Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

Stability analysis of discrete-time BAM neural networks based on standard neural network models*

ZHANG Sen-lin (张森林), LIU Mei-qin (刘妹琴)‡

(*School of Electrical Engineering*, *Zhejiang University*, *Hangzhou 310027*, *China*)

E-mail: slzhang@mail.hz.zj.cn; liumeiqin@cee.zju.edu.cn

Received Feb. 18, 2005; revision accepted Apr. 8, 2005

Abstract: To facilitate stability analysis of discrete-time bidirectional associative memory (BAM) neural networks, they were converted into novel neural network models, termed standard neural network models (SNNMs), which interconnect linear dynamic systems and bounded static nonlinear operators. By combining a number of different Lyapunov functionals with S-procedure, some useful criteria of global asymptotic stability and global exponential stability of the equilibrium points of SNNMs were derived. These stability conditions were formulated as linear matrix inequalities (LMIs). So global stability of the discrete-time BAM neural networks could be analyzed by using the stability results of the SNNMs. Compared to the existing stability analysis methods, the proposed approach is easy to implement, less conservative, and is applicable to other recurrent neural networks.

Key words: Standard neural network model (SNNM), Bidirectional associative memory (BAM), Linear matrix inequality (LMI), Stability, Generalized eigenvalue problem (GEVP)

doi:10.1631/jzus.2005.A0689 **Document code:** A **CLC number:** TP183

INTRODUCTION

Bidirectional associative memory (BAM) models are a class of commonly used neural network models for information memory and association. The distributed information processing mode can restore an incomplete pattern or a pattern contaminated by noise by using BAMs. The bidirectional associative memory model proposed by Kosko (1987) generalizes Cohen-Grossberg's model by extending single-layer networks to two-layer ones. Since then, BAM networks have been widely investigated and many interesting results have been reported (Jin, 1999; Xu *et al*., 1999; Wang and Don, 1995). Stability is of utmost importance in applying BAM networks. Many stability analysis methods have been suggested (Jin, 1999; Xu *et al*., 1999; Wang and Don, 1995; Cao and Wang, 2002; Zhang *et al*., 1993; Xu *et al*., 1992). Unfortunately, proof of stability is often highly complicated and the results are very conservative, which prevents them being applied to wider engineering fields.

Jin (1999) treated the discrete-time Hopfield BAM neural network as a special Hopfield network model. Constraints on the connection matrix have been found under which the neural network has a unique and asymptotically stable equilibrium point. Sufficient conditions for the global asymptotic stability of equilibrium points were derived using the existence of the positive diagonal solutions of the Lyapunov equations. However, these solutions are usually found by trial-and-error. Thus, difficulties will be encountered when the theoretical results on stability suggested by Jin (1999) are applied to engineering problems. On the other hand, Jin (1999)'s proposed conditions for global exponential stability of BAM networks are wrong. We will see that in the fifth section.

‡Corresponding author

*Project (No. 60074008) supported by the National Natural Science Foundation of China

In this paper, we suggest a new discrete-time BAM network model inspired by the work of Jin (1999). Conditions for guaranteeing the existence of equilibrium points are given. Next, we propose a novel network model termed standard neural network model (SNNM). Based on the Lyapunov method, sufficient conditions for global asymptotic stability and global exponential stability of the SNNM are given, assuming that the equilibrium point locates at the origin. The stability conditions are formulated in linear matrix inequalities (LMIs), which are easy for verification and less conservative. Then, we convert the discrete-time BAM network into the SNNM. In this way, the global asymptotic stability and global exponential stability of the equilibrium point of the discrete-time BAM network can be analyzed by solving the LMIs. Compared to the existing stability analysis methods, the proposed approach is more straightforward for analysis and the results are less conservative.

NOTATIONS AND PROBLEM FORMULATION

Throughout the paper, the following notations are used. $\Re^n$ denotes n dimensional Euclidean space, $\Re^{n\times m}$ is the set of all $n\times m$ real matrices, $\boldsymbol{E}$ denotes identity matrix of appropriate order, $*$ denotes the symmetric parts, $\lambda_M(\boldsymbol{A})$ and $\lambda_m(\boldsymbol{A})$ denote the maximal and minimal eigenvalue of a square matrix $\boldsymbol{A}$, respectively. $\|\boldsymbol{x}\|$ denotes the Euclidean norm of the vector $\boldsymbol{x}$, and $\|\boldsymbol{A}\|$ denotes the induced norm of the matrix $\boldsymbol{A}$, that is $\|\boldsymbol{A}\|=\sqrt{\lambda_M(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})}$. The notation $\boldsymbol{X}>\boldsymbol{Y}$ and $\boldsymbol{X}\geq\boldsymbol{Y}$, respectively, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are matrices of the same dimensions, means that the matrix $\boldsymbol{X}-\boldsymbol{Y}$ is positive definite and semi-positive definite, respectively. If $\boldsymbol{X}\in\Re^p$ and $\boldsymbol{Y}\in\Re^q$, $\boldsymbol{C}(\boldsymbol{X};\boldsymbol{Y})$ denotes the space of all continuous functions mapping $\Re^p\rightarrow\Re^q$.

Consider the following discrete-time BAM neural network (Jin, 1999)

$$\begin{cases}\boldsymbol{x}(k+1)=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{W}\boldsymbol{f}(\boldsymbol{y}(k))+\boldsymbol{I}\\ \boldsymbol{y}(k+1)=\boldsymbol{B}\boldsymbol{y}(k)+\boldsymbol{V}\boldsymbol{g}(\boldsymbol{x}(k))+\boldsymbol{J}\end{cases}\quad(1)$$

where $\boldsymbol{x}(k)=(x_1(k), x_2(k), \ldots, x_n(k))^{\mathrm{T}}\in\Re^n$, $\boldsymbol{y}(k)=(y_1(k), y_2(k), \ldots, y_m(k))^{\mathrm{T}}\in\Re^m$ are state vectors, $\boldsymbol{f}(\boldsymbol{y}(k))=(f_1(y_1(k)), f_2(y_2(k)), \ldots, f_m(y_m(k)))^{\mathrm{T}}$ and $\boldsymbol{g}(\boldsymbol{x}(k))=(g_1(x_1(k)), g_2(x_2(k)), \ldots, g_n(x_n(k)))^{\mathrm{T}}$ are function vectors, $g_i\in\boldsymbol{C}(\Re;\Re)$ $(i=1,\ldots,n)$ and $f_j\in\boldsymbol{C}(\Re;\Re)$ $(j=1,\ldots,m)$ are continuously differentiable and monotonically increasing sigmoid functions satisfying $f_i(0)=g_j(0)=0$, $\boldsymbol{I}=(I_1,I_2,\ldots,I_n)^{\mathrm{T}}$ and $\boldsymbol{J}=(J_1,J_2,\ldots,J_m)^{\mathrm{T}}$ are external input vectors, I_i $(i=1,\ldots,n)$ and J_j $(j=1,\ldots,m)$ are constant, $\boldsymbol{W}$ and $\boldsymbol{V}$ are real $n\times m$ and $m\times n$ matrices, respectively, $\boldsymbol{A}=\mathrm{diag}(a_1,a_2,\ldots,a_n)$, $\boldsymbol{B}=\mathrm{diag}(b_1,b_2,\ldots,b_n)$.

Let $\boldsymbol{z}(k)=(x_1(k), x_2(k), \ldots, x_n(k), y_1(k), y_2(k), \ldots, y_m(k))^{\mathrm{T}}\in\Re^{n+m}$, $\boldsymbol{\phi}(\boldsymbol{z}(k))=(g_1(x_1(k)), g_2(x_2(k)), \ldots, g_n(x_n(k)), f_1(y_1(k)), f_2(y_2(k)), \ldots, f_m(y_m(k)))^{\mathrm{T}}$, then, the BAM network Eq.(1) can be rewritten as

$$\boldsymbol{z}(k+1)=\boldsymbol{R}\boldsymbol{z}(k)+\boldsymbol{S}\boldsymbol{\phi}(\boldsymbol{z}(k))+\boldsymbol{H}\quad(2)$$

where

$$\boldsymbol{R}=\mathrm{diag}(\boldsymbol{A},\boldsymbol{B})\ (i=1,\ldots,n+m),$$
$$\boldsymbol{S}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{W}\\ \boldsymbol{V} & \boldsymbol{0}\end{bmatrix},\quad \boldsymbol{H}=(\boldsymbol{I},\boldsymbol{J})^{\mathrm{T}}.$$

If g_i $(i=1,\ldots,n)$ and f_j $(j=1,\ldots,m)$ are hyperbolic tangent functions, $\phi_i(z_i(k))$ $(i=1,\ldots,n+m)$ satisfies $\phi_i(z_i(k))\in[-1,1]$, $\phi_i(z_i(k))/z_i(k)\in[0,1]$ and $[\phi_i(z_i(k+1))-\phi_i(z_i(k))]/[z_i(k+1)-z_i(k)]\in[0,1]$. In this paper, we assume that the training of BAM network was completed when we did the stability analysis. Thus, the weights remain constant in the process of stability analysis. Note that the equilibrium points of the discrete-time BAM network are dependent on the input pattern $\boldsymbol{H}$. The questions now are under which conditions the dynamic system described by Eq.(2) has equilibrium points and if the weights $\boldsymbol{R}$ and $\boldsymbol{S}$ can guarantee that all trajectories of the system converge to the equilibrium points.

Theorem 1 If the activation function $\boldsymbol{\phi}(\boldsymbol{z}(k))$ is bounded and $\boldsymbol{E}-\boldsymbol{R}\neq 0$, then the neural network system Eq.(2) has at least one equilibrium point.

Proof The equilibrium point of the system Eq.(2) satisfies $\boldsymbol{z}(k+1)=\boldsymbol{z}(k)$. Thus, $\boldsymbol{z}(k)=\boldsymbol{R}\boldsymbol{z}(k)+\boldsymbol{S}\boldsymbol{\phi}(\boldsymbol{z}(k))+\boldsymbol{H}$ and then

$$\boldsymbol{z}(k)=(\boldsymbol{E}-\boldsymbol{R})^{-1}\boldsymbol{S}\boldsymbol{\phi}(\boldsymbol{z}(k))+(\boldsymbol{E}-\boldsymbol{R})^{-1}\boldsymbol{H}$$

where $\boldsymbol{E}$ is a $(n+m)\times(n+m)$ identity matrix. Let $(\boldsymbol{E}-\boldsymbol{R})^{-1}\boldsymbol{H}=\boldsymbol{H}'$, and we have $\boldsymbol{z}(k)=(\boldsymbol{E}-\boldsymbol{R})^{-1}\boldsymbol{S}\boldsymbol{\phi}(\boldsymbol{z}(k))+\boldsymbol{H}'$. We denote this function mapping with $G(\boldsymbol{z})$. Note that the equilibrium point of system Eq.(2) is the fixed point of $G(\boldsymbol{z})$, and vice versa. For any $\boldsymbol{H}'\in\Re^{n+m}$, let

$$\boldsymbol{D}=\left\{z:\|z-\boldsymbol{H}'\|\le\frac{(n+m)\alpha}{1-\|\boldsymbol{R}\|}\|\boldsymbol{S}\|\right\},$$

where $\alpha=\max\{|\phi_i(z_i(k))|\}$ (i=1, …, $n+m$).

Since $\|G(z)-\boldsymbol{H}'\|=\|(\boldsymbol{E}-\boldsymbol{R})^{-1}\boldsymbol{S}\boldsymbol{\phi}(z(k))\|\le\|(\boldsymbol{E}-\boldsymbol{R})^{-1}\|\cdot\|\boldsymbol{S}\|\cdot\|\boldsymbol{\phi}(z(k))\|\le\frac{(n+m)\alpha}{1-\|\boldsymbol{R}\|}\|\boldsymbol{S}\|$ holds for any $z\in\boldsymbol{D}$, mapping $G(z)$ is continuous from $\boldsymbol{D}$ to itself. Since $\boldsymbol{D}$ is a convex, closed and bounded set, $G(z)$ has at least one fixed point in $\boldsymbol{D}$ for arbitrary weight matrices according to the Brouwer's fixed point theorem (Smart, 1980). Therefore, the system Eq.(2) [i.e. discrete-time BAM neural network Eq.(1)] has at least one equilibrium point, say z_{eq}. Theorem 1 is thus proven.

A STANDARD NEURAL NETWORK MODEL

The architecture of the discrete-time SNNM is shown in Fig.1. The model is composed of a linear dynamic system and nonlinear bounded activation functions. Here, we discuss only the discrete-time SNNM. Similar conclusions can be drawn for continuous-time SNNMs (Zhang and Liu, 2005). In the SNNM, $\boldsymbol{\Phi}$ is a block diagonal operator consisting of nonlinear activation functions $\phi_i(\xi_i(k))$, which generally are continuously differential, monotonically increasing, and bounded. Besides that, the derivative of $\phi_i(\xi_i(k))$ is also assumed to be bounded. Vectors $\boldsymbol{\xi}(k)$ and $\boldsymbol{\phi}(\boldsymbol{\xi}(k))$ are the input and output of $\boldsymbol{\Phi}$, and matrix $\boldsymbol{N}$ represents a linear mapping.

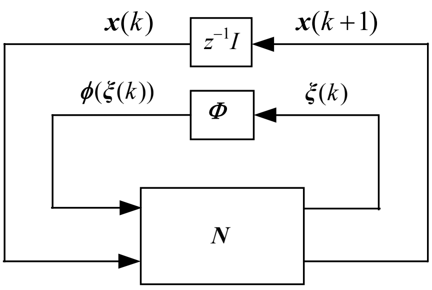

Fig.1 The discrete-time standard neural network model

If $\boldsymbol{N}$ in Fig.1 is partitioned as $\boldsymbol{N}=\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{bmatrix}$, then the SNNM can be depicted as a linear differential inclusion (LDI)

$$\begin{cases}\boldsymbol{x}(k+1)=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{B}\boldsymbol{\phi}(\boldsymbol{\xi}(k))\\ \boldsymbol{\xi}(k)=\boldsymbol{C}\boldsymbol{x}(k)+\boldsymbol{D}\boldsymbol{\phi}(\boldsymbol{\xi}(k))\\ \boldsymbol{\phi}(\boldsymbol{\xi}(k))=\boldsymbol{\Phi}(\boldsymbol{\xi}(k))\end{cases}\tag{3}$$

where $\boldsymbol{x}(k)\in\Re^n$ is the state vector, $\boldsymbol{A}\in\Re^{n\times n}$, $\boldsymbol{B}\in\Re^{n\times L}$, $\boldsymbol{C}\in\Re^{L\times n}$, and $\boldsymbol{D}\in\Re^{L\times L}$, are the corresponding state-space matrices, $\boldsymbol{\xi}\in\Re^L$ is the input vector of nonlinear operator $\boldsymbol{\Phi}$, $\boldsymbol{\phi}\in\boldsymbol{C}(\Re^L;\ \Re^L)$ is the output vector of nonlinear operator $\boldsymbol{\Phi}$ satisfying $\boldsymbol{\phi}(0)=0$, and $L\in\Re$ is the number of the nonlinear activation functions in the network model (in other words, the total number of the neurons in the hidden and output layer of the neural network). Since $\boldsymbol{x}=0$ satisfies Eq.(3), there exists at least one equilibrium point of SNNM Eq.(3) locating at the origin, i.e. $\boldsymbol{x}_{\text{eq}}=0$.

Before proceeding further, we first need the following definition.

Definition 1 (Jin, 1999) If there exist scalars $\gamma>0$, $c>0$ and any initial states $\boldsymbol{x}(0)$ such that

$$\|\boldsymbol{x}(k)-\boldsymbol{x}_{\text{eq}}\|\le c\|\boldsymbol{x}(0)\|\text{e}^{-\gamma k},\ \forall k>0,$$

the SNNM Eq.(3) is said to be exponentially stable at the equilibrium point $\boldsymbol{x}_{\text{eq}}$, where γ is called the degree of exponential stability.

If the activation functions in SNNM Eq.(3) satisfy the sector condition $\phi_i(\xi_i(k))/\xi_i(k)\in[q_i, u_i]$, i.e., $[\phi_i(\xi_i(k))-q_i\xi_i(k)]\cdot[\phi_i(\xi_i(k))-u_i\xi_i(k)]\le0$, $0\le q_i<u_i$ (i=1, …, L), the following theorems can be given.

Theorem 2 The origin of the discrete-time SNNM Eq.(3) is globally asymptotically stable, if there exist a symmetric positive definite matrix $\boldsymbol{P}$, and diagonal semi-positive definite matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{T}$, such that the following LMI holds

$$\boldsymbol{G}=\begin{bmatrix}\boldsymbol{G}_1 & \boldsymbol{G}_2\\ * & \boldsymbol{G}_3\end{bmatrix}\tag{4}$$

where

$$\boldsymbol{G}_1=\boldsymbol{A}^{\text{T}}\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P}-2\boldsymbol{C}^{\text{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{C},$$

$$\boldsymbol{G}_2=\boldsymbol{A}^{\text{T}}\boldsymbol{P}\boldsymbol{B}+\boldsymbol{C}^{\text{T}}\boldsymbol{\Lambda}-2\boldsymbol{C}^{\text{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{D}+\boldsymbol{C}^{\text{T}}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T},$$

$$\boldsymbol{G}_3=\boldsymbol{B}^{\text{T}}\boldsymbol{P}\boldsymbol{B}+\boldsymbol{\Lambda}\boldsymbol{D}+\boldsymbol{D}^{\text{T}}\boldsymbol{\Lambda}-2\boldsymbol{D}^{\text{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{D}-2\boldsymbol{T}$$

$$+\boldsymbol{D}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T}+\boldsymbol{T}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{D}$$

$$\boldsymbol{Q}=\mathrm{diag}(q_1,q_2,\cdots,q_L)\,,$$

$$\boldsymbol{U}=\mathrm{diag}(u_1,u_2,\cdots,u_L)\,.$$

Proof For simplicity, we denote $\boldsymbol{x}(k)$ as $\boldsymbol{x}_k$, $\xi_i(k)$ as $\xi_{k,i}$, $\phi_i[\xi_i(k)]$ as $\phi_{k,i}$, $\boldsymbol{\phi}[\boldsymbol{\xi}(k)]$ as $\boldsymbol{\phi}_k$. Consider the following positive definite Lyapunov function

$$V(\boldsymbol{x}_k)=\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k+2\sum_{i=1}^{L}\lambda_i\sum_{j=0}^{k-1}\phi_i[\xi_i(j)]\xi_i(j)\,,$$

where $\boldsymbol{P}>0$, $\lambda_i\geq 0$. Thus, $\forall \boldsymbol{x}_k\neq 0$, $V(\boldsymbol{x}_k)>0$ and $V(\boldsymbol{x}_k)=0$ iff $\boldsymbol{x}_k=0$. The difference along the solution of the SNNM Eq.(3) is

$$\begin{aligned}\Delta V(\boldsymbol{x}_k)&=V(\boldsymbol{x}_{k+1})-V(\boldsymbol{x}_k)\\&=\boldsymbol{x}_{k+1}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_{k+1}-\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k+2\sum_{i=1}^{L}\lambda_i\phi_{k,i}\xi_{k,i}\\&=\left(\boldsymbol{A}\boldsymbol{x}_k+\boldsymbol{B}\boldsymbol{\phi}_k\right)^{\mathrm{T}}\boldsymbol{P}\left(\boldsymbol{A}\boldsymbol{x}_k+\boldsymbol{B}\boldsymbol{\phi}_k\right)\\&\quad-\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k+2\sum_{i=0}^{L}\lambda_i\phi_{k,i}\left(\boldsymbol{C}_i\boldsymbol{x}_k+\boldsymbol{D}_i\boldsymbol{\phi}_k\right)\\&=\boldsymbol{x}_k^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P})\boldsymbol{x}_k+\boldsymbol{x}_k^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Lambda})\boldsymbol{\phi}_k\\&\quad+\boldsymbol{\phi}_k^{\mathrm{T}}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}+\boldsymbol{\Lambda}\boldsymbol{C})\boldsymbol{x}_k+\boldsymbol{\phi}_k^{\mathrm{T}}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}+\boldsymbol{\Lambda}\boldsymbol{D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Lambda})\boldsymbol{\phi}_k\\&=\begin{bmatrix}\boldsymbol{x}_k\\\boldsymbol{\phi}_k\end{bmatrix}^{\mathrm{T}}\underbrace{\begin{bmatrix}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P} & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Lambda}\\\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}+\boldsymbol{\Lambda}\boldsymbol{C} & \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}+\boldsymbol{\Lambda}\boldsymbol{D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Lambda}\end{bmatrix}}_{T_0}\begin{bmatrix}\boldsymbol{x}_k\\\boldsymbol{\phi}_k\end{bmatrix}\end{aligned}$$

where $\boldsymbol{C}_i$ is the ith row of matrix $\boldsymbol{C}$, $\boldsymbol{D}_i$ is the ith row of matrix $\boldsymbol{D}$, $\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_L)$ and $\boldsymbol{\Lambda}\geq 0$.

The sector conditions, $(\phi_{k,i}-q_i\xi_{k,i})(\phi_{k,i}-u_i\xi_{k,i})\leq 0$, can be rewritten as follows

$$(\phi_{k,i}-q_i\boldsymbol{C}_i\boldsymbol{x}_k-q_i\boldsymbol{D}_i\boldsymbol{\phi}_k)(\phi_{k,i}-u_i\boldsymbol{C}_i\boldsymbol{x}_k-u_i\boldsymbol{D}_i\boldsymbol{\phi}_k)\leq 0\,,$$

which is equivalent to

$$\begin{aligned}&2\phi_{k,i}^2-2\phi_{k,i}(q_i+u_i)\boldsymbol{C}_i\boldsymbol{x}_k-2\phi_{k,i}(q_i+u_i)\boldsymbol{D}_i\boldsymbol{\phi}_k\\&+2\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{C}_i^{\mathrm{T}}q_iu_i\boldsymbol{C}_i\boldsymbol{x}_k+2\boldsymbol{\phi}_k^{\mathrm{T}}\boldsymbol{D}_i^{\mathrm{T}}q_iu_i\boldsymbol{D}_i\boldsymbol{\phi}_k\\&+2\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{C}_i^{\mathrm{T}}q_iu_i\boldsymbol{D}_i\boldsymbol{\phi}_k+2\boldsymbol{\phi}_k^{\mathrm{T}}\boldsymbol{D}_i^{\mathrm{T}}q_iu_i\boldsymbol{C}_i\boldsymbol{x}_k\leq 0\end{aligned}\tag{5}$$

Rewrite Eq.(5) in the matrix form as Eq.(6),

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{x}_k\\\phi_{k,1}\\\vdots\\\phi_{k,i-1}\\\phi_{k,i}\\\phi_{k,i+1}\\\vdots\\\phi_{k,L}\end{bmatrix}^{\mathrm{T}}\underbrace{\begin{bmatrix}0&0&\cdots&0&-\boldsymbol{C}_i^{\mathrm{T}}(q_i+u_i)&0&\cdots&0\\0&0&\cdots&0&-d_{i,1}(q_i+u_i)&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0&-d_{i,i-1}(q_i+u_i)&0&\cdots&0\\-(q_i+u_i)\boldsymbol{C}_i&-(q_i+u_i)d_{i,1}&\cdots&-(q_i+u_i)d_{i,i-1}&2-2(q_i+u_i)d_{i,i}&-(q_i+u_i)d_{i,i+1}&\cdots&-(q_i+u_i)d_{i,L}\\0&0&\cdots&0&-d_{i,i+1}(q_i+u_i)&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0&-d_{i,L}(q_i+u_i)&0&\cdots&0\end{bmatrix}}_{T_i^1}\begin{bmatrix}\boldsymbol{x}_k\\\phi_{k,1}\\\vdots\\\phi_{k,i-1}\\\phi_{k,i}\\\phi_{k,i+1}\\\vdots\\\phi_{k,L}\end{bmatrix}\\&+\begin{bmatrix}\boldsymbol{x}_k\\\boldsymbol{\phi}_k\end{bmatrix}^{\mathrm{T}}\underbrace{\begin{bmatrix}2\boldsymbol{C}_i^{\mathrm{T}}q_iu_i\boldsymbol{C}_i&2\boldsymbol{C}_i^{\mathrm{T}}q_iu_i\boldsymbol{D}_i\\2\boldsymbol{D}_i^{\mathrm{T}}q_iu_i\boldsymbol{C}_i&2\boldsymbol{D}_i^{\mathrm{T}}q_iu_i\boldsymbol{D}_i\end{bmatrix}}_{T_i^2}\begin{bmatrix}\boldsymbol{x}_k\\\boldsymbol{\phi}_k\end{bmatrix}\leq 0\,,\end{aligned}\tag{6}$$

where $d_{i,j}$ is the element of matrix $\boldsymbol{D}$ at ith row and jth column. By the S-procedure (Boyd *et al*., 1994), if there exists $\tau_i\geq 0$ (i=1, …, L), such that the following inequality holds

$$\boldsymbol{T}_0-\sum_{i=1}^{L}\tau_i(\boldsymbol{T}_i^1+\boldsymbol{T}_i^2)=\begin{bmatrix}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P}&\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Lambda}\\\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}+\boldsymbol{\Lambda}\boldsymbol{C}&\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}+\boldsymbol{\Lambda}\boldsymbol{D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Lambda}\end{bmatrix}-\begin{bmatrix}2\boldsymbol{C}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{C}&2\boldsymbol{C}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{D}-\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T}\\ * &\begin{pmatrix}2\boldsymbol{D}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{D}+2\boldsymbol{T}\\-\boldsymbol{D}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T}-\boldsymbol{T}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{D}\end{pmatrix}\end{bmatrix}=\boldsymbol{G}<0$$

where $\boldsymbol{T}$=diag(τ_1, τ_2, …, τ_L) and $\boldsymbol{T}\geq 0$, then $T_0<0$, that is, $\forall \boldsymbol{x}_k\neq 0$, $\Delta V(\boldsymbol{x}_k)<0$ and $\Delta V(\boldsymbol{x}_k)=0$ iff $\boldsymbol{x}_k=0$. We can conclude that the origin of the discrete-time SNNM Eq.(3) is globally asymptotically stable.

Theorem 3 If there exist symmetric positive definite matrices $\boldsymbol{R}$ and $\boldsymbol{K}$, and diagonal semi-positive definite matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{T}$, and a positive scalar $0<\beta<1$ that satisfy the following generalized eigenvalue problem (GEVP)

$$\text{minimize} \quad \beta, \tag{7}$$

$$\text{subject to} \quad \hat{\boldsymbol{G}}=\begin{bmatrix}\hat{\boldsymbol{G}}_1 & \hat{\boldsymbol{G}}_2\\ * & \hat{\boldsymbol{G}}_3\end{bmatrix}, \tag{8}$$

$$\boldsymbol{K}<\beta\boldsymbol{R}, \tag{9}$$

where

$$\hat{\boldsymbol{G}}_1=\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}-\boldsymbol{K}-2\boldsymbol{C}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{C},$$

$$\hat{\boldsymbol{G}}_2=\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Lambda}-2\boldsymbol{C}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{D}+\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T},$$

$$\hat{\boldsymbol{G}}_3=\boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}+\boldsymbol{\Lambda}\boldsymbol{D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Lambda}-2\boldsymbol{D}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{D}$$
$$-2\boldsymbol{T}+\boldsymbol{D}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T}+\boldsymbol{T}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{D},$$

$$\boldsymbol{Q}=\mathrm{diag}(q_1,q_2,\cdots,q_L),$$

$$\boldsymbol{U}=\mathrm{diag}(u_1,u_2,\cdots,u_L),$$

then the origin of the discrete-time SNNM Eq.(3) is globally exponentially stable. Moreover,

$$\|\boldsymbol{x}(k)\|\leq\sqrt{\frac{\lambda_M(\boldsymbol{P})}{\lambda_m(\boldsymbol{P})}}\|\boldsymbol{x}(0)\|\mathrm{e}^{-\gamma k}, \tag{10}$$

where $\boldsymbol{P}=\mathrm{e}^{-2\gamma}\boldsymbol{R}$, $\beta=\mathrm{e}^{-2\gamma}$.

Proof Construct the following positive definite Lyapunov function

$$\hat{V}(\boldsymbol{x}_k)=\mathrm{e}^{2\gamma k}\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k+2\sum_{i=1}^{L}\lambda_i\sum_{j=0}^{k-1}\mathrm{e}^{2\gamma j}\phi_i[\xi_i(j)]\xi_i(j),$$

where $\boldsymbol{P}>0$, $\lambda_i\geq 0$. Thus, $\forall \boldsymbol{x}_k\neq 0$, $\hat{V}(\boldsymbol{x}_k)>0$ and $\hat{V}(\boldsymbol{x}_k)=0$ iff $\boldsymbol{x}_k=0$. The difference along the solution of the SNNM Eq.(3) is

$$\Delta\hat{V}(\boldsymbol{x}_k)=\mathrm{e}^{2\gamma(k+1)}\boldsymbol{x}_{k+1}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_{k+1}-\mathrm{e}^{2\gamma k}\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k$$
$$+2\mathrm{e}^{2\gamma k}\sum_{i=1}^{L}\lambda_i\phi_{k,i}\xi_{k,i}$$

$$=\mathrm{e}^{2\gamma k}\Big[\left(\boldsymbol{A}\boldsymbol{x}_k+\boldsymbol{B}\boldsymbol{\phi}_k\right)^{\mathrm{T}}\mathrm{e}^{2\gamma}\boldsymbol{P}\left(\boldsymbol{A}\boldsymbol{x}_k+\boldsymbol{B}\boldsymbol{\phi}_k\right)$$
$$-\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k+2\sum_{i=0}^{L}\lambda_i\phi_{k,i}\left(\boldsymbol{C}_i\boldsymbol{x}_k+\boldsymbol{D}_i\boldsymbol{\phi}_k\right)\Big] \tag{11}$$

Let $\boldsymbol{R}=\mathrm{e}^{2\gamma}\boldsymbol{P}$, Eq.(11) can be written as

$$\Delta\hat{V}(\boldsymbol{x}_k)=\boldsymbol{x}_k^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}-\mathrm{e}^{-2\gamma}\boldsymbol{R})\boldsymbol{x}_k$$
$$+\boldsymbol{x}_k^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Lambda})\boldsymbol{\phi}_k+\boldsymbol{\phi}_k^{\mathrm{T}}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}+\boldsymbol{\Lambda}\boldsymbol{C})\boldsymbol{x}_k$$
$$+\boldsymbol{\phi}_k^{\mathrm{T}}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}+\boldsymbol{\Lambda}\boldsymbol{D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Lambda})\boldsymbol{\phi}_k. \tag{12}$$

In light of Eq.(9), we can get

$$\Delta\hat{V}(\boldsymbol{x}_k)<\boldsymbol{x}_k^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}-\boldsymbol{K})\boldsymbol{x}_k+\boldsymbol{x}_k^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Lambda})\boldsymbol{\phi}_k$$
$$+\boldsymbol{\phi}_k^{\mathrm{T}}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}+\boldsymbol{\Lambda}\boldsymbol{C})\boldsymbol{x}_k+\boldsymbol{\phi}_k^{\mathrm{T}}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}+\boldsymbol{\Lambda}\boldsymbol{D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Lambda})\boldsymbol{\phi}_k^{\mathrm{T}}$$
$$=\begin{bmatrix}\boldsymbol{x}_k\\ \boldsymbol{\phi}_k\end{bmatrix}^{\mathrm{T}}\underbrace{\begin{bmatrix}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}-\boldsymbol{K} & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Lambda}\\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}+\boldsymbol{\Lambda}\boldsymbol{C} & \boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}+\boldsymbol{\Lambda}\boldsymbol{D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Lambda}\end{bmatrix}}_{\hat{T}_0}\begin{bmatrix}\boldsymbol{x}_k\\ \boldsymbol{\phi}_k\end{bmatrix}. \tag{13}$$

By the S-procedure (Boyd *et al.*, 1994) and Eq.(6), if there exists $\tau_i\geq 0$ (i=1, …, L), such that the following inequality holds

$$\hat{\boldsymbol{T}}_0-\sum_{i=1}^{L}\tau_i(\boldsymbol{T}_i^1+\boldsymbol{T}_i^2)$$
$$=\begin{bmatrix}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}-\boldsymbol{K} & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Lambda}\\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}+\boldsymbol{\Lambda}\boldsymbol{C} & \boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}+\boldsymbol{\Lambda}\boldsymbol{D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Lambda}\end{bmatrix}$$
$$-\begin{bmatrix}2\boldsymbol{C}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{C} & 2\boldsymbol{C}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{D}-\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T}\\ * & \begin{pmatrix}2\boldsymbol{D}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{D}+2\boldsymbol{T}\\ -\boldsymbol{D}^{T}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T}-\boldsymbol{T}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{D}\end{pmatrix}\end{bmatrix}$$
$$=\hat{\boldsymbol{G}}<0$$

where $\boldsymbol{T}$=diag(τ_1, τ_2, …, τ_L) and $\boldsymbol{T}\geq 0$, then $\hat{\boldsymbol{T}}_0<0$, that is, $\forall \boldsymbol{x}_k\neq 0$, $\Delta\hat{V}(\boldsymbol{x}_k)<0$ and $\Delta\hat{V}(\boldsymbol{x}_k)=0$ iff $\boldsymbol{x}_k=0$. So from Eq.(13), we have

$$\hat{V}(\boldsymbol{x}(k))\leq\hat{V}(\boldsymbol{x}(0))$$

However, $\hat{V}(\boldsymbol{x}(0))=\boldsymbol{x}(0)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(0)\leq\lambda_M(\boldsymbol{P})\|\boldsymbol{x}(0)\|^2$ and

$\hat{V}(\boldsymbol{x}(k)) \geq e^{2\gamma k}\boldsymbol{x}(k)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(k) \geq e^{2\gamma k}\lambda_m(\boldsymbol{P})\|\boldsymbol{x}(k)\|^2$, therefore we can get the convergence rates of the SNNM's states, i.e. Eq.(10). From Definition 1, we conclude that the origin of the SNNM Eq.(3) is global exponentially stable. We hope that the degree of exponential stability γ is maximal (or β is minimal) such that the SNNM Eq.(3) converges to the equilibrium point as fast as possible. It requires solving the generalized eigenvalue minimization problem Eqs.(7)~(9), which is a quasi-convex optimization problem and can be solved by using the MATLAB LMI Toolbox (Gahinet *et al*., 1995). Theorem 3 provides a simple method to determine the exponential stability of the SNNM Eq.(3) and get the optimal exponential convergence rate, and can be widely applied to stability analysis.

STABILITY ANALYSIS OF DISCRETE-TIME BAM NEURAL NETWORKS

To apply Theorem 2 and Theorem 3 to stability analysis of the discrete-time BAM network, we need to transform the BAM neural network Eq.(2) into the SNNM Eq.(3) and move the equilibrium point to the origin. System Eq.(2) can be reformulated by

$$z(k+1)=\boldsymbol{R}z(k)+\boldsymbol{S}\phi(\xi(k))+\boldsymbol{H},\ \xi(k)=z(k) \tag{14}$$

If z_{eq} is the equilibrium point of system Eq.(14), then it satisfies

$$z_{\mathrm{eq}}=\boldsymbol{R}z_{\mathrm{eq}}+\boldsymbol{S}\phi(z_{\mathrm{eq}})+\boldsymbol{H}.$$

To conduct a linear transformation on system Eq.(14), let $z'(k)=z(k)-z_{\mathrm{eq}}$, then we have

$$\begin{aligned} &z'(k+1)=\boldsymbol{R}z'(k)+\boldsymbol{S}\eta(\sigma(k)),\\ &\eta(\sigma(k))=\phi(\sigma(k)+z_{\mathrm{eq}})-\phi(z_{\mathrm{eq}}),\\ &\sigma(k)=z'(k)\end{aligned} \tag{15}$$

System Eq.(15) has formulation similar to that of system Eq.(14), but has an equilibrium point at the origin. The components of the nonlinear activation functions η

$$\eta_i[\sigma_i(k)]=\phi_i[\sigma_i(k)+z_{\mathrm{eq}i}]-\phi_i(z_{\mathrm{eq}i}) \quad (i=1, \ldots, n+m)$$

are different if $z_{\mathrm{eq}i}$ are different. Nevertheless, η_i keeps some properties of ϕ_i. In system Eq.(14), if the hyperbolic tangent is adopted for activation function ϕ_i, then $\eta_i[\sigma_i(k)]=\tanh[\sigma_i(k)+z_{\mathrm{eq}i}]-\tanh(z_{\mathrm{eq}i})$. If $z_{\mathrm{eq}}=0$, the sector for each function ϕ_i is [0, 1]. When $z_{\mathrm{eq}}\neq 0$, the sector becomes a subset of the former.

Let $\varphi_i(s)=\tanh(s+z_{\mathrm{eq}i})-\tanh(z_{\mathrm{eq}i})$, and according to the paper by Barabanov and Prokhorov (2002), the upper boundary of the sector can be calculated by

$$\begin{aligned}&u_i=\max\{\varphi_i(s)/s{:}s\neq 0\},\\ &U=\mathrm{diag}\{u_i\},\end{aligned}$$

and the lower boundary can be set to zero.

In this way, system Eq.(15) can be transformed into the form of the SNNM Eq.(3), where $\boldsymbol{A}=\boldsymbol{R}$, $\boldsymbol{B}=\boldsymbol{S}$, $\boldsymbol{C}=\boldsymbol{E}_{(n+m)\times(n+m)}$, $\boldsymbol{D}=0$, $L=n+m$. Furthermore, system Eq.(15) satisfies $\eta_i(\sigma_i(k))/\sigma_i(k)\in[q_i, u_i]$, $0\leq q_i<u_i$. So we can use Theorem 2 and Theorem 3 to analyze the global stability of system Eq.(15) or BAM neural network Eq.(2).

NUMERICAL EXAMPLES

To demonstrate the effectiveness of the proposed approach, we analyze the asymptotic stability and exponential stability of a discrete-time BAM neural network with 4 neurons. The dynamics of the network is as follows

$$\begin{cases} x_1(k+1)=0.7000x_1(k)+0.1500\tanh(y_1(k))\\ \qquad -0.0857\tanh(y_2(k))+0.3000\\ x_2(k+1)=0.7000x_2(k)-0.2000\tanh(y_1(k))\\ \qquad +0.1000\tanh(y_2(k))-0.3000\\ y_1(k+1)=0.7000y_1(k)-0.0750\tanh(x_1(k))\\ \qquad +0.1000\tanh(x_2(k))+0.6000\\ y_2(k+1)=0.7000y_2(k)+0.0429\tanh(x_1(k))\\ \qquad -0.0500\tanh(x_2(k))-0.6000 \end{cases} \tag{16}$$

We first convert system Eq.(16) into the form of Eq.(2), where

$$\begin{aligned}&z(k)=(x_1(k), x_2(k), y_1(k), y_2(k))^{\mathrm{T}},\\ &\boldsymbol{R}=\mathrm{diag}(0.7000, 0.7000, 0.7000, 0.7000),\\ &\boldsymbol{H}=(0.3, -0.3, 0.6, -0.6)^{\mathrm{T}},\end{aligned}$$

$$\boldsymbol{S}=\begin{bmatrix} 0 & 0 & 0.1500 & -0.0857 \\ 0 & 0 & -0.2000 & 0.1000 \\ -0.0750 & 0.1000 & 0 & 0 \\ 0.0429 & -0.0500 & 0 & 0 \end{bmatrix}.$$

Since $\boldsymbol{R}$ satisfies $\boldsymbol{E}-\boldsymbol{R}\neq 0$, this system has an equilibrium point $z_{eq}=(1.7150, -1.9088, 1.4467, -1.7066)^{\mathrm{T}}$ by Theorem 1. Calculate the boundaries of the sectors, $\boldsymbol{U}=\mathrm{diag}\{0.6384, 0.5988, 0.6998, 0.6402\}$, $\boldsymbol{Q}=\boldsymbol{0}_{4\times 4}$. Then with the help of the MATLAB LMI Toolbox (Gahinet *et al.*, 1995) we solve Eq.(4) where $\boldsymbol{A}=\boldsymbol{R}$, $\boldsymbol{B}=\boldsymbol{S}$, $\boldsymbol{C}=\boldsymbol{E}_{4\times 4}$, $\boldsymbol{D}=\boldsymbol{0}$, $L=4$, and have the solutions as

$$\boldsymbol{P}=\begin{bmatrix} 284.7176 & 28.9940 & 9.7500 & -8.4545 \\ 28.9940 & 240.4943 & -7.9115 & 7.6343 \\ 9.7500 & -7.9115 & 467.1979 & -44.1626 \\ -8.4545 & 7.6343 & -44.1626 & 401.7075 \end{bmatrix},$$

$$\boldsymbol{\Lambda}=\mathrm{diag}\{7.4250, 5.5691, 7.4701, 14.1602\},$$

$$\boldsymbol{T}=\mathrm{diag}\{59.6202, 62.5651, 105.4940, 95.03203\}.$$

Then we can state that the equilibrium point z_{eq} of the BAM network Eq.(16) is globally asymptotically stable according to Theorem 2. The state trajectory is shown in Fig.2. The result is independent of the initial states of the system. Compared with Theorem 2 and Theorem 3 in the paper by Jin (1999), our result is easier to obtain than theirs, since it is more difficult to find the solutions of the Lyapunov equations (parameter β, the matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$) in the results of the paper by Jin (1999).

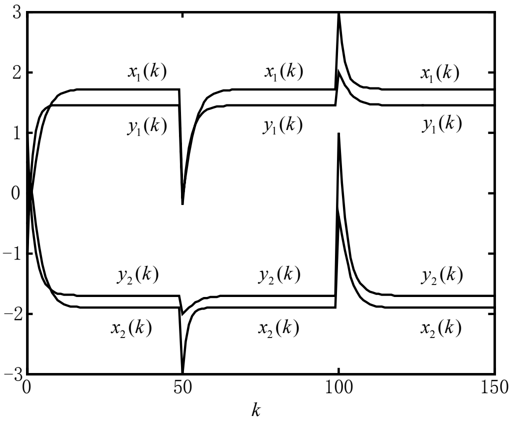

Fig.2 The state trajectories of the discrete-time BAM neural network with 4 neurons described by Eq.(16)
$x_1(k)$, $x_2(k)$, $y_1(k)$ and $y_2(k)$ are initialized arbitrarily at k=0, k=50 and k=100, respectively

Next, we solve the GEVP Eqs.(7)~(9), and obtain the solutions as

$$\beta=0.6019,$$

$$\boldsymbol{R}=10^{-5}\times\begin{bmatrix} 0.1457 & 0.0083 & 0.0012 & -0.0015 \\ 0.0083 & 0.1305 & -0.0003 & 0.0008 \\ 0.0012 & -0.0003 & 0.2765 & -0.0165 \\ -0.0015 & 0.0008 & -0.0165 & 0.2396 \end{bmatrix},$$

$$\boldsymbol{K}=10^{-5}\times\begin{bmatrix} 0.0847 & 0.0030 & 0.0007 & -0.0009 \\ 0.0030 & 0.0766 & -0.0002 & 0.0005 \\ 0.0007 & -0.0002 & 0.1645 & -0.0117 \\ -0.0009 & 0.0005 & -0.0117 & 0.1398 \end{bmatrix},$$

$$\boldsymbol{\Lambda}=10^{-7}\times\mathrm{diag}\{0.1376, 0.0972, 0.16041, 0.3766\},$$

$$\boldsymbol{T}=10^{-6}\times\mathrm{diag}\{0.3420, 0.3633, 0.6223, 0.6275\}.$$

So the equilibrium point z_{eq} of the BAM network Eq.(16) is also globally exponentially stable according to our Theorem 3. We can see it from the output waveforms of system Eq.(16) in Fig.2. However, Theorem 4 in the paper by Jin (1999) states that if the parameters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{W}$, and $\boldsymbol{V}$ in BAM neural network Eq.(1) satisfy the following condition

$$LF=\ln\left(\left\|\mathrm{diag}(\boldsymbol{A},\boldsymbol{B})\right\|\right)+\max\left\{\|\boldsymbol{W}\|,\|\boldsymbol{V}\|\right\}<0,$$

then the equilibrium point of Eq.(1) is globally exponentially stable. For example, we consider the following BAM network with 4 neurons

$$\begin{cases} x_1(k+1)=-0.7000x_1(k)+0.1500\tanh(y_1(k)) \\ \qquad -0.0857\tanh(y_2(k))+0.3000 \\ x_2(k+1)=-0.7000x_2(k)-0.2000\tanh(y_1(k)) \\ \qquad +0.1000\tanh(y_2(k))-0.3000 \\ y_1(k+1)=-0.7000y_1(k)-0.0750\tanh(x_1(k)) \\ \qquad +0.1000\tanh(x_2(k))+0.6000 \\ y_2(k+1)=-0.7000y_2(k)+0.0429\tanh(x_1(k)) \\ \qquad -0.0500\tanh(x_2(k))-0.6000 \end{cases} \tag{17}$$

Computing LF=−0.0742<0 of system Eq.(17), we can conclude that the system Eq.(17) is globally exponentially stable by Theorem 4 in the paper by Jin (1999). But we can see it is not exponentially stable from the output waveforms of system Eq.(17) in Fig.3. So Theorem 4 in the paper by Jin (1999) is wrong.

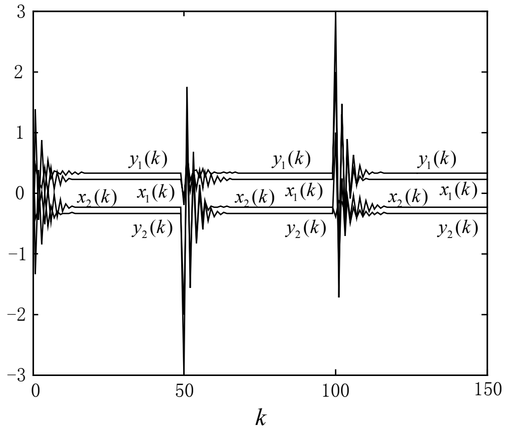

Fig.3 The state trajectories of the discrete-time BAM neural network with 4 neurons described by Eq.(17)
$x_1(k)$, $x_2(k)$, $y_1(k)$ and $y_2(k)$ are initialized arbitrarily at k=0, k=50 and k=100, respectively

CONCLUSION

The standard neural network model (SNNM) proposed in this paper provides an easier approach to stability analysis of the BAM neural network. We transform the discrete-time BAM neural network into the form of an SNNM. By using the MATLAB LMI Toolbox, we can confirm the global asymptotic stability and global exponential stability of the SNNM and then of the discrete-time BAM neural network. The method is easy to apply and less conservative, which makes it feasible for engineering applications. In principle, this approach is extendable to other types of recurrent neural networks, such as recurrent multiplayer perceptrons (Liu and Zhang, 2003). Furthermore, we give some exponential stability conditions represented as GEVP, and estimate the exponential convergence rates for these RNNs. It should be pointed out that our Theorem 2 and Theorem 3 provide only sufficient conditions for global stability. This means that if we cannot get a feasible solution to the LMIs, the stability of the system is unknown. In this case, we can reduce the intensity of the hetero-association or the size of the sector area to get feasible solutions of LMIs. Unfortunately, this also degrades the performance of the discrete-time BAM neural networks. Therefore, our research will be directed to enhancing both stability and performance of the BAM neural networks.

References

Barabanov, N.E., Prokhorov, D.V., 2002. Stability analysis of discrete-time recurrent neural networks. *IEEE Trans. on Neural Networks*, **13**(2):292-303.

Boyd, S.P., Ghaoui, L.E., Feron, E., Balakrishnan, V., 1994. Linear Matrix Inequalities in System and Control Theory. SIAM, Philadelphia, PA.

Cao, J.D., Wang, L., 2002. Exponential stability and periodic oscillatory solution in BAM networks with delays. *IEEE Trans. on Neural Networks*, **13**(2):457-463.

Gahinet, P., Nemirovski, A., Laub, A.J., Chilali, M., 1995. LMI Control Toolbox−For Use with Matlab. The Math Works Inc., Natick, MA.

Jin, C., 1999. Stability analysis of discrete-time Hopfield BAM neural networks. *Acta Automatica Sinica*, **25**(5):606-612 (in Chinese).

Kosko, B., 1987. Adaptive bidirectional associative memories. *Appl. Opt.*, **26**(23):4947-4960.

Liu, M.Q., Zhang, S.L., 2003. Stability analysis of a class of discrete-time recurrent neural networks: an LMI approach. *Journal of Zhejiang University* (*Engineering Science*), **37**(1):19-23 (in Chinese).

Smart, D.R., 1980. Fixed Point Theorems. Cambridge University Press, Cambridge.

Wang, C.C., Don, H.S., 1995. An analysis of high-capacity discrete exponential BAM. *IEEE Trans. on Neural Networks*, **6**(2):492-496.

Xu, B.Z., Zhang, B.L., Kwong, C.P., 1992. Asymptotic Stability Analysis of Continuous Bidirectional Associative Memory Networks. IEEE International Conference on Systems Engineering, Kobe, Japan, p.572-575.

Xu, Z.X., Zheng, C.Y., Ye, Z., Xie, M.P., 1999. Complex-valued multistate bidirectional associative memory. *Acta Electronica Sinica*, **27**(5):118-120 (in Chinese).

Zhang, S.L., Liu, M.Q., 2005. LMI-based approach for global asymptotical stability analysis of continuous BAM neural networks. *Journal of Zhejiang University SCIENCE*, **6A**(1):32-37.

Zhang, B.L., Xu, B.Z., Kwong, P.K., 1993. Performance analysis of the bidirectional associative memory and an improved model from the matched-filtering viewpoint. *IEEE Trans. on Neural Networks*, **4**(5):864-872.