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A class of not max-stable extreme value distributions*

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Abstract: The sequences $\{Z_{i,n}, 1\le i\le n\}$, $n\ge1$ have multi-nomial distribution among i.i.d. random variables $\{X_{1,i}, i\ge1\}$, $\{X_{2,i}, i\ge1\}$, …, $\{X_{m,i}, i\ge1\}$. The extreme value distribution $G_Z(x)$ of this particular triangular array of i.i.d. random variables $Z_{1,n}, Z_{2,n}, \ldots, Z_{n,n}$ is discussed in this paper. We found a new type of not max-stable extreme value distributions, i) $G_Z(x)=\prod_{i=1}^{r-1}\Phi_{\alpha_i}^{A_i}(x)\times\Phi_{\alpha_r}(x)$; ii) $G_Z(x)=\prod_{i=1}^{r-1}\Psi_{\alpha_i}^{A_i}(x)\times\Psi_{\alpha_r}(x)$; iii) $G_Z(x)=\prod_{i=1}^{r-1}\Lambda^{A_i}(\lambda_i x)\times\Lambda(x)$, $r\ge2$, $0<\alpha_1\le\alpha_2\le\ldots\le\alpha_r$ and $\lambda_i\in(0,1]$ for i, $1\le i\le r-1$ which occur if $F_j, \ldots, F_m$ belong to the same MDA.

Key words: Extreme value distribution, Maximum domain of attraction (MDA), Mixed distribution functions
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INTRODUCTION

In Jiang (2004a; 2004b) we considered the case of mixing two sequences, with $m=2$, and proved that the limit distribution is often still an extreme value distribution, hence max-stable. But in some cases mixtures of extreme value distributions such as i) $\Phi_{\alpha_1}^{A}(x)\Phi_{\alpha_2}(x)$ and ii) $\Psi_{\alpha_1}^{A}(x)\Psi_{\alpha_2}(x)$ $(\alpha_1<\alpha_2)$, iii) $\Lambda^A(\rho x)\Lambda(x)$ $(0<\rho<1)$ occur. The question was posed whether with the mixture of more than 2 distributions, a more general class of limit distributions can be observed which contains even mixtures of different types of extreme value distributions. In this paper we will investigate this situation.

The question can be formulated into the following mathematical model: Let $\{X_{1,i}, i\ge1\}$, $\{X_{2,i}, i\ge1\}$, …, $\{X_{m,i}, i\ge1\}$ be m sequences of independent and identically distributed random variables with distribution functions $F_j(x)$ for j, $1\le j\le m$ (Jiang, 2002). Assume $F_j(x)\in MDA(G_j)$ i.e. for every j, there exist normalizing sequences $\alpha_{j,n}$, $\beta_{j,n}$ such that

$$\lim_{n\to\infty}F_j^n(\alpha_{j,n}x+\beta_{j,n})=G_j(x).$$

Consider the sequences $\{Z_{i,n}, 1\le i\le n\}$, $n\ge1$, which are defined by:

$$Z_{i,n}=\begin{cases}X_{1,i} & \text{with probability } p_{1,n}\\ \quad\ldots & \\ X_{j,i} & \text{with probability } p_{j,n}\\ \quad\ldots & \\ X_{m,i} & \text{with probability } p_{m,n}\end{cases}$$

where $\sum_{j=1}^{m}p_{j,n}=1$ and $p_{j,n}\to p_j\ge0$ $(1\le j\le m)$.

Hence,

$$\overline{F}_{Z,n}(x)=\sum_{j=1}^{m}p_{j,n}\overline{F}_j(x), \tag{1}$$

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Consider the limit $G_Z(x)=\lim_{n\to\infty}\Pr\{M_n(Z)\le\alpha_n x+\beta_n\}$ with $M_n(Z)=\max(Z_{1,n}, Z_{2,n}, \ldots, Z_{n,n})$ and for some sequences $0\le p_{j,n}\le 1$, for $1\le j\le m$ and normalizing constants α_n, β_n.

QUESTIONS TO BE ANALYZED

If $p_{j,n}\to p_j\in[0,1)$, does the above extreme value distribution $G_Z(x)$ exist? In which case, is it a max-stable distribution? If not, what $G_Z(x)$ distributions are possible? In which case, is $G_Z(x)$ Fréchet, Gumbel or Weibull? What are the relationships between $G_Z(x)$ and $G_1(x)$, …, $G_m(x)$? Does the probability $p_{j,n}$ influence the form of the extreme value distribution $G_Z(x)$?

This model is motivated for instance by the idea that the extreme values can be contaminated by some more than 2 other random variables. In other situations, extreme values are based on observations which stem from several sources with rather different distributions.

Let x_{F_j} be the right endpoint of distribution function $F_j(x)$ for j, $1\le j\le m$. In this paper we discuss the case in which $F_j(x)$ ($1\le j\le m$) have the same right endpoint and belong to the same MDA with $p_{j,n}\to 0$.

RESULTS FOR EQUAL RIGHT ENDPOINTS

In this section we derive the limit distribution $G_Z(x)$ with $p_{j,n}\to 0$ for j with $1\le j\le m-1$, and $p_{m,n}\to 1$, assuming that $x_{F_1}=\ldots=x_{F_m}=x_F\le\infty$. If $\lim_{x\to x_F}\dfrac{\bar{F}_r(x)}{\bar{F}_m(x)}=d_r\in[0,\infty)$, by Khintchine theorem (Resnick, 1987) there exists a function $c(x)\in[0,\infty)$ such that

$$np_{m,n}\bar{F}_m(\alpha_n x+\beta_n)\to-\log G_m(ax+b)=c(x)$$

for some $a>0$ and $b\in(0,\infty)$.

For large n

$$\begin{aligned}np_{r,n}\bar{F}_r(\alpha_n x+\beta_n)&\sim p_{r,n}d_r(n\bar{F}_m(\alpha_n x+\beta_n))\\&\sim p_{r,n}d_r c(x)\to 0,\end{aligned}$$

thus the terms of the r.v. X_r have no influence on the limit distribution $G_Z(x)$. Hence, w.l.o.g. we can suppose that every $F_j(x)$ ($1\le j\le m-1$) in our model satisfies

$$\lim_{x\to x_F}\frac{\bar{F}_j(x)}{\bar{F}_m(x)}=\infty. \tag{2}$$

On the other hand, if for large n, $np_{r,n}<\infty$, then $np_{r,n}\bar{F}_r(\alpha_n x+\beta_n)\to 0$, the term of the r.v. X_r, has also no influence on the limit distribution $G_Z(x)$. w.l.o.g. in our model we can also suppose that for every j with $1\le j\le m$

$$np_{r,n}\to\infty, \tag{3}$$

and thus the normalizing sequence $\alpha'_{j,n}=\alpha_{j,[np_j,n]}$ and $\beta'_{j,n}=\beta_{j,[np_j,n]}$ exist for every j with $1\le j\le m$.

Furthermore, we can order the sequences $\{X_{i,J}, J\ge 1\}$, $1\le i\le m-1$. w.l.o.g. we can suppose

$$\lim_{x\to x_F}\frac{\bar{F}_i(x)}{\bar{F}_1(x)}=c_i,\ \text{and } 1=c_1\ge c_2\ge\ldots\ge c_{m-1}\ge c_m=0, \tag{4}$$

where if $c_i=c_j$ and $i<j$, then assume $\lim_{x\to x_F}\dfrac{\bar{F}_i(x)}{\bar{F}_j(x)}\ge 1$ and if $\lim_{x\to x_F}\dfrac{\bar{F}_i(x)}{\bar{F}_j(x)}=1$ we can order them as we want.

Lemma 1 Suppose that continuous $F_j(x)\in MDA(G_j)$ satisfy Eqs.(2) and (4) for j with $1\le j\le m$.

i) If $F_m(x)\in MDA(\Phi_{\alpha_m})$, then $F_j(x)\in MDA(\Phi_{\alpha_j})$ for j with $1\le j\le m-1$ and $\alpha_1\le\alpha_2\le\ldots\le\alpha_{m-1}\le\alpha_m$.

ii) If $F_m(x)\in MDA(\Psi_{\alpha_m})$, then $F_j(x)\in MDA(\Psi_{\alpha_j})$ for j with $1\le j\le m-1$ and $\alpha_1\le\alpha_2\le\ldots\le\alpha_{m-1}\le\alpha_m$.

Proof i) Since $F_m(x)\in MDA(\Phi_{\alpha_m})$ and $x_F=\infty$, $F_j(x)\in MDA(\Phi_{\alpha_j})$ or $F_j(x)\in MDA(\Lambda)$ for j with $1\le j\le m-1$. But if $F_j(x)\in MDA(\Lambda)$, then by Lemma 1 in Jiang (2004a) we get $\lim_{x\to x_F}\dfrac{\bar{F}_j(x)}{\bar{F}_m(x)}=0$. This is contradictory to Eq.(2). Thus, $F_j(x)\in MDA(\Phi_{\alpha_j})$ for j with $1\le j\le m-1$. Hence, the statement follows by Eq.(4) and

Lemma 1 in Jiang (2004a).

ii) The proof is similar to that in i).

If $F_m(x)\in MDA(\Lambda)$ and $x_F=\infty$, then for every j with $1\leq j\leq m-1$, $F_j(x)\in MDA(\Phi_\alpha)$ or $MDA(\Lambda)$. Since $F_j(x)$ satisfies Eq.(2) for every j with $1\leq j\leq m-1$, by Lemma 1 in Jiang (2004a), w.l.o.g. we can suppose $F_j(x)\in MDA(\Phi_{\alpha_j})$ for j with $1\leq j\leq k$, and $F_j(x)\in MDA(\Lambda)$ for j with $k+1\leq j\leq m$, respectively, or $F_j(x)\in MDA(\Lambda)$ for all j with $1\leq j\leq m$. Similarly, if $F_m(x)\in MDA(\Lambda)$ and $x_F<\infty$, w.l.o.g. by Lemma 1 in Jiang (2004a) we can suppose $F_j(x)\in MDA(\Psi_{\alpha_j})$ for j with $1\leq j\leq k$, and $F_j(x)\in MDA(\Lambda)$ for j with $k+1\leq j\leq m$, respectively, or $F_j(x)\in MDA(\Lambda)$ for all j with $1\leq j\leq m$. Since the remaining cases were treated in Lemma 1, the following cases will be dealt with:

i) $x_F=\infty$ and $F_j(x)\in MDA(\Lambda)$ for j with $1\leq j\leq m$.

ii) $x_F=\infty$, $F_j(x)\in MDA(\Phi_{\alpha_j})$ for j with $1\leq j\leq k$ and $F_j(x)\in MDA(\Lambda)$ for j with $k+1\leq j\leq m$.

iii) $x_F<\infty$ and $F_j(x)\in MDA(\Lambda)$ for j with $1\leq j\leq m$.

iv) $x_F<\infty$ and $F_j(x)\in MDA(\Psi_{\alpha_j})$ for j with $1\leq j\leq k$ and $F_j(x)\in MDA(\Lambda)$ for j with $k+1\leq j\leq m$.

Lemma 2 Assume that $F_j(x)\in MDA(\Lambda)$ satisfy Eqs.(2) and (4) for j with $1\leq j\leq m$. Let

$$\lambda_{j/s}=\lim_{t\to\infty}\frac{f_s(t)}{f_j(t)}$$

exist for every j with $1\leq j\leq s\leq m$. Then $\lambda_{j/s}\leq 1$ and $\lambda_{1/s}\leq\lambda_{2/s}\leq\ldots\leq\lambda_{s-1/s}$.

Proof Since $F_j(x)\in MDA(\Lambda)$ satisfy Eqs.(2) and (4) for j with $1\leq j\leq m$, we have for $j<s$

$$\lim_{x\to x_F}\frac{\bar{F}_s(x)}{\bar{F}_j(x)}\leq 1.$$

Hence, by Lemma 1 in Jiang (2004a) we get $\lambda_{j/s}\leq 1$. On the other hand, for $i<j\leq s$

$$\lambda_{i/s}=\lim_{t\to\infty}\frac{f_s(t)}{f_i(t)}=\lim_{t\to\infty}\frac{f_s(t)}{f_j(t)}\frac{f_j(t)}{f_i(t)}=\lambda_{j/s}\lambda_{i/j}\leq\lambda_{j/s}$$

Now we can deal with the above situations.

Lemma 3 Assume that $F_j(x)\in MDA(\Phi_{\alpha_j})$ satisfy Eqs.(2), (3) and (4) for j with $1\leq j\leq m$. If for all j, s with $1\leq j,s\leq m$, $np_{j,n}\bar{F}_j(\alpha'_{s,n})\to A_{j,s}\in[0,\infty)$, then there exists an index r, such that $r=\max\{s\geq 1: A_{j,s}<\infty$ for every j with $1\leq j\leq s\}$.

Proof By Lemma 1, we get $\alpha_1\leq\alpha_2\leq\ldots\leq\alpha_m$.

If $np_{j,n}\bar{F}_j(\alpha'_{m,n})\to A_{j,m}<\infty$ for $1\leq j\leq m-1$, then $r=m$; if there exist some j, such that $A_{j,m}=\infty$, let $s_1=\max\{1\leq j\leq m-1: A_{j,m}=\infty\}$. If $np_{j,n}\bar{F}_j(\alpha'_{s_1,n})\to A_{j,s_1}<\infty$ for $1\leq j\leq s_1-1$, then $r=s_1$. Otherwise, we can continue this step. Since $A_{1,1}=1<\infty$, then $r\geq 1$. This completes the proof.

Theorem 1 Under the assumptions of Lemma 3, then the limit distribution of $M_Z(x)$ with the normalizing sequences $\alpha_n=\alpha'_{r,n}$ and $\beta_n=0$ is

$$G_Z(x)=\Phi_{\alpha_1}^{A_{1,r}}(x)\ldots\Phi_{\alpha_{r-1}}^{A_{r-1,r}}(x)\times\Phi_{\alpha_r}(x)$$

where r is given in Lemma 3.

Proof By Lemma 1, we get $\alpha_1\leq\alpha_2\leq\ldots\leq\alpha_m$.

For r given in Lemma 3 and every l with $r<l\leq m$ there exists $s<l$, such that $np_{s,n}\bar{F}_s(\alpha'_{l,n})\to A_{s,l}=\infty$. Hence,

$$\lim_{n\to\infty}\frac{\bar{F}_s(\alpha'_{l,n})}{\bar{F}_s(\alpha'_{s,n})}=\infty\quad\text{or}\quad\lim_{n\to\infty}\frac{\bar{F}_s(\alpha'_{s,n})}{\bar{F}_s(\alpha'_{l,n})}=0\tag{5}$$

and by Lemma 2 in Jiang (2004b) for j with $s<j\leq m$,

$$\lim_{n\to\infty}\frac{\bar{F}_j(\alpha'_{s,n})}{\bar{F}_j(\alpha'_{l,n})}=0.\tag{6}$$

1) If $s\leq r$, then $A_{s,r}<\infty$. Thus

$$\lim_{n\to\infty}\frac{\bar{F}_s(\alpha'_{r,n})}{\bar{F}_s(\alpha'_{l,n})}=\lim_{n\to\infty}\frac{\bar{F}_s(\alpha'_{r,n})}{\bar{F}_s(\alpha'_{s,n})}\frac{\bar{F}_s(\alpha'_{s,n})}{\bar{F}_s(\alpha'_{l,n})}=A_{s,r}\lim_{n\to\infty}\frac{\bar{F}_s(\alpha'_{s,n})}{\bar{F}_s(\alpha'_{l,n})}=0.\tag{7}$$

and by Lemma 2 in Jiang (2004b) for j with $s<j\leq m$,

$$\lim_{n\to\infty}\frac{\bar{F}_j(\alpha'_{r,n})}{F_j(\alpha'_{l,n})}=0.\tag{8}$$

Hence,

$$\lim_{n\to\infty} np_{l,n}\bar{F}_l(\alpha'_{r,n}) = \lim_{n\to\infty} np_{l,n}\bar{F}_l(\alpha'_{r,n})\frac{\bar{F}_l(\alpha'_{r,n})}{\bar{F}_l(\alpha'_{l,n})}$$
$$=\lim_{n\to\infty}\frac{\bar{F}_l(\alpha'_{r,n})}{\bar{F}_l(\alpha'_{l,n})}=0.$$

2) If $s>r$, there exists $s_1<s$, such that $np_{s_1,n}\bar{F}_{s_1}(\alpha'_{s,n})\to A_{s_1,s}=\infty$. Hence, as in the derivation in Eqs.(5) and (6) we have for j with $s_1\leq j\leq m$,

$$\lim_{n\to\infty}\frac{\bar{F}_j(\alpha'_{s_1,n})}{\bar{F}_j(\alpha'_{s,n})}=0\,.$$

If $s_1\leq r$, then $A_{s_1,r}<\infty$. Thus, as in the derivation in Eqs.(7) and (8) we have for j with $s_1\leq j\leq m$

$$\lim_{n\to\infty}\frac{\bar{F}_j(\alpha'_{r,n})}{\bar{F}_j(\alpha'_{s_1,n})}=0\,.$$

This results in

$$\lim_{n\to\infty}\frac{\bar{F}_s(\alpha'_{r,n})}{\bar{F}_s(\alpha'_{l,n})}=\lim_{n\to\infty}\frac{\bar{F}_s(\alpha'_{r,n})}{\bar{F}_s(\alpha'_{s_1,n})}\frac{\bar{F}_s(\alpha'_{s_1,n})}{\bar{F}_s(\alpha'_{s,n})}\frac{\bar{F}_s(\alpha'_{s,n})}{\bar{F}_s(\alpha'_{l,n})}=0.$$

and by Lemma 2 in Jiang (2004b) for j with $s<j\leq m$,

$$\lim_{n\to\infty}\frac{\bar{F}_j(\alpha'_{r,n})}{\bar{F}_j(\alpha'_{l,n})}=0\,.$$

Hence, again we get

$$\lim_{n\to\infty} np_{l,n}\bar{F}_l(\alpha'_{r,n}) = \lim_{n\to\infty} np_{l,n}\bar{F}_l(\alpha'_{l,n})\frac{\bar{F}_l(\alpha'_{r,n})}{\bar{F}_l(\alpha'_{l,n})}$$
$$=\lim_{n\to\infty}\frac{\bar{F}_l(\alpha'_{r,n})}{\bar{F}_l(\alpha'_{l,n})}=0.$$

If $s_1>r$, we repeat the step and get $s_2>s_3>\ldots>s_i$. Since $m<\infty$, after some limited number of steps we get $s_i\leq r$ which shows that $\lim_{n\to\infty} np_{l,n}\bar{F}_l(\alpha'_{r,n})=0$, for all $l>r$. Thus also $np_{l,n}\bar{F}_l(\alpha'_{r,n}x)\sim np_{l,n}\bar{F}_l(\alpha'_{r,n})x^{-\alpha_l}\to 0$, which completes the proof.

We can call the extreme value distribution $G_Z(x)=\Phi_{\alpha_1}^{A_{1,r}}(x)\ldots\Phi_{\alpha_{r-1}}^{A_{r-1,r}}(x)\times\Phi_{\alpha_r}(x)$ Fréchet mixture distribution. Similarly, $G_Z(x)=\Psi_{\alpha_1}^{A_{1,r}}(x)\ldots\Psi_{\alpha_{r-1}}^{A_{r-1,r}}(x)\times\Psi_{\alpha_r}(x)$ can be called Weibull mixture distribution and we have the following theorem.

Theorem 2 Assume that $F_j(x)\in MDA(\Psi_{\alpha_j})$ satisfy Eqs.(2), (3) and (4) for $1\leq j\leq m$ with $x_F<\infty$. If for all j, s with $1\leq j,s\leq m$, $np_{j,n}\bar{F}_j(\gamma'_{s,n})\to A_{j,s}\in[0,\infty]$.

Then

i) there exists an index r, such that $r=\max\{s\geq 1{:}A_{j,s}<\infty$ for every j with $1\leq j\leq s\}$;

ii) the limit distribution of $M_Z(x)$ with the normalizing sequences $\alpha_n=\alpha'_{r,n}$ and $\beta_n=x_F$ is

$$G_Z(x)=\Psi_{\alpha_1}^{A_{1,r}}(x)\ldots\Psi_{\alpha_{r-1}}^{A_{r-1,r}}(x)\times\Psi_{\alpha_r}(x).$$

Lemma 4 Assume that $F_j(x)\in MDA(\Lambda)$ satisfy Eqs.(2), (3) and (4) for j with $1\leq j\leq m$. If for all j, s with $1\leq j,s\leq m$, $np_{j,n}\bar{F}_j(\beta'_{s,n})\to A_{j,s}\in[0,\infty]$, then there exists an index r, such that $r=\max\{s\geq 1{:}A_{j,s}<\infty$ if $\lambda_{j/s}>0$ or $A_{j,s}=0$ if $\lambda_{j/s}=0$ for every j with $1\leq j\leq s\}$.

Proof If $np_{j,n}\bar{F}_j(\beta'_{m,n})\to A_{j,m}<\infty$ if $\lambda_{j/m}>0$ or $A_{j,m}=0$ if $\lambda_{j/s}=0$ for $1\leq j\leq m-1$, then $r=m$. If there exist some j, such that $A_{j,m}=\infty$ if $\lambda_{j/m}>0$ or $A_{j,m}>0$ if $\lambda_{j/m}=0$, let $s_1=\max\{1\leq j\leq m-1{:}A_{j,m}=\infty$ if $\lambda_{j/m}>0$ or $A_{j,m}>0$ if $\lambda_{j/m}=0\}$.

If $np_{j,n}\bar{F}_j(\beta'_{s_1,n})\to A_{j,s_1}<\infty$ if $\lambda_{j/s_1}>0$ or $A_{j,s_1}=0$ if $\lambda_{j/s_1}=0$ for $1\leq j\leq s_1-1$, then $r=s_1$. Otherwise, we continue this step. Since $A_{1,1}=1<\infty$, then $r\geq 1$. This completes the proof.

Theorem 3 Under the assumptions of Lemma 4, then for any x the limit distribution of $M_Z(x)$ with the normalizing sequences $\alpha_n=\alpha'_{r,n}$ and $\beta_n=\beta'_{r,n}$ is

$$G_Z(x)=\Lambda^{A_{1,r}}(\lambda_{1/r}x)\ldots\Lambda^{A_{r-1,r}}(\lambda_{r-1/r}x)\times\Lambda(x)\,,$$

where r is given in Lemma 4.

Proof We discuss the following two cases.

A) $x_F=\infty$:

For r given in Lemma 4 and every l with $r<l\leq m$ there exists $s<l$, such that $np_{s,n}\bar{F}_s(\beta'_{l,n})\to A_{s,l}=\infty$ if

$\lambda_{s/l}>0$ or $A_{s/l}>0$ if $\lambda_{s/l}=0$. Thus,

$$\lim_{n\to\infty}\frac{\overline{F}_s(\beta'_{l,n})}{\overline{F}_s(\beta'_{s,n})}=A_{s,l}\,,$$

by Lemma 2 in Jiang (2004b) for j with $s<j\le m$

$$\lim_{n\to\infty}\frac{\overline{F}_j(\beta'_{s,n})}{\overline{F}_j(\beta'_{l,n})}=0\,. \tag{9}$$

1) If $s\le r$, then $A_{s,r}<\infty$ or $\lim_{n\to\infty}\frac{\overline{F}_s(\beta'_{r,n})}{\overline{F}_s(\beta'_{s,n})}=A_{s,r}<\infty$, which implies by Lemma 2 in Jiang (2004b) for j with $s<j\le m$ and for any ε and large n,

$$\frac{\overline{F}_r(\beta'_{r,n})}{\overline{F}_r(\beta'_{s,n})}\le A_{s,r}^{1-\varepsilon}<\infty\,.$$

Hence,

$$\lim_{n\to\infty}\frac{\overline{F}_r(\beta'_{r,n})}{\overline{F}_r(\beta'_{l,n})}=\lim_{n\to\infty}\frac{\overline{F}_r(\beta'_{s,n})}{\overline{F}_r(\beta'_{l,n})}\frac{\overline{F}_r(\beta'_{r,n})}{\overline{F}_r(\beta'_{s,n})}=0\,.$$

Also by Lemma 2 in Jiang (2004b) for j with $r<j\le m$

$$\lim_{n\to\infty}\frac{\overline{F}_j(\beta'_{r,n})}{\overline{F}_j(\beta'_{l,n})}=0\,, \tag{10}$$

implying for l and any x

$$\lim_{n\to\infty}\frac{\overline{F}_l(\alpha'_{r,n}x+\beta'_{r,n})}{\overline{F}_l(\beta'_{l,n})}=\lim_{n\to\infty}\frac{\overline{F}_r(\alpha'_{r,n}x+\beta'_{r,n})}{\overline{F}_l(\beta'_{r,n})}\frac{\overline{F}_l(\beta'_{r,n})}{\overline{F}_l(\beta'_{l,n})}$$
$$=\mathrm{e}^{-\lambda_{l/r}x}\lim_{n\to\infty}\frac{\overline{F}_l(\beta'_{r,n})}{\overline{F}_l(\beta'_{l,n})}=0. \tag{11}$$

Hence,

$$\lim_{n\to\infty}np_{l,n}\overline{F}_l(\alpha'_{r,n}x+\beta'_{r,n})$$
$$=\lim_{n\to\infty}np_{l,n}\overline{F}_l(\beta'_{l,n})\frac{\overline{F}_l(\alpha'_{r,n}x+\beta'_{r,n})}{\overline{F}_l(\beta'_{l,n})}=0 \tag{12}$$

2) If $s>r$, there exists s_1 with $s_1<s$, such that $np_{s_1,n}\overline{F}_{s_1}(\beta'_{s,n})\to A_{s_1,s}=\infty$ if $\lambda_{s_1/s}>0$ or $A_{s_1,s}>0$ if $\lambda_{s_1/s}=0$. Thus,

$$\lim_{n\to\infty}\frac{\overline{F}_{s_1}(\beta'_{s,n})}{\overline{F}_{s_1}(\beta'_{s_1,n})}=A_{s_1,s}\,.$$

Hence, as in the derivation in Eq.(9) we have for j with $s_1<j\le m$, $\lim_{n\to\infty}\frac{\overline{F}_j(\beta'_{s_1,n})}{\overline{F}_j(\beta'_{s,n})}=0$.

If $s_1\le r$, then $\lim_{n\to\infty}\frac{\overline{F}_{s_1}(\beta'_{r,n})}{\overline{F}_{s_1}(\beta'_{s_1,n})}=A_{s_1,r}<\infty$, which implies by Lemma 2 in Jiang (2004b) for j with $s_1<j\le m$ and for any ε and large n,

$$\frac{\overline{F}_j(\beta'_{r,n})}{\overline{F}_j(\beta'_{s_1,n})}\le A_{s_1,r}^{1-\varepsilon}<\infty\,.$$

Hence,

$$\lim_{n\to\infty}\frac{\overline{F}_s(\beta'_{r,n})}{\overline{F}_s(\beta'_{l,n})}=\lim_{n\to\infty}\frac{\overline{F}_s(\beta'_{s,n})}{\overline{F}_s(\beta'_{l,n})}\frac{\overline{F}_s(\beta'_{s_1,n})}{\overline{F}_s(\beta'_{s,n})}\frac{\overline{F}_s(\beta'_{r,n})}{\overline{F}_s(\beta'_{s_1,n})}=0\,.$$

Thus, as in the derivation in Eqs.(10), (11) and (12) we have for j with $s\le j\le m$,

$$\lim_{n\to\infty}\frac{\overline{F}_j(\beta'_{r,n})}{\overline{F}_j(\beta'_{l,n})}=0 \quad\text{and}\quad \lim_{n\to\infty}\frac{\overline{F}_l(\alpha'_{r,n}x+\beta'_{r,n})}{\overline{F}_l(\beta'_{l,n})}=0\,.$$

Hence we also get,

$$\lim_{n\to\infty}np_{l,n}\overline{F}_l(\alpha'_{r,n}x+\beta'_{r,n})=0\,.$$

If $s_1>r$, we repeat the step and get $s_2>s_3>\ldots>s_i$. Since $m<\infty$, after some limited number of steps we get $s_i\le r$. This completes the proof.

B) $x_F<\infty$:

The proof is similar to that in i).

The extreme value distribution $G_Z(x)=\Lambda^{A_{1,r}}(\lambda_{1/r}x)\ldots\Lambda^{A_{r-1,r}}(\lambda_{r-1/r}x)\times\Lambda(x)$ can also be called Gumbel mixture distribution.

CONCLUSION

Fréchet mixture, Gumbel mixture and Weibull mixture can be not max-stable distribution.

Theorem 4 i) If $\alpha_1=\alpha_r$, then a Fréchet mixture is Fréchet distribution.

ii) If $\alpha_1<\alpha_r$, then a Fréchet mixture is not max-stable.

Proof i) If $\alpha_1=\alpha_r$, then $\alpha_i=\alpha_r$ for all $1\le i\le r-1$ and

$$\prod_{i=1}^{r-1}\Phi_{\alpha_i}^{A_i}(x)\times\Phi_{\alpha_r}(x)=\Phi_{\alpha_r}^{\beta}(x)=\Phi_{\alpha_r}(B^{-1/\alpha_r}x),$$

where $B=1+\sum_{i=1}^{r-1}A_i$ which implies our statement.

ii) Let $\alpha_1<\alpha_r$. If $\alpha_i=\alpha_j$ for some $1\le i<j\le r$, then

$$\Phi_{\alpha_i}^{A_i}(x)\Phi_{\alpha_j}^{A_j}(x)=\Phi_{\alpha_j}^{(A_i+A_j)}(x).$$

Hence, w.l.o.g. we suppose that $\alpha_1\le\alpha_2\le\ldots\le\alpha_r$.

Now suppose $G(x)=\prod_{i=1}^{r-1}\Phi_{\alpha_i}^{A_i}(x)\times\Phi_{\alpha_r}(x)$ is max-stable. Thus, there exist constants $\alpha_r>0$ and b_k such that

$$\prod_{i=1}^{r-1}\Phi_{\alpha_i}^{kA_i}(a_kx+b_k)\times\Phi_{\alpha_r}^{k}(a_kx+b_k)=\prod_{i=1}^{r-1}\Phi_{\alpha_i}^{A_i}(x)\times\Phi_{\alpha_r}(x)$$

By taking logarithms, this is equivalent to

$$k\sum_{i=1}^{r-1}A_i(a_kx+b_k)^{-\alpha_i}+k(a_kx+b_k)^{-\alpha_r}=\sum_{i=1}^{r-1}A_ix^{-\alpha_i}+x^{-\alpha_r}$$

where $\alpha_1<\alpha_2<\ldots<\alpha_r$.

For fixed k, let $x\searrow 0$, then the right hand side of the above equality tends to approach ∞, but if $b_k\neq 0$, then the left hand side of the above equality is bounded, this is a contradiction. Hence $b_k=0$ and

$$k\sum_{i=1}^{r-1}A_i(a_kx)^{-\alpha_i}+k(a_kx)^{-\alpha_r}=\sum_{i=1}^{r-1}A_ix^{-\alpha_i}+x^{-\alpha_r}. \quad (13)$$

This is equivalent to

$$kA_1(a_kx)^{-\alpha_1}\left(1+\sum_{i=2}^{r-1}\frac{A_i}{A_1}(a_kx)^{\alpha_1-\alpha_i}+\frac{1}{A_1}(a_kx)^{\alpha_1-\alpha_r}\right)$$
$$=A_1(x)^{-\alpha_1}\left(1+\sum_{i=2}^{r-1}\frac{A_i}{A_1}x^{\alpha_1-\alpha_i}+\frac{1}{A_1}x^{\alpha_1-\alpha_r}\right).$$

Let $x\to\infty$, for any fixed k we get $k\alpha_k^{-\alpha_1}=1$, and hence $\alpha_k=k^{1/\alpha_1}$. Inserting it into Eq.(13)

$$\sum_{i=1}^{r-1}A_ik^{1-\alpha_i/\alpha_1}x^{-\alpha_i}+k^{1-\alpha_r/\alpha_1}x^{-\alpha_r}=\sum_{i=1}^{r-1}A_ix^{-\alpha_i}+x^{-\alpha_r}$$

By multiplying both sides with x^{α_r} we get

$$\sum_{i=1}^{r-1}A_ik^{1-\alpha_i/\alpha_1}x^{\alpha_r-\alpha_i}+k^{1-\alpha_r/\alpha_1}=\sum_{i=1}^{r-1}A_ix^{\alpha_r-\alpha_i}+1.$$

Let $x\to 0$, for any k we get $k^{1-\alpha_r/\alpha_1}=1$. Hence, $\alpha_1=\alpha_r$, this is a contradiction.

Theorem 5 i) If $\alpha_1=\alpha_r$, then a Weibull mixture is Weibull distribution.

ii) If $\alpha_1<\alpha_r$, then a Weibull mixture is not max-stable.

Proof The proof is similar to that in Theorem 4.

Theorem 6 i) If $\lambda_1=\lambda_2=\ldots=\lambda_{r-1}=1$, then a Gumbel mixture is Gumbel distribution.

ii) If there exist at least one parameter $\lambda_i<1$, $1\le i\le r-1$, then a Gumbel mixture is not max-stable.

Proof i) If $\lambda_i=1$ for all $1\le i\le r-1$, then

$$\prod_{i=1}^{r-1}\Lambda^{A_i}(\lambda_ix)\times\Lambda(x)=\Lambda^{B}(x)=\Lambda(x-\log B),$$

where $B=1+\sum_{i=1}^{r-1}A_i$, which implies our statement.

ii) If $\lambda_i=\lambda_j$ for some $1\le i<j\le r$, then

$$\Lambda^{A_i}(\lambda_ix)\times\Lambda^{A_j}(\lambda_jx)=\Lambda^{A_i+A_j}(\lambda_jx).$$

Hence, w.l.o.g. suppose that $0<\lambda_1<\lambda_2<\ldots<\lambda_{r-1}<1=\lambda_r$.

Let $A_r=1$. Now suppose

$$G_Z(x)=\prod_{i=1}^{r-1}\Lambda^{A_i}(\lambda_ix)\times\Lambda(x)$$

is max-stable, it means that there exist constants $a_k>0$ and b_k such that:

$$\prod_{i=1}^{r-1}\Lambda^{kA_i}(\lambda_i(a_k x+b_k)\times\Lambda(x)=\prod_{i=1}^{r-1}\Lambda^{A_i}(\lambda_i x)\times\Lambda(x).$$

By taking logarithms, this is equivalent to

$$k\sum_{i=1}^{r}A_i\mathrm{e}^{-\lambda_i(a_k x+b_k)}=\sum_{i=1}^{r}A_i\mathrm{e}^{-\lambda_i x}. \tag{14}$$

Thus,

$$A_1\mathrm{e}^{-\lambda_1\left(a_k x+b_k-\frac{\log k}{\lambda_1}\right)}\left(1+\sum_{i=2}^{r}\frac{A_i}{A_1}\mathrm{e}^{-(\lambda_i-\lambda_1)(a_k x+b_k)}\right)$$
$$=A_1\mathrm{e}^{-\lambda_1 x}\left(1+\sum_{i=2}^{r}\frac{A_i}{A_1}\mathrm{e}^{-(\lambda_i-\lambda_1)x}\right).$$

Let $x\to\infty$, we have for any fixed k,

$$A_1\mathrm{e}^{-\lambda_1\left(a_k x+b_k-\frac{\log k}{\lambda_1}\right)}\sim A_1\mathrm{e}^{-\lambda_1 x}.$$

This results in

$$a_k=1,\ \ b_k=\log k/\lambda_1.$$

Inserting it into Eq.(14)

$$\sum_{i=1}^{r}A_i k^{1-\lambda_i/\lambda_1}\mathrm{e}^{-\lambda_i x}=\sum_{i=1}^{r}A_i\mathrm{e}^{-\lambda_i x}.$$

By multiplying both sides with e^x, we get

$$\sum_{i=2}^{r-1}A_i k^{1-\lambda_i/\lambda_1}\mathrm{e}^{(1-\lambda_i)x}+k^{1-1/\lambda_1}=\sum_{i=2}^{r-1}A_i\mathrm{e}^{(1-\lambda_i)x}+1,$$

Let $x\to-\infty$, we have for any fixed k,

$$k^{1-1/\lambda_1}=1,$$

which implies $\lambda_1=1$ and

$$\sum_{i=2}^{r-1}A_i k^{1-\lambda_i}\mathrm{e}^{-\lambda_i x}=\sum_{i=2}^{r-1}A_i\mathrm{e}^{-\lambda_i x}.$$

By multiplying both sides with $\mathrm{e}^{\lambda_2 x}$, we get

$$\sum_{i=3}^{r-1}A_i k^{1-\lambda_i}\mathrm{e}^{(\lambda_2-\lambda_i)x}+k^{1-\lambda_2}=\sum_{i=3}^{r-1}A_i\mathrm{e}^{(\lambda_2-\lambda_i)x}+1,$$

Let $x\to-\infty$, we have for any fixed k,

$$k^{1-\lambda_2}=1,$$

which implies $\lambda_2=1$ and similarly, for i with $3\le i\le r-1$, $\lambda_i=1$.

Hence, $\lambda_1=\lambda_2=\ldots=\lambda_{r-1}=1$, this is a contradiction.

Fréchet mixture, Gumbel mixture and Weibull mixture in Theorems 1, 2, 3 can be represented as mixed generalized extreme value distribution (MGEV) $H_{\rho_1,\ldots,\rho_r}(x)$ as follows:

$$H_{\rho_1,\ldots,\rho_r}(x)=\begin{cases}\prod_{i=1}^{r}\exp\{-A_i(1+\rho_r x)^{-1/\rho_i}\} \\ \qquad \text{if } \rho_r>0 \text{ or } \rho_r<0, \\ \prod_{i=1}^{r}\exp\{-A_i\exp\{-\lambda_i x\}\} \\ \qquad \text{if } \rho_i=0 \ \text{ for all } i\le r\end{cases} \tag{15}$$

where $1+\rho_r x>0$, $A_i\in(0,\infty)$ and $\lambda_i\in[0,1)$ for $1\le i\le r-1$, $\lambda_r=A_r=1$ and $\rho_1\ge\rho_2\ge\ldots\ge\rho_r\ge0$ or $\rho_1\le\rho_2\le\ldots\le\rho_r\le0$.

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