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On Turán type inequality with doubling weights and A^* weights

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Abstract: Let H_n be the set of real algebraic polynomials of degree n, whose zeros all lie in the interval $[-1,1]$. The well known Turán type inequalities tell us that for $f(x)\in H_n$, it holds $\|f'\|\geq C\sqrt{n}\|f\|$. This note deals with the weighted Turán type inequalities with the weights having inner singularities under L^p norm for $0<p\leq\infty$. Our results essentially extend the result of Wang and Zhou (2002), and the method used in this paper is simpler and more direct than that of Wang and Zhou (2002). The results and methods have their own values in approximation theory and computation.

Key words: Turán type inequality, Doubling weights, A^* weights
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INTRODUCTION

Let H_n be the set of real algebraic polynomials of degree n, whose zeros all lie in the interval $[-1,1]$. As we know, when the magnitude of the derivative of a real polynomial is considered, the so-called Bernstein inequality, Markov inequality, Turán type inequality, etc. are very useful and important.

It was Turán who first established the so-called Turán type inequality in 1939. He proved that for $f\in H_n$, $\|f'\|\geq C\sqrt{n}\|f\|$. It should be noted that the Turán inequality gives lower estimate for derivative of polynomials, while Bernstein inequality and Markov inequality give the upper estimate. Since Turán's initial work, there was great development in this direction (Bakenko and Pichugov, 1986a; 1986b; Milovanović and Rassias, 1998; Eröd, 1939; Varma, 1976; 1978; 1983; Wang and Zhou, 2002); Zhou (1992; 1993; 1995). Among them, Zhou (1995) established the following good results:

Theorem Zh (Theorem 4 in Zhou (1995)) If $f\in H_n$, then for $0<p\leq q\leq\infty$, $1-1/p+1/q\geq 0$,

$$\left(\int_{-1}^{1}|f'(x)|^p\mathrm{d}x\right)^{1/p}\geq C^*(\sqrt{n})^{1-1/p+1/q}\left(\int_{-1}^{1}|f(x)|^q\mathrm{d}x\right)^{1/q}$$

where C^* denotes a positive absolute constant in the case $1\leq p\leq\infty$, or a positive constant which may depend on p but not on n for $0<p<1$.

Let $f(x)=(1-x^2)^k$, $k=[n/2]$, then it is obvious that the above inequality cannot be improved as the degree is considered.

It is very interesting and important to consider the extension of the Turán inequality to the weighted cases. Xiao and Zhou (1999) did the initial work by establishing the weighted Turán inequality under uniform norm for some special weights-continuous piecewise monotone functions. The most recent work in this direction was that of Wang and Zhou (2002), who established the following

Theorem WZ Let $W(x)\in GJW$, $0<p<\infty$. If $f\in H_n$, then there exists a positive constant $C_{W,p}$ only depending upon $W(x)$ and p such that

$$\left(\int_{-1}^{1}|f'(x)|^p W(x)\mathrm{d}x\right)^{1/p} \geq C_{W,p}\sqrt{n}\left(\int_{-1}^{1}|f(x)|^p W(x)\mathrm{d}x\right)^{1/p}$$

holds for sufficiently large n.

GJW in Theorem WZ denotes the class of so-called generalized Jacobi weights, that is, we say $W(x)\in GJW$, if $W(x)\geq 0$, $\int_{-1}^{1}W(x)\mathrm{d}x<\infty$ and $W(x_1)\sim W(x_2)$ for any $-1<x_1<x_2\leq 0$ and $|x_2-x_1|<1+x_1$ or for any $0\leq x_2<x_1\leq 1$ and $|x_2-x_1|<1-x_1$. It is obvious that all Jacobi weights $W(x)=(1+x)^{\alpha}(1-x)^{\beta}$, $\alpha,\beta>-1$ belong to *GJW*. However, if $W(x)$ has a zero in $(-1,1)$, then $W(x)$ will vanish in the whole $(-1,1)$. Therefore, this type *GJW* can even not include another kind of so-called generalized Jacobi weights of the form

$$W(x)=h(x)\prod_{j=1}^{k}|x-x_j|^{\gamma_j},\ \gamma_j>-1,\ x,x_j\in[-1,1], \quad (1)$$

where h is a positive measurable function bounded away from zero and infinity.

Recently, weighted polynomial inequalities with weights having inner singularity have been researched extensively because of its important roles in approximation theory and harmonic analysis. For example, Mastroianni and Totik (2000), and Erdélyi (1999), consider weighted inequalities such as Bernstein, Nikolskii, Remez, etc., inequalities with the doubling weights and A_∞ weights under L^p norm. In this note, we consider the Turán type inequality with doubling weights and A^* weights. We say an integrable nonnegative function W is a doubling weight if it satisfies the so-called doubling condition

$$W(2I)\leq LW(I) \quad (2)$$

for all intervals I, where L is a constant independent of I, $2I$ is the interval twice the length of I and with midpoint at the midpoint of I (note that parts of $2I$ may lie outside $[-1,1]$, where we set W=0), and $W(I)=\int_I W(u)\mathrm{d}u$.

We also need the following average of $W(x)$:

$$W_n(x)=\frac{1}{\Delta_n(x)}\int_{x-\Delta_n(x)}^{x+\Delta_n(x)}W(u)\mathrm{d}u,$$

where $\Delta_n(x)=\frac{1}{n^2}+\frac{\sqrt{1-x^2}}{n}$.

It is well known that the generalized Jacobi weights of Eq.(1) satisfy the doubling condition. Furthermore, there are nonzero doubling weights that vanish on a set of positive measure (Mastroianni and Totik, 2000). Obviously, any nonzero doubling weight having inner zeros cannot be a *GJW* weight. Thus, there comes a natural problem whether any $W(x)\in GJW$ must be a doubling weight? It is still an open problem.

MAIN RESULT

Our main results are as follows:

Theorem 1 Let W be a doubling weight defined as in Eq.(2), $0<p<\infty$, then for any $f(x)\in H_n$, there is a positive constant $C_{L,p}$, such that

$$\left(\int_{-1}^{1}|f'(x)|^p W_n(x)\mathrm{d}x\right)^{1/p} \geq C_{L,p}\sqrt{n}\left(\int_{-1}^{1}|f(x)|^p W_n(x)\mathrm{d}x\right)^{1/p}$$

holds for sufficiently large n. Here and throughout the paper, $C_{L,p}$ denotes a positive constant only depending on L in the case $1\leq p<\infty$, or a positive constant which may depend on L and p for $0<p<1$. As usual, their values may be different in different situation.

By Theorem 7.2 in (Mastroianni and Totik, 2000) for $1\leq p<\infty$ and the related results and discussion by Erdélyi (1999) for $0<p<1$, we have

Corollary 1 Let W be a doubling weight defined as in Eq.(2), $0<p<\infty$, then for any $f(x)\in H_n$, there is a positive constant $C_{L,p}$ such that

$$\left(\int_{-1}^{1}|f'(x)|^p W(x)\mathrm{d}x\right)^{1/p} \geq C_{L,p}\sqrt{n}\left(\int_{-1}^{1}|f(x)|^p W(x)\mathrm{d}x\right)^{1/p}$$

holds for sufficiently large n.

Theorem 1 is an essential extension of Theorem WZ and the method used in the present paper is simpler and more direct than that of Wang and Zhou (2002). For L^∞ case, a natural assumption is that W is bounded from above. As Mastroianni and Totik (2000) discussed for Bernstein inequality, we introduce the so-called A^* weights which are defined for those satisfying the property: there is a constant L such that for all interval $I\subset[-1,1]$ and $x\in[-1,1]$ we have

$$W(x)\le L\frac{1}{|I|}\int_I W(u)\mathrm{d}u.$$

It is obvious that any A^* weight must be a doubling weight (Mastroianni and Totik, 2000).

Theorem 2 Let W be an A^* weight, then for any $f(x)\in H_n$, there is a positive constant C_L only depending on L such that

$$\|f'W_n\|\ge C_L\sqrt{n}\|fW_n\|$$

holds for sufficiently large n.

By Theorem 2 and Eq.(7.27) in (Mastroianni and Totik, 2000), we have

Corollary 2 Let W be an A^* weight, then for any $f(x)\in H_n$, there is a positive constant C_L only depending on L such that

$$\|f'W\|\ge C_L\sqrt{n}\|fW\|$$

holds for sufficiently large n.

PROOF OF RESULTS

Denote by $-1\le x_1<x_2<\ldots<x_k\le 1$ all the distinct zeros of $f(x)\in H_n$, and by l_i the multiplicity of x_i, $1\le i\le k$. Let α_j be the maximum point of $|f(x)|$ between (x_j, x_{j+1}), $1\le j<k$. It is easy and useful to observe that, for $x\in[x_j, \alpha_j]$ (or $x\in[\alpha_j, x_{j+1}]$), $|f(x)|$ is increasing (or decreasing). Write

$$m(x)=\frac{f'(x)}{f(x)}=\sum_{i=1}^{k}\frac{l_i}{x-x_i},\quad d_j=|m'(\alpha_j)|^{-1}.$$

Lemma 1 (Lemma 1(3) in (Wang and Zhou, 2002))

For $1\le j\le k-1$, it holds

$$\sqrt{d_j}\le\min_{1\le i\le k}\left\{\left|\alpha_j-x_i\right|,2n^{-1/2}\right\}. \tag{3}$$

If $x\in\left(x_j,\alpha_j-\sqrt{d_j}/8\right)\cup\left(\alpha_j+\sqrt{d_j}/8,x_{j+1}\right)$, then

$$|m(x)|\ge\frac{2}{25}\sqrt{d_j}^{-1} \tag{4}$$

If $x\in\left(\alpha_j-\sqrt{d_j}/4,\alpha_j+\sqrt{d_j}/4\right)$, then

$$|m(x)|\le\frac{4}{9}\sqrt{d_j}^{-1} \tag{5}$$

Proof of Theorem 1 We divide the proof of Theorem 1 into the following Lemmas.

Lemma 2 For $j=1, 2, \ldots, k-1$, it holds that

$$\int_{x_j}^{\alpha_j}|f'(x)|^p W_n(x)\mathrm{d}x \ge\left(\frac{8}{225}\right)^p L^{-2}n^{p/2}\int_{x_j}^{\alpha_j}|f(x)|^p W_n(x)\mathrm{d}x.$$

Proof By Eqs.(3) and (4), we get

$$\begin{aligned}\int_{x_j}^{\alpha_j-\sqrt{d_j}/4}|f'(x)|^p W_n(x)\mathrm{d}x &=\int_{x_j}^{\alpha_j-\sqrt{d_j}/4}|f(x)m(x)|^p W_n(x)\mathrm{d}x\\ &\ge\left(\frac{2}{25}\right)^p\sqrt{d_j}^{-p}\int_{x_j}^{\alpha_j-\sqrt{d_j}/4}|f(x)|^p W_n(x)\mathrm{d}x\\ &\ge\left(\frac{1}{25}\right)^p n^{p/2}\int_{x_j}^{\alpha_j-\sqrt{d_j}/4}|f(x)|^p W_n(x)\mathrm{d}x\end{aligned} \tag{6}$$

For any $x\in\left[\alpha_j-\sqrt{d_j}/4,\alpha_j+\sqrt{d_j}/4\right]$, without loss of generality, assume that $x\in\left[\alpha_j-\sqrt{d_j}/4,\alpha_j\right]$, then by mean value theorem, there is a $\xi\in(x,\alpha_j)\subset\left[\alpha_j-\sqrt{d_j}/4,\alpha_j\right]$ such that

$$\left|f(\alpha_j)-f(x)\right|=\left|f(\xi)m(\xi)\right|\left|\alpha_j-x\right|\le\frac{1}{9}\left|f(\alpha_j)\right| \tag{7}$$

where Eq.(5) is used in the last inequality. By Eq.(7), we have

$$\left|f(x)\right|\ge\frac{8}{9}\left|f(\alpha_j)\right|. \tag{8}$$

By Eqs.(4) and (8), we get

$$\int_{\alpha_j-\sqrt{d_j}/4}^{\alpha_j+\sqrt{d_j}/8}\left|f'(x)\right|^p W_n(x)\mathrm{d}x$$
$$=\int_{\alpha_j-\sqrt{d_j}/4}^{\alpha_j+\sqrt{d_j}/8}\left|f(x)m(x)\right|^p W_n(x)\mathrm{d}x$$
$$\geq\left(\frac{16}{225}\right)^p \sqrt{d_j}^{-p}\left|f(\alpha_j)\right|^p\int_{\alpha_j-\sqrt{d_j}/4}^{\alpha_j-\sqrt{d_j}/8}W_n(x)\mathrm{d}x$$
$$\geq\left(\frac{16}{225}\right)^p L^{-2}\sqrt{d_j}^{-p}\left|f(\alpha_j)\right|^p\int_{\alpha_j-\sqrt{d_j}/4}^{\alpha_j}W_n(x)\mathrm{d}x$$
$$\geq\left(\frac{8}{225}\right)^p L^{-2}n^{p/2}\int_{\alpha_j-\sqrt{d_j}/4}^{\alpha_j}\left|f(x)\right|^p W_n(x)\mathrm{d}x \qquad (9)$$

where in the second inequality, we used the property: If $W(x)$ is a doubling weight, then so is W_n for every n with a doubling constant independent of n (Mastroianni and Totik, 2001).

Combining Eqs.(6)~(9) leads to Lemma 2.

Exactly as the way in Lemma 2, we have

Lemma 3 For j=1, 2, …, k−1, it holds

$$\int_{\alpha_j}^{x_{j+1}}\left|f'(x)\right|^p W_n(x)dx$$
$$\geq\left(\frac{8}{225}\right)^p L^{-2}n^{p/2}\int_{\alpha_j}^{x_{j+1}}\left|f(x)\right|^p W_n(x)\mathrm{d}x$$

Lemma 4 If $f(1)\neq 0$, then

$$\int_{x_k}^{1}\left|f'(x)\right|^p W_n(x)\mathrm{d}x\geq\left(\frac{n}{2}\right)^p\int_{x_k}^{1}\left|f(x)\right|^p W_n(x)\mathrm{d}x.$$

If $f(-1)\neq 0$, then

$$\int_{-1}^{x_1}\left|f'(x)\right|^p W_n(x)\mathrm{d}x\geq\left(\frac{n}{2}\right)^p\int_{-1}^{x_1}\left|f(x)\right|^p W_n(x)\mathrm{d}x.$$

Proof It can be obtained directly by noting that for $x\in[-1, x_1]$ or $x\in(x_k, 1)$, it holds $|m(x)|\geq n/2$.

Combining Lemma 2~Lemma 4, completes Theorem 1.

Proof of Theorem 2 Assume that

$$|f(\alpha)W_n(\alpha)|=\|fW_n\|,\ \alpha\in[-1,1],$$

α_j, d_j, j=1, 2, …, k−1 defined as before.

We consider the following many cases.

Case 1 There is a j, $1\leq j\leq k-1$, such that $\alpha\in[x_j, x_{j+1}]$.

Case 1.1 If $\alpha\in\left[\alpha_j-\sqrt{d_j}/4,\alpha_j\right]$, then by Eqs.(4), (8), the doubling property and the A^* property, we get

$$\|f'W_n\|\geq\frac{8}{\sqrt{d_j}}\int_{\alpha_j-\sqrt{d_j}/4}^{\alpha_j-\sqrt{d_j}/8}\left|f'(x)\right|W_n(x)\mathrm{d}x$$
$$\geq C\sqrt{d_j}^{-2}|f(\alpha_j)|\int_{\alpha_j-\sqrt{d_j}/4}^{\alpha_j-\sqrt{d_j}/8}W_n(x)\mathrm{d}x$$
$$\geq C_L\sqrt{d_j}^{-2}|f(\alpha_j)|\int_{\alpha_j-\sqrt{d_j}/4}^{\alpha_j}W_n(x)\mathrm{d}x$$
$$\geq C_L\sqrt{d_j}^{-1}|f(\alpha_j)|W_n(\alpha)$$
$$\geq C_L\sqrt{d_j}^{-1}|f(\alpha)|W_n(\alpha)$$
$$\geq C_L\sqrt{n}|f(\alpha)|W_n(\alpha),$$

where we also used the obvious fact (Mastroianni and Totik, 2001): If $W(x)$ is an A^* weight, then $W_n(x)$ is an A^* weight too.

Case 1.2 If $\alpha\in\left[\alpha_j,\alpha_j+\sqrt{d_j}/4\right]$, then it can be treated in a similar way as that for Case 1.1.

Case 1.3 If $\alpha\in\left[x_j,\alpha_j-\sqrt{d_j}/4\right]$, then by Eq.(4), we get

$$\|f'W_n\|\geq|f'(\alpha)|W_n(\alpha)$$
$$=|m(\alpha)f(\alpha)|W_n(\alpha)\geq\frac{1}{25}\sqrt{n}\|fW_n\|.$$

Case 1.4 If $\alpha\in\left(\alpha_j+\sqrt{d_j}/4,x_{j+1}\right)$, similar to the Cases 1.3, we can deduce that the wanted result also holds.

Case 2 If $x_k\neq 1$ and $\alpha\in(x_k, 1)$ or $x_1\neq 1$ and $\alpha\in(-1, x_1)$, then by the obvious fact

$$|m(x)|\geq n/2,\quad x\in(x_k,1]\ \text{or}\ x\in[-1,x_1),$$

again, we have

$$\|f'W_n\|\geq|f'(\alpha)|W_n(\alpha)\geq\frac{n}{2}\|fW_n\|.$$

We finished Theorem 2 by combining Case 1 and Case 2.

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