lemma belongs to Anoussis and Katsoulis (1996b).

Lemma 4 Let $A \in \mathcal{B}(\mathcal{H})_1$ and $X \in \mathcal{B}(\mathcal{H})_{\frac{1}{2}}$, then

$$\| A \pm (I - AA^*)^{\frac{1}{2}} X (I - A^*A)^{\frac{1}{2}} \| \leq 1.$$

Lemma 5 Let $A \in \mathcal{B}(\mathcal{H})_1$, and suppose that B is a nonzero contractive perturbation of A. For any $Y \in \mathcal{B}(\mathcal{H})$, there exists a nonzero complex number λ_Y such that $\lambda_Y BYB$ is also a contractive perturbation of A. Moreover, A has a nonzero contractive perturbation of the form $(I - AA^*)^{\frac{1}{2}} X (I - A^*A)^{\frac{1}{2}}$ with X an operator in $\mathcal{B}(\mathcal{H})$.

Proof By Lemma 3, there exist operators S and T for which $B = S(I - A^*A)^{\frac{1}{2}} = (I - AA^*)^{\frac{1}{2}} T$. Thus

$$BYB = (I - AA^*)^{\frac{1}{2}} TYS (I - A^*A)^{\frac{1}{2}}.$$

If we choose λ_Y such that $\|\lambda_Y TYS\| \leq 1/2$, then the first statement follows Lemma 4. Set $Y = B^*$. Since $B \neq 0$, we have $BB^*B \neq 0$. Thus, there exists some operator X such that $(I - AA^*)^{\frac{1}{2}} X (I - A^*A)^{\frac{1}{2}}$ is a nonzero contractive perturbation of A.

For a norm closed nest algebra module $\mathcal{U}$, we define an order homomorphism τ from $\mathcal{N}$ into $\mathcal{N}$:

$$\tau(N) = [\mathcal{U} N], \ \forall N \in \mathcal{N}.$$

Also, we let $\mathcal{U}_\tau$ be the weakly closed nest algebra module determined by the order homomorphism τ. The following lemma (Erdos and Power, 1982) is essential to our paper.

Lemma 6 Suppose that $\mathcal{U}$ is a norm closed nest algebra module. Then $\mathcal{U}$ and $\mathcal{U}_\tau$ contain the same set of rank one operators.

Proposition 1 Let $\mathcal{U}$ be a norm closed nest algebra module. Then, a rank one operator $x \otimes y \in \mathcal{U}_\perp$ if and only if, for some $N \in \mathcal{N}$, $x \in \mathcal{N}$ and $y \in \tau(N)^\perp$.

Proof Suppose that $x \otimes y \in \mathcal{U}_\perp$. Set $N = [Alg(\mathcal{N})x]$. Thus $N \in LatAlg(\mathcal{N}) = \mathcal{N}$, and

$$\tau(N) = [\mathcal{U}N] = [\mathcal{U}[Alg(\mathcal{N})x]] = [\mathcal{U}x].$$

For any $T \in \mathcal{U}$, since $x \otimes y \in \mathcal{U}_\perp$, we have that

$$0 = tr(T(x \otimes y)) = (Tx, y).$$

This shows that $y \in [\mathcal{U}x]^\perp = \tau(N)^\perp$.

Conversely, suppose that there exists an element $N \in \mathcal{N}$ such that $x \in N$ and $y \in \tau(N)^\perp$. For any $T \in \mathcal{U}$, it follows from the definition of $\tau(N)$ that $\tau(N)^\perp TN = 0$. Hence,

$$\begin{aligned} tr(T(x \otimes y)) &= tr(TN(x \otimes y)\tau(N)^\perp) \\ &= tr(\tau(N)^\perp TN(x \otimes y)) = 0. \end{aligned}$$

This shows that $x \otimes y \in \mathcal{U}_\perp$.

EXTREME POINTS OF THE UNIT BALL OF $\mathcal{U}$

In this section, we will characterize the extreme points of the unit ball of $\mathcal{U}$.

Proposition 2 Suppose that $\mathcal{U}$ is a norm closed nest algebra module, and that $B \in \mathcal{B}(\mathcal{H})$. If $BXB = 0$ for every rank one operator X in $\mathcal{U}_\perp$, then there exists an element $N_0 \in \mathcal{N}$ such that $B = \tau_*(N_0) B N_0^\perp$.

Proof For any $N\in\mathcal{N}\backslash\{0\}$, if $B|_N\neq 0$, we may choose $x\in N$ with $Bx\neq 0$. If $\tau(N)=\mathcal{H}$, $B^*|_{\tau(N)^\perp}=0$. If $\tau(N)\neq\mathcal{H}$, for any nonzero $y\in\tau(N)^\perp$, it follows from Proposition 1 that the rank one operator $x\otimes y\in\mathcal{U}_\perp$. Hence,

$$Bx\otimes B^*y=B(x\otimes y)B=0.$$

We obtain $B^*y=0$ and $B^*|_{\tau(N)^\perp}=0$. Therefore, for any $N\in\mathcal{N}$, either $B|_N=0$ or $B^*|_{\tau(N)^\perp}=0$.

Set

$$N_0=\vee\{N'\in\mathcal{N}:B|_{N'}=0\}$$

Thus,

$$B|_{N_0}=0 \text{ and } B|_{N'}\neq 0\quad\forall N'>N_0.$$

So

$$B=B\,N_0^\perp \text{ and } B^*|_{\tau(N)^\perp}=0,\quad\forall N'>N_0$$

Thus, for any $N'>N_0$, we have $B^*=B^*\tau(N')$ and $B=\tau(N')B$. So,

$$B=\tau(N')BN_0^\perp,\ \forall N'>N_0.$$

Following the definition of $\tau_*(N_0)$, we obtain $B=\tau_*(N_0)BN_0^\perp$ by taking the limit in the above equation.

Proposition 3 Suppose that $\mathcal{U}$ is a norm closed nest algebra module. If $A\in\mathcal{U}_1$ and A is not an extreme point in $\mathcal{U}_1$, then there exists in $\mathcal{U}$ a rank one contractive perturbation of A.

Proof By Lemma 2, there exists a nonzero operator B in $\mathcal{U}$ such that

$$\|A\pm B\|\leq 1.$$

Hence by Lemma 5, for any $X\in\mathcal{U}_\perp$, there exists a nonzero complex number λ_X such that

$$\|A\pm\lambda_X BXB\|\leq 1.$$

If there is a rank one operator $X=x\otimes y\in\mathcal{U}_\perp$ such that $\lambda_X BXB\neq 0$, we will prove that the rank one operator $\lambda_X BXB\in\mathcal{U}$.

It follows from Proposition 1 that there exists $N_0\in\mathcal{N}$ such that $x\in N_0$ and $y\in\tau(N_0)^\perp$. For any $N\in\mathcal{N}$, we consider separately two cases.

Case 1 $N\leq N_0$. We have $\tau(N)\leq\tau(N_0)$, and $BN\subseteq\tau(N)\subseteq\tau(N_0)$. Hence

$$\begin{aligned}\lambda_X BXBN&=\lambda_X BN_0(x\otimes y)\tau(N_0)^\perp BN\\&\subseteq\lambda_X BN_0(x\otimes y)\tau(N_0)^\perp\tau(N_0)=(0)\subseteq\tau(N).\end{aligned}$$

Case 2 $N>N_0$. We have $\tau(N)\geq\tau(N_0)$, so

$$\begin{aligned}\lambda_X BXBN&=\lambda_X BN_0(x\otimes y)\tau(N_0)^\perp BN\\&\subseteq BN_0\subseteq\tau(N_0)\subseteq\tau(N).\end{aligned}$$

By Case 1 and Case 2, we obtain that $\lambda_X BXBN\subseteq\tau(N)$, for any $N\in\mathcal{N}$. Hence the rank one operator $\lambda_X BXB\in\mathcal{U}$, and $\lambda_X BXB$ is a rank one contractive perturbation of A in $\mathcal{U}$.

Suppose, on the contrary, that $BXB=0$ for each rank one X in $\mathcal{U}_\perp$. By Proposition 2, there exists an element $N_0\in\mathcal{N}$ such that

$$B=\tau_*(N_0)B\,N_0^\perp.$$

Since $B\neq 0$, we may choose a nonzero vector x with $Bx\neq 0$. Likewise, we may choose a nonzero vector y with $B^*y\neq 0$. Therefore, the rank one operator $C=B(x\otimes y)B=Bx\otimes B^*y\neq 0$ and by Lemma 1 and Lemma 6, we have

$$C=B(x\otimes y)B=\tau_*(N_0)B(x\otimes y)BN_0^\perp\in\mathcal{U}.$$

It follows from Lemma 5 that there exists a nonzero complex number $\lambda_{x\otimes y}$ such that $\lambda_{x\otimes y}C$ is a rank one contractive perturbation of A in $\mathcal{U}$. The proof is complete.

Now we are in a position to characterize the extreme point structure of $\mathcal{U}_1$ completely.

Theorem 1 Suppose that $\mathcal{U}$ is a norm closed nest algebra module. If $A\in\mathcal{U}_1$, then A is an extreme point in $\mathcal{U}_1$ if and only if for any $N\in\mathcal{N}$, either

$$N\cap ran\,(I-AA^*)^{\frac{1}{2}}=(0)$$

or

$$\tau_\sim(N)^\perp\cap ran\,(I-A^*A)^{\frac{1}{2}}=(0).$$

Proof Sufficiency. Suppose that A is not an extreme point in $\mathcal{U}_1$. By Proposition 3, there exists a rank one operator $x\otimes y$ of $\mathcal{U}$ such that

$$\| A\pm x\otimes y\| \leq 1.$$

By Lemma 3, there exist bounded operators S and T such that

$$x\otimes y=S(I-A^*A)^{\frac{1}{2}}=(I-AA^*)^{\frac{1}{2}}T$$

Hence, $x\in ran(I-AA^*)^{\frac{1}{2}}$ and $y\in ran(I-A^*A)^{\frac{1}{2}}$. Furthermore, since $x\otimes y\in\mathcal{U}$ and by Lemma 1 and Lemma 6, there exists an element $N\in\mathcal{N}$ with $x\in N$, $y\in \tau_{\sim}(N)^{\perp}$. So

$$x\in N\cap ran(I-AA^*)^{\frac{1}{2}}$$

and

$$y\in \tau_{\sim}(N)^{\perp}\cap ran(I-A^*A)^{\frac{1}{2}}.$$

This is a contradiction, so A is an extreme point of $\mathcal{U}_1$.

Necessity. Suppose that there exists an element $N\in\mathcal{N}$ with

$$N\cap ran(I-AA^*)^{\frac{1}{2}}\neq(0)$$

and

$$\tau_{\sim}(N)^{\perp}\cap ran(I-A^*A)^{\frac{1}{2}}\neq(0).$$

Thus we may choose nonzero vectors x and y such that

$$0\neq(I-AA^*)^{\frac{1}{2}}x\in N$$

and

$$0\neq(I-A^*A)^{\frac{1}{2}}y\in\tau_{\sim}(N)^{\perp}.$$

Therefore, by Lemma 1 and Lemma 6, the nonzero rank one operator

$$B=(I-AA^*)^{\frac{1}{2}}x\otimes(I-A^*A)^{\frac{1}{2}}y$$
$$=(I-AA^*)^{\frac{1}{2}}(x\otimes y)(I-A^*A)^{\frac{1}{2}}\in\mathcal{U}.$$

It follows from Lemma 4 that B is a nonzero contractive perturbation of A in $\mathcal{U}$, provided the norms of x and y are sufficiently small. By Lemma 2, A is not an extreme point of $\mathcal{U}_1$. This is a contradiction.

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