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**Guest Review:**

# Potential theory method for 3D crack and contact problems of multi-field coupled media: A survey*

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**Abstract:** This paper presents an overview of the recent progress of potential theory method in the analysis of mixed boundary value problems mainly stemming from three-dimensional crack or contact problems of multi-field coupled media. This method was used to derive a series of exact three dimensional solutions which should be of great theoretical significance because most of them usually cannot be derived by other methods such as the transform method and the trial-and-error method. Further, many solutions are obtained in terms of elementary functions that enable us to treat more complicated problems easily. It is pointed out here that the method is usually only applicable to media characterizing transverse isotropy, from which, however, the results for the isotropic case can be readily obtained.

**Key words:** Potential theory method, Mixed boundary value problem, Multi-field coupled media
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## INTRODUCTION

The term "potential theory" should be rooted in the mathematical treatment of the potential-energy functions used in physics to study gravitation and electromagnetism. Potential theory has become a profound branch of mathematics with much abstract content; it still has the essential function for solving boundary value problems related to our physical world. In particular, the potential theory methods have played an important role in solving problems related to elasticity theory. In fact, every effort has been made to reduce the governing equations of elasticity to those expressed in terms of potential (harmonic) functions only (Wang, 2002). Thus, the original elasticity problem is transformed to a problem in potential theory. There are two representative monographs in this respect, which have influenced greatly later researchers who devoted themselves to the mixed boundary value problems in elasticity. Muskhelishvili (1953) developed a systematic method for two-dimensional (2D) problems based on the theory of Cauchy integrals. Sneddon (1966), on the other hand, focused on the theory of dual/triple integral equations and the theory of dual series equations, etc. The application of potential theory methods in solving crack problems in elasticity was illustrated extensively by Sneddon and Lowengrub (1969). Here, however, we will pay our attention to the potential theory method originally proposed by Fabrikant (1989; 1991) for three-dimensional (3D) elasticity problems.

After nearly twenty years of research, Fabrikant published his first monograph, named "Applications of Potential Theory in Mechanics: A Selection of New Results" in 1989 (Fabrikant,

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1989). In this book, Fabrikant first discussed the limitations of the conventional methods such as Green's functions and the integral transform method. The rigorous mathematical development of potential theory method was then presented and many mixed boundary value problems associated with contact mechanics and crack mechanics in transversely isotropic elasticity, were then solved systematically using his new treatment. His second book published two years later, deals with more advanced mixed boundary value problems not confined to elasticity, as revealed by the title "Mixed Boundary Value Problem of Potential Theory and Their Applications in Engineering". These two books provide a fundamental tool for solving mixed boundary value problems. As stated in Fabrikant (1989), the main advantage of the new method is its ability to deal with non-classical problems that usually cannot be easily solved by other known methods. Fabrikant's work on potential theory was further extended afterwards by himself (Fabrikant, 2000; 2001). Using his results, other researchers found that exact solutions of some more complicated problems could be obtained (Hanson, 1992b; 1994). In recent years, the present authors generalized Fabrikant's method for solving mixed boundary value problems of multi-field coupled media (Chen, 1999c; Ding and Chen, 2001; Chen *et al*., 2004b). Various exact three-dimensional solutions were obtained, which enable us to gain deeper insight into the effect of coupling phenomena on material behavior and structural response.

This article is aimed at presenting the state-of-the-art of recent developments of the potential theory method along the lines of Fabrikant. The paper is organized as follows. First, the key idea of Fabrikant's method is explained, and some recent relevant studies on transversely isotropic elasticity are briefly reviewed. The authors' work on multi-field media then constitutes the main body of this article, arranged in two different sections. Some important results are repeated here. The paper concludes with some remarks concerning additional information and future directions.

Throughout this paper, we use the following terminology. If crack problem is considered, then the term "normal crack" means that the crack is subjected to normal loading at its surfaces, which is symmetric with respect to the crack plane; while the term "tangential crack" corresponds to a tangential loading applied antisymmetrically with respect to the crack plane. If contact problem is considered, the term "smooth contact" stands for the case when only normal displacement is prescribed and the tangential stresses vanish in the domain of contact; the term "tangential contact" stands for the case when only the tangential displacements are prescribed and the normal stress vanishes in the domain of contact; and the term "adhesive contact" for the case that both normal and tangential displacements are prescribed in the domain of contact. In the following, $a$ indicates either the radius of a circular crack or the radius of a circular contact area. It is emphasized that for the mixed boundary value problem of a transversely isotropic half-space, the potential theory method to be presented only applies to the case when the plane of isotropy is parallel to the plane surface.

## FABRIKANT'S POTENTIAL THEORY METHOD FOR ELASTICITY

The key point of Fabrikant's method is to express the reciprocal of the distance between two points as (Fabrikant, 1989)

$$\begin{aligned}\frac{1}{R}&=\frac{1}{\sqrt{r^2+r_0^2-2rr_0\cos(\phi-\phi_0)+z^2}}\\&=\frac{2}{\pi}\int_0^{l_1(r_0)}\frac{\lambda(\frac{x^2}{rr_0},\ \phi-\phi_0)\,\mathrm{d}x}{\sqrt{l_1^2(r_0)-x^2}\sqrt{l_2^2(r_0)-x^2}}\\&=\frac{2}{\pi}\int_{l_2(r_0)}^{\infty}\frac{\lambda(\frac{rr_0}{x^2},\ \phi-\phi_0)\,\mathrm{d}x}{\sqrt{x^2-l_1^2(r_0)}\sqrt{x^2-l_2^2(r_0)}},\end{aligned}\tag{1}$$

where $M(r,\phi,z)$ and $N(r_0,\phi_0)$ are two points in the space ($N$ is on $z$=0),

$$l_1(r_0)\equiv l_1(r_0,\ r,\ z)$$

$$=[\sqrt{(r+r_0)^2+z^2}-\sqrt{(r-r_0)^2+z^2}]/2$$

$$l_2(r_0)\equiv l_2(r_0,\ r,\ z) \qquad (2)$$
$$=[\sqrt{(r+r_0)^2+z^2}+\sqrt{(r-r_0)^2+z^2}]/2$$

and

$$\lambda(k,\psi)=\frac{1-k^2}{1+k^2-2k\cos\psi} \qquad (3)$$

It seems rather cumbersome to rewrite the simple expression $1/R$ in such a complicated integral form. However, this new form greatly facilitates the acquirement of exact analytical solutions of mixed boundary value problems. In particular, the two new parameters $l_1$ and $l_2$ as introduced in Eq.(2) can make the expression of solutions neater and more compact. As an illustration, let us consider the problem of a potential function $V$, which satisfies the Laplace equation in the half-space $z\geq 0$ and vanishes at infinity, subject to the following mixed boundary conditions on $z=0$:

$$V=v(r,\ \phi),\ \text{for } r\leq a,\ 0\leq\phi\leq 2\pi,$$
$$\frac{\partial V}{\partial z}=0,\ \text{for } r>a,\ 0\leq\phi\leq 2\pi. \qquad (4)$$

According to the potential theory (Courant and Hilbert, 1953), the function $V$ can be written as the potential of a simple layer as

$$V=\int_0^{2\pi}\int_0^{\infty}\frac{s(r_0,\phi_0)}{R}r_0\,\mathrm{d}r_0\,\mathrm{d}\phi_0. \qquad (5)$$

Using Eq.(1) and the boundary conditions in Eq.(4), we can obtain the governing integral equation (Fabrikant, 1986)

$$4\int_0^r\frac{\mathrm{d}x}{\sqrt{r^2-x^2}}\int_0^a\frac{r_0\,\mathrm{d}r_0}{\sqrt{r_0^2-x^2}}L\left(\frac{x^2}{rr_0}\right)s(r_0,\phi)=v(r,\phi) \qquad (6)$$

where $L(\cdot)$ is an operator (Poisson operator) defined as

$$L(k)f(\phi)=\frac{1}{2\pi}\int_0^{2\pi}\lambda(k,\ \phi-\phi_0)f(\phi_0)\,\mathrm{d}\phi_0. \qquad (7)$$

It can be seen that the left-hand side of Eq.(6) contains a sequence of two Abel-type operators and one Poisson operator. By applying proper combinations of Abel operator and Poisson operator to both sides of Eq.(6), Fabrikant (1986; 1989) obtained

$$V(r,\phi,z)=\frac{2}{\pi}\int_0^a\frac{\mathrm{d}l_1(t)}{\sqrt{r^2-l_1^2(t)}}L\left(\frac{l_1^2(t)}{rt^2}\right)\cdot$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{r_0\,\mathrm{d}r_0}{\sqrt{t^2-r_0^2}}L(r_0)v(r_0,\phi)$$
$$=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^a\left[\frac{R}{h}+\tan^{-1}\left(\frac{h}{R}\right)\right]\frac{z}{R^3}\cdot$$
$$v(r_0,\phi_0)r_0\,\mathrm{d}r_0\,\mathrm{d}\phi_0 \qquad (8)$$

where $h=\sqrt{a^2-l_1^2(a)}\sqrt{a^2-r_0^2}/a$. The first equality in Eq.(8) is very useful when an explicit evaluation of the integrals is possible. For example, when $v(r,\phi)=v_nr^n\cos(n\phi)$, here $v_n$ is a constant, the potential function $V$ can be obtained as

$$V(r,\phi,z)=v_nr^n\cos(n\phi)\left[1-\frac{2\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+1/2)}\frac{\sqrt{l_2^2(a)-a^2}}{l_2(a)}\cdot\right.$$
$$\left.F\left(\frac{1}{2}-n,\ \frac{1}{2};\ \frac{3}{2};\ \frac{l_2^2(a)-a^2}{l_2^2(a)}\right)\right] \qquad (9)$$

where $\Gamma$ is the Gamma function and

$$F\left(\frac{1}{2}-n,\ \frac{1}{2};\ \frac{3}{2};\ \zeta\right)=\frac{(1-\zeta)^{n+1/2}}{\Gamma(n+1)}\cdot$$
$$\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left[\frac{\zeta^{n-1/2}}{\sqrt{1-\zeta}}\sin^{-1}(\sqrt{\zeta})\right] \qquad (10)$$

is the hypergeometric function. It can be readily seen that the expression for $V(r,\phi,z)$ in the full space is obtained in terms of elementary functions.

The above key procedure was applied with appropriate extensions by Fabrikant to analyze various mixed boundary value problems in mechanics and other engineering areas (Fabrikant, 1989; 1991). In particular, he obtained many exact and complete three-dimensional solutions of crack and contact problems in terms of elementary functions for transversely isotropic elastic materials. These include the solution of a penny-shaped or external circular crack under symmetric normal point forces or antisymmetric tangential point forces that are applied at the crack surface; the solution of upright or inclined circular punch with a flat end on an elastic half-space, and the solution of the problem of a tangential loading underneath a smooth circular punch, etc. Apart from the exact solutions, efficient approximate analyses were also suggested, again based on the new method, to investigate the more complicated problems such as close interaction of pressurized coplanar circular cracks, flat crack of general shape, flat punch of arbitrary planform, interaction between punches, curved punch, rough punch, and interaction between loading and bonded punch, etc. It is noted that Fabrikant's results are usually derived for materials characterizing transverse isotropy, while the results for isotropic materials are obtained by using the L'Hospital's rule.

New solutions and insightful findings have been being reported continuously by Fabrikant in the latest decade. This indicates that his creative method has laid a solid foundation for future research. Here, we just mention several of his recent papers. In 1997, he considered three major types of contact problems (smooth contact, tangential contact and adhesive contact), for which relationships between the resultant forces and moments were established (Fabrikant, 1997a). Along with the generalized method of images, the cases of flat cracks of arbitrary shape inside a transversely isotropic elastic half-space and inside a layer were considered (Fabrikant, 1997b). It was shown that the solutions of normal and tangential crack problems have an intrinsic relationship; the significance of this finding is clear: the normal problem is simpler and through the established relationship one can avoid solving the more complicated tangential problem (Fabrikant, 1998a). Fabrikant (1998b) derived the expression for elastic field variation due to the variation of domain of crack or contact, and an interesting term "crack and contact calculus" was put forward. A full space solution of the problem of a penny-shaped crack interacting with two arbitrarily located normal forces was derived in Fabrikant (1999), while the case for two tangential forces was considered in another paper (Fabrikant, 2000). Exact solution for the external tangential contact problem was derived in Fabrikant (2001). Recently, he considered the contact problem for a transversely isotropic elastic layer by using the generalized images method (Fabrikant, 2004).

The unique results obtained by Fabrikant were also employed by other researchers to deal with problems involving more complicated effects. Hanson (1992a) obtained a Green's function for a point shear dislocation coplanar with a penny-shaped crack. Hanson (1992b; 1992c; 1994) extended Fabrikant's contact analysis to include the sliding friction of Coulomb type for conical, spherical or cylindrical circular indentation. Hanson (1993) and Hanson and Johnson (1993) also discussed the particular cases for isotropy. It is noted that for the normal contact problems, Hanson provided an alternative way to derive the solutions by integrating the point force solution for a half-space over the contact region. This method is straightforward but the contact pressure under the punch should be known *a priori*. Hanson (1992d) obtained closed form analytical expressions in terms of elementary functions for the potential functions for an infinitesimal prismatic coplanar dislocation loop interacting with a circular crack. Yong and Hanson (1992) showed that the coupled two-dimensional integral equations relevant to mixed boundary value problems with annular type regions could be converted to two non-coupled integral equations, whose solutions were then obtained in series form. Yong and Hanson (1994a) proposed an efficient numerical method for studying the circular crack system containing a penny-shaped crack and a concentric, coplanar external crack under arbitrary normal loading in a transversely isotropic body. Yong and Hanson (1994b) developed a method, which is based on point set theory and properties of orthogonal functions, for determining exact solutions for three-dimensional crack and contact problems with complicated geometric configurations. For elliptical Hertzian contact, Hanson and Puja (1997a) derived the elastic field in transversely isotropic half-space in closed-form expressions for both normal and shear loading. The elastic field in a transversely isotropic half-space caused by a circular flat bonded punch under torsion loading was presented by Hanson and Puja (1997b). Analytical expressions were derived for the elastic fields in a transversely isotropic half-space with various loadings applied over a circular area on the surface (Hanson and Puja, 1998a) and two special cases were investigated in Hanson and Puja (1998b). Karapetian and Hanson (1994) presented an evaluation of crack opening displacement and stress intensity factors for the problem of a concentrated load outside a circular crack. Karapetian and Kachanov (1996) derived exact solutions in elementary functions for the stress intensity factors of circular cracks interacting with various stress sources including dipoles, moments, centers of dilatation and rotation. Later, they extended their results (Karapetian and Kachanov, 1996) to the case of a half-plane crack (Kachanov and Karapetian, 1997). Karapetian and Kachanov (1998) further obtained Green's functions (in integral form) for a transversely isotropic space containing a circular crack by using the principle of superposition. Xiao *et al.*(1995) considered the interaction between two skew-parallel penny-shaped cracks in a 3-D transversely isotropic solid and obtained a closed-form solution for the stress intensity factors by the approach of series expansion in terms of the distance between the centers of the cracks. Kaczyński and Matysiak (2003) studied thermal stresses due to a plane crack lying on an interface in a microperiodic two-layered composite under uniform heat flow. Popova and Gorbatikh (2004) discussed the problem of a planar frictional sliding that initiates in the vicinity of reduced friction, and then uniformly propagates under the increased shear load as a penny-shaped zone.

## POTENTIAL THEORY METHOD FOR PIEZOELECTRIC MATERIALS

### General solution

The general solution for transversely isotropic piezoelectric materials was first presented by Wang and Zheng (1995) by extending the derivation for transversely isotropic elasticity (Elliott, 1948). Ding *et al.*(1996) suggested a rigorous mathematical derivation using the operator theory and obtained a simpler general solution, which can be rewritten in a complex form (Chen, 1999c; Ding and Chen, 2001)

$$U=\Lambda\left(\sum_{i=1}^{3}F_i+\mathrm{i}F_4\right),\quad w=\sum_{i=1}^{3}\alpha_{i1}\frac{\partial F_i}{\partial z_i},\quad \Phi=\sum_{i=1}^{3}\alpha_{i2}\frac{\partial F_i}{\partial z_i}\tag{11}$$

where $\Lambda=\partial/\partial x+\mathrm{i}\partial/\partial y$, $U=u+\mathrm{i}v$, $u(v,w)$ and $\Phi$ are displacements and electric potential, respectively, $\alpha_{ij}$ are constants defined in Ding *et al.*(1997a), and $F_i$ are quasi-harmonic functions satisfying

$$\left(\Delta+\frac{\partial^2}{\partial z_i^2}\right)F_i=0,\qquad (i=1,2,3,4)\tag{12}$$

in which $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the two-dimensional Laplace operator, $z_i=s_iz$ with $s_4=\sqrt{c_{66}/c_{44}}$, and $s_i$ ($i$=1,2,3) are the roots with positive real part of a third-order algebraic equation in $s^2$ (Ding *et al.*, 1996).

The linear constitutive relations can be used to derive the following expressions for stresses and electric displacements:

$$\sigma_1=2\sum_{i=1}^{3}[(c_{66}-c_{11})+c_{13}s_i\alpha_{i1}+e_{31}s_i\alpha_{i2}]\frac{\partial^2}{\partial z_i^2}F_i$$

$$\sigma_2=2c_{66}\Lambda^2\left(\sum_{i=1}^{3}F_i+\mathrm{i}F_4\right),\quad \sigma_z=\sum_{i=1}^{3}\gamma_{1i}\frac{\partial^2}{\partial z_i^2}F_i$$

$$\tau_z=\Lambda\left(\sum_{i=1}^{3}[c_{44}(s_i+\alpha_{i1})+e_{15}\alpha_{i2}]\frac{\partial}{\partial z_i}F_i+\mathrm{i}s_4c_{44}\frac{\partial}{\partial z_4}F_4\right)$$

$$D=\Lambda\left(\sum_{i=1}^{3}[e_{15}(s_i+\alpha_{i1})-\varepsilon_{11}\alpha_{i2}]\frac{\partial}{\partial z_i}F_i+\mathrm{i}s_4e_{15}\frac{\partial}{\partial z_4}F_4\right)$$

$$D_z=\sum_{i=1}^{3}\gamma_{2i}\frac{\partial^2}{\partial z_i^2}F_i\tag{13}$$

where, $\sigma_1=\sigma_x+\sigma_y$, $\sigma_2=\sigma_x-\sigma_y+2\mathrm{i}\tau_{xy}$, $\tau_z=\tau_{xz}+\mathrm{i}\tau_{yz}$ and $D=D_x+\mathrm{i}D_y$, $\sigma_i(\tau_{ij})$ are the normal stress (shear stress) components, $D_i$ are components of electric displacement, $c_{ij}$, $e_{ij}$ and $\varepsilon_{ij}$ are elastic, piezoelectric and dielectric constants, respectively, and $\gamma_{ij}$ are constants defined in Chen (1999c).

It is noted here that the general solution given in Eqs.(11) and (13) is only valid for distinct eigenvalues ($s_i^2$), while for other cases, different forms should be adopted (Ding *et al*., 1997a; Ding and Chen, 2001). In this paper, we will just pay our attention to the case of distinct eigenvalues.

It is clear that the solution presented above is expressed in terms of (quasi) harmonic functions or potential functions. Hence, it becomes natural to solve related problems in the category of potential theory. Chen (1999c) showed that Fabrikant's method could be readily applied in analyzing mixed boundary value problems of transversely isotropic piezoelectric materials. This can be achieved by introducing a new potential function which corresponds to the electric field. We will present the basic analysis procedure for the normal and tangential crack problems as well as contact problems in the following subsections.

**Normal crack problems**

Suppose that a transversely isotropic piezoelectric space is weakened by a flat crack $S$ in the plane $z$=0, with arbitrary pressure $p$ and surface charge $q$ applied symmetrically to the upper and lower crack faces. The above crack problem can be transformed into a mixed boundary value problem for a half-space $z\geq 0$. Chen and Shioya (1999a) assumed that

$$F_i(z)=c_iG(z_i)+d_iH(z_i),\ (i=1,2,3);\ F_4(z)=0 \quad (14)$$

where $G$ and $H$ are given as two potentials of a simple layer:

$$G(r,\phi,z)=\iint_S \frac{\omega(N)}{R(M,N)}\mathrm{d}S$$
$$H(r,\phi,z)=\iint_S \frac{\varphi(N)}{R(M,N)}\mathrm{d}S \quad (15)$$

in which $\omega$ and $\varphi$ stand for the crack face displacement and electric potential, respectively, $R(M,N)$ is the distance between the points $M(r,\phi,z)$ and $N(\rho,\psi,0)$. From the boundary conditions at $z$=0, the constants $c_i$ and $d_i$ in Eq.(14) were determined and the governing equations were finally derived as (Chen and Shioya, 1999a)

$$g_4\omega(N_0)-g_2\varphi(N_0)=A\iint_S \frac{p(N)}{R(N_0,N)}\mathrm{d}S,$$
$$g_1\varphi(N_0)-g_3\omega(N_0)=A\iint_S \frac{q(N)}{R(N_0,N)}\mathrm{d}S, \quad (16)$$

where $g_i$ and $A$ are constants defined in Chen and Shioya (1999a). The above two equations have the same structure as the corresponding equation obtained by Fabrikant (1989) for transversely isotropic elasticity, and hence the results presented therein can be utilized directly. If a penny-shaped crack of radius $a$ is considered, then for any polynomial form distributions of $p$ and $q$, all elasto-electric field variables can be expressed in elementary functions. In particular, the fundamental solution was derived by Chen and Shioya (1999a) for point loading also in terms of elementary functions. If the point force $P$ and point electric charge $Q$ are applied at the points $(r_1,\phi_1,0)$ and $(r_2,\phi_2,0)$, the intensity factors can be written as (Chen and Shioya, 1999a)

$$k_\sigma=\lim_{r\to a}\left(\sqrt{r-a}\,\sigma_z\big|_{z=0}\right)=\frac{P}{\pi^2\sqrt{2a}}\frac{\sqrt{a^2-r_1^2}}{a^2+r_1^2-2ar_1\cos(\phi-\phi_1)},$$

$$k_D=\lim_{r\to a}\left(\sqrt{r-a}\,D_z\big|_{z=0}\right)=-\frac{Q}{\pi^2\sqrt{2a}}\frac{\sqrt{a^2-r_2^2}}{a^2+r_2^2-2ar_2\cos(\phi-\phi_2)}. \quad (17)$$

This expression is very simple and had not been derived by other researchers before (Wang, 1992; Huang, 1997). It clearly shows that both factors are independent of material constants and are only related to the corresponding field variables applied at the crack faces. The stress intensity factor (SIF) and electric displacement intensity factor (EDIF) for arbitrarily distributed loadings can be obtained simply through integration over the crack surface (Chen and Shioya, 1999a).

Chen and Shioya (1999b) also presented the fundamental solution of an external circular crack. For a penny-shaped crack subjected to far-field uniform mechanical and electric loading, Chen *et al.*(2000) derived the exact solution in terms of elementary functions by using the principle of superposition. The exact solution for a semi-infinite crack was derived from the solution obtained in Chen and Shioya (1999a) by using a limit procedure (Chen, 1999a), which was originally suggested by Fabrikant *et al.*(1993).

**Tangential crack problems**

For a flat crack subjected to a complex shear loading $\tau$ that is antisymmetric with respect to the crack plane, it can be assumed (Chen and Shioya, 2000)

$$F_i(z)=c_i[\Lambda\bar{F}(z_i)+\bar{\Lambda}F(z_i)],\ (i=1,2,3);\qquad F_4(z)=c_4[\Lambda\bar{F}(z_4)-\bar{\Lambda}F(z_4)]. \tag{18}$$

where an overbar indicates the complex conjugate value, and $F$ is given by

$$F(r,\phi,z)=\iint_S \ln[R(M,N)+z]U(N)\,\mathrm{d}S_N. \tag{19}$$

Here $U(N)$ represents the crack surface complex displacement. Like the normal crack problem, the constants $c_i$ in Eq.(18) were determined from the boundary conditions and the following governing equation was also derived

$$\tau(N_0)=-\frac{1}{2\pi^2(G_1^2-G_2^2)}\left[G_1\Delta\iint_S\frac{U(N)}{R(N,N_0)}\mathrm{d}S+G_2\Lambda^2\iint_S\frac{\bar{U}(N)}{R(N,N_0)}\mathrm{d}S\right] \tag{20}$$

where $G_1$ and $G_2$ are two constants defined in Chen and Shioya (2000). Again, Eq.(20) is exactly the same as that for elasticity (Fabrikant, 1989) and can be solved when the crack domain is circular. The exact solution for a penny-shaped crack subjected to uniform shear loading was presented in Chen and Ding (1999a), whose results showed good agreement with those obtained by Kogan *et al.*(1996). The fundamental solution for a penny-shaped crack subjected to an antisymmetric point shear force was obtained in Chen and Shioya (2000) with the mode II and III SIFs being

$$\begin{aligned}k_{\mathrm{II}}+\mathrm{i}\,k_{\mathrm{III}}&=\lim_{r\to a}\left(\sqrt{r-a}\,\mathrm{e}^{-\mathrm{i}\phi}\,\tau_z\big|_{z=0}\right)\\&=\frac{1}{\pi^2}\sqrt{\frac{a^2-r_0^2}{2a}}\left[\frac{T\mathrm{e}^{-\mathrm{i}\phi}}{r_0^2+a^2-2ar_0\cos(\phi-\phi_0)}+\frac{G_2}{G_1}\frac{(3a-r_0\mathrm{e}^{\mathrm{i}(\phi-\phi_0)})\bar{T}\mathrm{e}^{\mathrm{i}\phi}}{a(a-r_0\mathrm{e}^{\mathrm{i}(\phi-\phi_0)})^2}\right].\end{aligned} \tag{21}$$

Here it is assumed that the point complex shear force $T=T_x+\mathrm{i}T_y$ is applied at the point $(r_0,\phi_0,0)$. Unlike the normal crack, the mode II and III SIFs depend on the material properties. Having obtained Eq.(21), the SIFs for an arbitrarily distributed shear loading can be obtained through integration and in fact, for the uniform shear loading, the result in Chen and Ding (1999a) was recovered.

The exact solution of the problem of an external circular crack subjected to an antisymmetric shear loading was obtained by Chen *et al.*(2001a). Hou *et al.*(2001b) presented the corresponding solution for a half-plane crack.

**Other crack problems**

Based on the Green's functions for two-phase piezoelectric materials (Ding *et al.*, 1997a), Chen *et al.*(1999a) presented integral equations for general boundary value problems of a transversely isotropic half-space. As an example, the exact solution of the normal crack problem was derived in a way different from Chen and Shioya (1999a). Further, Chen *et al.*(2001b) derived an exact solution for a penny-shaped crack subjected to point normal loading that is antisymmetric with respect to the crack plane by virtue of Fabrikant's results. In connection with the principle of superposition, Hou *et al.*(2001a) obtained the solution for the case when a concentrated normal force is applied to one crack surface while the other surface is stress-free. The interaction between arbitrarily located point forces and a circular crack was considered by Hou *et al*.(2001c) by extending the work of Karapetian and Hanson (1994). Hou *et al*.(2002) presented the Green's functions for an infinite transversely isotropic piezoelectric space containing a circular crack. A systematic study was carried out by Hou (2000) for various interaction problems.

It is noted that Karapetian *et al*.(2000) presented exact solutions for normal and tangential crack problems using a method almost identical with that proposed by Chen *et al*.(Chen and Shioya, 1999a; 2000; Chen, 1999a). Later, Karapetian *et al*.(2002) established a correspondence principle between elastic and piezoelectric problems for transversely isotropic materials, using which the solution for piezoelectric materials can be directly derived if the corresponding solution for elastic materials is known and written in a proper form.

**Contact problems**

Fabrikant's potential theory method for elastic contact mechanics has also been extended successfully for piezoelectric materials. For the normal contact problem with surface displacement $\omega$ and electric potential $\varphi$ prescribed within the contact area $S$, the potential functions $F_i$ still take the form as shown in Eq.(14), but with

$$
\begin{aligned}
G(r,\phi,z) &= \iint_S \ln[R(M,N)+z]\sigma_0(N)\,\mathrm{d}S,\\
H(r,\phi,z) &= \iint_S \ln[R(M,N)+z]D_0(N)\,\mathrm{d}S,
\end{aligned}
\tag{22}
$$

where $\sigma_0(N)$ and $D_0(N)$ stand for values of $\sigma_z$ and $D_z$ at point $N$. The satisfaction of the boundary conditions leads to the determination of the arbitrary constants $c_i$ and $d_i$ in Eq.(14) as well as the following governing equation (Chen, 2000a):

$$
\begin{aligned}
g_4\omega(N_0)-g_2\varphi(N_0) &= A\iint_S \frac{\sigma_0(N)}{R(N_0,N)}\,\mathrm{d}S,\\
g_1\varphi(N_0)-g_3\omega(N_0) &= A\iint_S \frac{D_0(N)}{R(N_0,N)}\,\mathrm{d}S,
\end{aligned}
\tag{23}
$$

where $g_i$ and $A$ are different from those in Eq.(16) (Chen, 2000a). Again, the above equation has the same structure as that for elastic contact problems (Fabrikant, 1989) and hence Fabrikant's results can be utilized. For an electrically inductive spherical rigid indenter of radius $R$ and constant electric potential $\Phi_0$, Chen and Ding (1999b) got the expressions for $G$ and $H$ as

$$
\begin{aligned}
G(r,\phi,z) = \frac{g_4}{\pi AR}\Big\{ & z\left(2a^2-r^2+\frac{2}{3}z^2\right)\sin^{-1}\left[\frac{a}{l_2(a)}\right]\\
& +\frac{4}{3}a^3\ln\left[l_2(a)+\sqrt{l_2^2(a)-r^2}\right]+\frac{1}{3}\left[5r^2-\frac{10}{3}a^2\right.\\
& \left.-2l_2^2(a)-\frac{11}{3}l_1^2(a)\right]\sqrt{a^2-l_1^2(a)}\Big\}\\
& -\frac{2g_2\Phi_0}{\pi A}\Big\{ z\sin^{-1}\left[\frac{a}{l_2(a)}\right]-\sqrt{a^2-l_1^2(a)}\\
& +a\ln\left[l_2(a)+\sqrt{l_2^2(a)-r^2}\right]\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
H(r,\phi,z) = -\frac{g_3}{\pi AR}\Big\{ & z\left(2a^2-r^2+\frac{2}{3}z^2\right)\sin^{-1}\left[\frac{a}{l_2(a)}\right]\\
& +\frac{4}{3}a^3\ln\left[l_2(a)+\sqrt{l_2^2(a)-r^2}\right]+\frac{1}{3}\left[5r^2-\frac{10}{3}a^2\right.\\
& \left.-2l_2^2(a)-\frac{11}{3}l_1^2(a)\right]\sqrt{a^2-l_1^2(a)}\Big\}\\
& +\frac{2g_1\Phi_0}{\pi A}\Big\{ z\sin^{-1}\left[\frac{a}{l_2(a)}\right]-\sqrt{a^2-l_1^2(a)}\\
& +a\ln\left[l_2(a)+\sqrt{l_2^2(a)-r^2}\right]\Big\}.
\end{aligned}
\tag{24}
$$

Here $a$ is the circular contact region which can be determined immediately if the total force applied on the indenter is known. The expressions for the full-space electro-elastic field can then be obtained from Eq.(24) by simple differentiation, with all results being in terms of elementary functions. It is noted that, due to the discontinuity of the electric potential across the contact border, singularities arise in the stress field and electric displacement field, resulting in the following SIF and EDIF at the edge of the contact area:

$$k_{\sigma} = -\frac{g_2\Phi_0}{\pi^2 A}\sqrt{\frac{1}{2a}},\quad k_D = \frac{g_1\Phi_0}{\pi^2 A}\sqrt{\frac{1}{2a}}. \tag{25}$$

Chen (1999b) presented exact and complete expressions for the electro-elastic field for the problem of a tilted circular flat punch indenting a transversely isotropic half-space, while Chen *et al.*(1999b) derived the exact solution for a rigid conical punch. For the tilted circular flat punch, the singularities come from the discontinuity of electric potential as well as the discontinuity of normal displacement across the contact border, while for the conical indentation; another type of singularity (logarithmic) emerges because of the sharp apex of the indenter.

It was noted earlier that the solution to elastic contact problem can also be obtained by integrating the point force solution (Hanson, 1992b; 1992c; 1993; 1994; Hanson and Johnson, 1993). This solution methodology has been extended to the case of transversely isotropic piezoelectric materials (Ding *et al.*, 1999; 2000; Hou, 2000). In particular, Ding *et al.*(2000) considered the general contact mechanics of two piezoelectric bodies and drew the conclusion that the contact stress and contact electric displacement have the same distribution as that for the elastic contact. The contact problems including Coulomb type of friction were also considered. The elliptical Hertzian contact for piezoelectric bodies was considered by Ding *et al.*(1999). This work was recently extended to the case for magneto-electro-elastic materials (Hou *et al.*, 2003).

## POTENTIAL THEORY METHOD FOR MULTI-FIELD MATERIALS WITH THERMAL EFFECT

Although the potential theory method proposed by Fabrikant (1989; 1991) for pure elasticity has been widely applied in analyzing various boundary value problems in contact and fracture mechanics, it is very surprising that no work involving thermal effect can be found yet by this paper's authors. This may boil down to the lack of a general solution for transversely isotropic thermoelasticity that is expressed in terms of (quasi) harmonic functions only. The general solution employed by Podil'chuk and Sokolovskii (1994) involves four potential functions, three of them are quasi-harmonic and the remaining one satisfies a differential equation with inhomogeneous term. Ding *et al.*(1997b) also presented a general solution consisting of a particular solution and a general solution. The particular solution can be solved from the heat conduction equation and the corresponding boundary conditions. The general solution is identical to those of the purely elastic one. In either the general solution employed by Podil'chuk and Sokolovskii (1994) or the one proposed by Ding *et al.*(1997b), the temperature field should be solved independently and *a priori*. For transient problems, Ashida *et al.*(1993) proposed a general solution technique in which the temperature filed also should be solved in advance. Employing such general solutions, however, we cannot use the many splendid results obtained by Fabrikant (1989; 1991).

To overcome the difficulty, Chen and Ding (2003) derived a general solution that does not include the particular solution by using the operator theory. The key point is to solve the steady state Fourier equation governing the temperature field simultaneously with the equilibrium equations. The general solution for transversely isotropic elasticity with thermal effect thus reads as (Chen and Ding, 2003)

$$\begin{aligned}
&U = -\Lambda\left(\sum_{i=1}^{3} F_i + \mathrm{i}F_4\right),\ w = \sum_{i=1}^{3}\alpha_{i1}\frac{\partial F_i}{\partial z_i},\\
&T = \sum_{i=1}^{3}\alpha_{i2}\frac{\partial^2 F_i}{\partial z_i^2},\ \sigma_z = \sum_{i=1}^{3}\gamma_{i1}\frac{\partial^2 F_i}{\partial z_i^2},\\
&\sigma_1 = \sum_{i=1}^{3}\gamma_{i2}\frac{\partial^2 F_i}{\partial z_i^2},\ \sigma_2 = -2c_{66}\Lambda^2\left(\sum_{i=1}^{3} F_i + \mathrm{i}F_4\right),\\
&\tau_z = \Lambda\left(\sum_{i=1}^{3}\gamma_{i3}\frac{\partial F_i}{\partial z_i} - \mathrm{i}s_4 c_{44}\frac{\partial F_4}{\partial z_4}\right),
\end{aligned} \tag{26}$$

where $F_i$ are the quasi-harmonic functions satisfying Eq.(12). It is noted that the definition of material constants in Eq.(26) is different from those in

Eqs.(11) and (13), and the reader is referred to Chen and Ding (2003) and Chen *et al.*(2004a) for details. The general solution in Eq.(26) is also valid for distinct eigenvalues $s_i$.

Consider an infinite transversely isotropic elastic body containing a flat crack $S$, the crack plane being parallel with the plane of isotropy. If the crack surfaces have symmetric temperature distribution $\Theta(x,y)$, we can still assume the potential functions $F_i$ in the form of Eq.(14). However, the two functions $G$ and $H$ now

$$G(r,\phi,z)=\iint_S \frac{\omega(N)}{R(M,N)}\,\mathrm{d}S,$$
$$H(r,\phi,z)=\iint_S \vartheta(N)\left\{z\ln\left[R(M,N)+z\right]-R(M,N)\right\}\mathrm{d}S \tag{27}$$

where $\omega$ and $\vartheta$ are the crack surface displacement and temperature gradient, respectively. Similarly, the constants in Eq.(14) were determined from the boundary conditions and the following governing equations were derived simultaneously (Chen and Ding, 2003)

$$\iint_S \frac{\vartheta(N)}{R(N_0,N)}\,\mathrm{d}S=-2\pi s_3\Theta(N_0),$$
$$\Delta\iint_S \frac{\omega(N)}{R(N_0,N)}\,\mathrm{d}S=-2\pi s_3 g_{12}\Theta(N_0)/g_{11}, \tag{28}$$

where $s_3=\sqrt{k_{11}/k_{33}}$ ; here $k_{ij}$ are the coefficients of thermal conductivity. The expressions for $g_{ij}$ are given in Chen and Ding (2003). It can be seen that the first equation in Eq.(28) is similar to the governing equation for elastic contact mechanics while the second equation is similar to the one for elastic crack mechanics (Fabrikant, 1989). Thus, the results of Fabrikant can be employed again for the followed analysis. For a penny-shaped crack with uniform temperature $T_0$ applied over the crack surface, Chen and Ding (2003) derived

$$G(r,\phi,z)=\frac{s_3 g_{12} T_0}{g_{11}}\left\{(2a^2+2z^2-r^2)\sin^{-1}\left[\frac{a}{l_2(a)}\right]\right.$$
$$\left.-\frac{2a^2-3l_1^2(a)}{a}\sqrt{l_2^2(a)-a^2}\right\},$$
$$H(r,\phi,z)=-2s_3T_0\left\{\left(z^2-a^2-\frac{r^2}{2}\right)\sin^{-1}\left[\frac{a}{l_2(a)}\right]\right.$$
$$-\frac{3[2a^2-l_1^2(a)]}{2a}\sqrt{l_2^2(a)-a^2}$$
$$\left.+2az\ln\left[l_2(a)+\sqrt{l_2^2(a)-r^2}\right]\right\}. \tag{29}$$

From Eq.(29) the full-space thermo-elastic field can be obtained through simple differentiation and all results are expressed in terms of elementary functions. The SIF was then derived as

$$k_\sigma=-2\sqrt{2a}s_3g_{12}T_0, \tag{30}$$

which agrees well with that obtained by Tsai (1983). Note that the definition of $k_\sigma$ in Eq.(17) is somewhat different from that in Chen and Ding (2003). The fundamental solution for the problem of a penny-shaped crack subjected to a pair of point temperature loads $\Theta_0$ symmetrically applied at the point $(r_0,\phi_0,0)$ was derived by Chen *et al.*(2004a). The expression for SIF is as follows

$$k_\sigma=-\frac{s_3g_{12}}{\pi}\sqrt{\frac{2}{a}}\Theta_0\frac{\sqrt{a^2-r_0^2}}{a^2+r_0^2-2ar_0\cos(\phi-\phi_0)}. \tag{31}$$

Then the SIF for an arbitrarily distributed temperature can be obtained by integrating Eq.(31) over the crack surface.

The general solution for thermo-piezo-elasticity was derived by Chen (2000b), who also obtained the exact solutions for a penny-shaped crack subjected to uniform temperature loading on the crack surface. The general solution for thermo-magneto-electro-elastic materials was presented by Chen *et al.*(2004b).

## CONCLUDING REMARKS

This paper presented a state-of-the-art survey of the recent advance in potential theory method which was originally proposed by Fabrikant. Much emphasis was put on the studies on multi-field coupled media that had been carried out by the authors themselves. It should be noted that, however, compared to the results for pure elasticity, the results for multi-field coupled theory are still very limited. For examples, most results were obtained for simple configurations (circular, half-plane or external circular region); the interaction between several punches and/or cracks has not yet been considered, the more complicated problems such as those studied by Kaczyński and Matysiak (2003) and Popova and Gorbatikh (2004) are still remaining untouched. Thus, continuous attention should be paid to these and other new topics.

In the case of isotropic elasticity, the two eigenvalues are equal for which Fabrikant (1989; 1991) adopted the L'Hospital rule to obtain the corresponding solutions. However, this method involves tedious mathematical manipulation. For multi-field coupled media, the task becomes even more awesome. In the appendices of Chen and Shioya (1999a) and Chen (2000a), an alternative way has been proposed that can overcome this difficulty. However, it has not been deeply explored yet.

When the thermal effect is taken into consideration, the problem becomes more complex. As shown in Chen *et al.*(2004), although the governing equations have the same structures as those studied by Fabrikant (1989, 1991), lengthy mathematical manipulation are involved in obtaining the solutions because of the different right-hand side content.

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