Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

Science Letters:

Analytical solution for fixed-end beam subjected to uniform load*

DING Hao-jiang (丁皓江)[†1], HUANG De-jin (黄德进)[1,2], WANG Hui-ming (王惠明)[1]

([1]Department of Civil Engineering, Zhejiang University, Hangzhou 310027, China)
([2]Faculty of Engineering, Ningbo University, Ningbo 315211, China)
[†]E-mail: dinghj@zju.edu.cn

Received June 17, 2005; revision accepted July 3, 2005

Abstract: A bi-harmonic stress function is constructed in this work. Ariy stress function methodology is used to obtain a set of analytical solutions for both ends fixed beams subjected to uniform load. The treatment for fixed-end boundary conditions is the same as that presented by Timoshenko and Goodier (1970). The solutions for propped cantilever beams and cantilever beams are also presented. All of the analytical plane-stress solutions can be obtained for a uniformly loaded isotropic beam with rectangular cross section under different types of classical boundary conditions.

Key words: Analytical solution, Fixed-end beam, Stress Function
doi:10.1631/jzus.2005.A0779 **Document code:** A **CLC number:** O343.1

INTRODUCTION

The plane stress problem of beams is very classical in elasticity theory and is encountered frequently in practical cases. Timoshenko and Goodier (1970) investigated isotropic beams for different cases, such as tension, shearing, pure bending, bending of a cantilever by transverse load at the end, bending of a simply supported beam by uniform load and other cases of continuously loaded beams. Lekhnitskii (1968) studied the anisotropic beams problem including tension, shearing, pure bending, bending of a cantilever loaded at the end, bending of simply supported beams and cantilever beams by uniform load or linearly distributed load. Jiang and Ding (2005) investigated orthotropic cantilever beams subjected to uniform load. For beams fixed at both ends subjected to uniform load, Gere and Timoshenko (1984) presented the deflection and stress expressions with Euler-Bernoulli beam theory. Ahmed *et al.*(1996) presented a numerical solution of fixed-end deep beams. To the authors' knowledge, no literature on the analytical elasticity solution for both ends fixed beams acted on uniform load has been published yet. Stress function methodology was used to investigate fixed-end beams in plane stress subjected to uniform load, and obtain the stress and displacements expressions. The solutions for propped cantilever beam and cantilever beam are also presented.

BASIC EQUATIONS IN PLANE STRESS STATE

In the absence of the body force, the stress components can be expressed with stress function ϕ as follows

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2},\quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2},\quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{1}$$

where stress function ϕ satisfies Eq.(2):

*Project (No. 10472102) supported by the National Natural Science Foundation of China

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=0 \tag{2}$$

The relations between displacement and stress are as Eq.(3):

$$\begin{aligned}&\frac{\partial u}{\partial x}=\frac{\sigma_x}{E}-\mu\frac{\sigma_y}{E},\ \frac{\partial v}{\partial y}=\frac{\sigma_y}{E}-\mu\frac{\sigma_x}{E},\\ &\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\frac{2(1+\mu)}{E}\tau_{xy}\end{aligned} \tag{3}$$

where E and μ are Young's modulus and Poisson ratio, respectively.

STRESS AND DISPLACMENT

Consider a fixed-end beam with unit width rectangular cross section subjected to a uniform load q as shown in Fig.1. The length of the beam is l and height h. Take the stress function in the following form of a bi-harmonic polynomial with 7 terms

$$\phi=a\left(\frac{1}{5}y^5-x^2y^3\right)+bxy^3+cy^3+dy^2+ex^2y+fxy+gx^2 \tag{4}$$

where a, b, c, d, e, f and g are unknown constants to be determined. The substitution of Eq.(4) into Eq.(1) gives

$$\sigma_x=2a(2y^3-3x^2y)+6bxy+6cy+2d \tag{5}$$

$$\sigma_y=-2ay^3+2ey+2g \tag{6}$$

$$\tau_{xy}=6axy^2-3by^2-2ex-f \tag{7}$$

Substituting Eqs.(5) and (6) into the first two equations in Eq.(3), and then integrating them with respect to x and y, respectively, yields the displacement components u and v as

$$\begin{aligned}u=&\frac{1}{E}\Big[2a(2+\mu)xy^3-(2ax^3-3bx^2-6cx\\&+2\mu ex)y+2dx-2\mu gx\Big]+A(y)\end{aligned} \tag{8}$$

$$\begin{aligned}v=&\frac{1}{E}\bigg[-\frac{a}{2}(1+2\mu)y^4+\left[3a\mu x^2-3b\mu x-3\mu c+e\right]y^2\\&-2d\mu y+2gy\bigg]+B(x)\end{aligned} \tag{9}$$

where $A(y)$ and $B(x)$ are functions of x and y, respectively. Substituting Eqs.(8), (9) and (7) into the third equation in Eq.(3), we have

$$\begin{aligned}&\frac{1}{E}\left[3b(2+\mu)y^2+2(1+\mu)f\right]+A'(y)\\&+\frac{1}{E}\left[-2ax^3+3bx^2+6cx+2(2+\mu)ex\right]+B'(x)=0\end{aligned} \tag{10}$$

Eqs.(11) and (12) can be obtained from Eq.(10),

$$\frac{1}{E}\left[3b(2+\mu)y^2+2(1+\mu)f\right]+A'(y)=\omega \tag{11}$$

$$\frac{1}{E}\left[-2ax^3+3bx^2+6cx+2(2+\mu)ex\right]+B'(x)=-\omega \tag{12}$$

where ω is an arbitrary constant. It is noted that, $A(y)$ and $B(x)$ can be obtained by integrating Eqs.(11) and (12). Substituting $A(y)$ and $B(x)$ into Eqs.(8) and (9), we have

$$\begin{aligned}u=&\frac{1}{E}\left[2a(2+\mu)xy^3-(2ax^3-3bx^2-6cx+2\mu ex)y\right.\\&\left.+2dx-2\mu gx\right]-\frac{1}{E}\left[b(2+\mu)y^3+2(1+\mu)fy\right]+\omega y+u_0,\end{aligned} \tag{13}$$

$$\begin{aligned}v=&\frac{1}{E}\bigg[-\frac{a}{2}(1+2\mu)y^4+\left[3a\mu x^2-3b\mu x-3\mu c+e\right]y^2\\&-2d\mu y+2gy\bigg]+\frac{1}{E}\left[\frac{1}{2}ax^4-bx^3-3cx^2-(2+\mu)ex^2\right]\\&-\omega x+v_0\end{aligned} \tag{14}$$

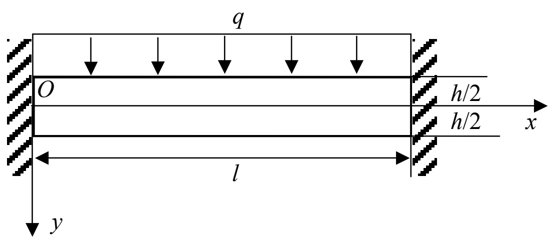

Fig.1 Fixed-end beam subjected to uniform load

where arbitrary constants u_0, v_0 and ω denote the translation and rotation of rigid body, respectively.

FIXED-END BEAM SUBJECTED TO UNIFORM LOAD

Timoshenko and Goodier (1970) presented two methods for dealing with the boundary conditions for fixed-end beams. Both of them will be considered. The first method is to treat the boundary conditions as, (1) $y=h/2$, $\sigma_y=0$, (2) $y=-h/2$, $\sigma_y=-q$, (3) $y=\pm h/2$, $\tau_{xy}=0$, (4) $x=0$, $y=0$ point and $x=l$, $y=0$ point, $u=v=0$, $\partial v/\partial x=0$. By substituting the stress components Eqs.(6), (7) and displacement components Eqs.(13), (14) into corresponding boundary conditions, 10 algebraic equations can be obtained and all the unknown constants can be determined as

$$a=q/h^3,\ b=ql/h^3,\ c=-q/2h-ql^2/6h^3-q\mu/4h,\ d=-q\mu/4,\ e=3q/4h,\ f=-3ql/4h,\ g=-q/4,\ u_0=0,\ v_0=0,\ \omega=0 \tag{15}$$

Substituting Eq.(15) into Eq.(5), (6), (7), (13) and (14), the stress and displacement components are then obtained

$$\sigma_x=-\frac{q}{2J}\left(x^2-lx+\frac{l^2}{6}\right)y+\frac{q}{24J}\left[8y^3-3(2+\mu)h^2y-\mu h^3\right] \tag{16}$$

$$\sigma_y=-\frac{q}{24J}(4y^3-3h^2y+h^3) \tag{17}$$

$$\tau_{xy}=\frac{q}{4J}(l-2x)\left(\frac{h^2}{4}-y^2\right) \tag{18}$$

$$u=-\frac{qxy}{12EJ}(l-x)(l-2x)+\frac{q(l-2x)y}{24EJ}\left[3(1+\mu)h^2-2(2+\mu)y^2\right] \tag{19}$$

$$v=\frac{q}{24EJ}(l-x)^2x^2+\frac{q}{48EJ}\left\{-2(1+2\mu)y^4+\left[(6x^2-6lx+l^2)2\mu+3h^2(\mu+1)^2\right]y^2+24J(\mu^2-1)y\right\}, \tag{20}$$

where $J=h^3/12$.

The stress components and deflection expressions obtained with Euler-Bernoulli beam theory are as follows (Gere and Timoshenko, 1984):

$$\sigma_x=-\frac{q}{2J}\left(x^2-lx+\frac{l^2}{6}\right)y \tag{21}$$

$$\tau=\frac{q}{4J}(l-2x)\left(\frac{h^2}{4}-y^2\right) \tag{22}$$

$$v=\frac{q}{24EJ}(l-x)^2x^2 \tag{23}$$

It can be found that the first part of the normal stress expression Eq.(16) coincides with Eq.(21) actually, and that the second part in Eq.(16) is the correction term. We also found that the shear stress Eq.(18) is the same as Eq.(22). In Eq.(20), letting $y=0$, we obtain the deflection expression which is the same as Eq.(23) of Euler-Bernoulli beam theory.

The second method to treat fixed-end boundary conditions is to substitute $\partial u/\partial y=0$ for $\partial v/\partial x=0$ at $x=0$, $y=0$ point and $x=l$, $y=0$ point. We thus resolve the problem to obtain

$$\sigma_x=-\frac{q}{2J}\left(x^2-lx+\frac{l^2}{6}\right)y+\frac{q}{3J}y^3+\frac{q}{24J}\mu h^2(3y-h) \tag{24}$$

$$\sigma_y=-\frac{q}{24J}(4y^3-3h^2y+h^3) \tag{25}$$

$$\tau_{xy}=\frac{q}{4J}(l-2x)\left(\frac{h^2}{4}-y^2\right) \tag{26}$$

$$u=\frac{qy}{Eh^3}(l-2x)[(x-l)x-(2+\mu)y^2] \tag{27}$$

$$v=\frac{q}{24EJ}(l-x)^2x^2+\frac{q}{8EJ}(2\mu y^2-h^2\mu-h^2)x^2+\frac{ql}{8EJ}(-2\mu y^2+h^2+h^2\mu)x+\frac{q}{48EJ}\left[-2(1+2\mu)y^4+(3h^2+2\mu l^2-3h^2\mu^2)y^2-2(1-\mu^2)h^3y\right] \tag{28}$$

Ahmed *et al.*(1996) investigated fixed-end deep beams subjected to uniform load with finite-difference technique. The material constants applied here are $E=2\times10^{11}$ N/m^2, $\mu=0.3$. The boundary conditions are treated as, (1) $x=0$, l, $-h/2\le y\le h/2$, $u=v=0$, (2) $y=h/2$, $\sigma_y=\tau_{xy}=0$, (3) $y=-h/2$, $\sigma_y=-6\times10^7$ N/m^2, $\tau_{xy}=0$. They presented the deflection curve at

y/h=0.04 for l/h=2. Fig.2 shows the deflection curve obtained by using Eqs.(20) and (28) at y/h=0.04, and presented the curves altogether in Fig.2, where the dashed line is the deflection curve of Eq.(28), the dash-dot line is that of Eq.(20) and the solid line is that obtained by Ahmed *et al.*(1996). We find that the numerical result by Ahmed *et al.*(1996) locates between the two analytical solution obtained in this paper. Correctly, the differences among the curves in Fig.2 denote that the different treatments for fixed end will display different constrain effects in physics. The two obtained analytical solutions are two important approximation for practical application.

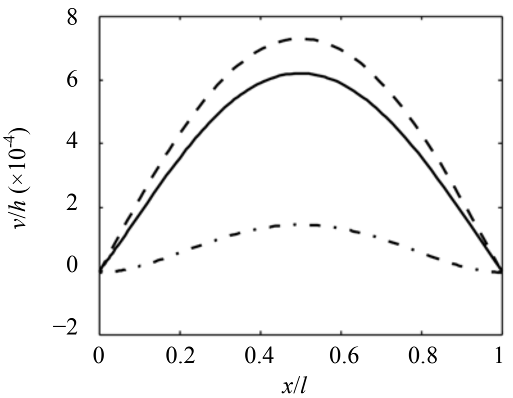

Fig.2 Deflection curves at y/h=0.04 in different boundary condition (l/h=2)

PROPPED CANTILEVER BEAM SUBJECTED TO UNIFORM LOAD

In this cases, boundary condition are, (1) $y=h/2$, $\sigma_y=0$; (2) $y=-h/2$, $\sigma_y=-q$; (3) $y=\pm h/2$, $\tau_{xy}=0$; (4) $x=0$, $N=0$, $M=0$, and $x=y=0$ point, $v=0$; (5) $x=l$, $y=0$ point, $u=v=0$, $\partial v/\partial x=0$, where

$$N=\int_{-h/2}^{h/2}\sigma_x\,\mathrm{d}y=2hd \tag{29}$$

$$M=\int_{-h/2}^{h/2}\sigma_x y\,\mathrm{d}y=\frac{h^3}{2}\left[\frac{h^2}{10}a+c-ax^2+bx\right], \tag{30}$$

By following the above-mentioned procedure, we obtain

$$a=q/h^3,\quad b=\frac{3q}{40h^3l}\left[10l^2-(5\mu+8)h^2\right]$$

$$c=-\frac{q}{10h},\ d=0,\ e=\frac{3q}{4h},\ g=-\frac{q}{4}$$

$$f=\frac{9q}{32hl}\left(\frac{5\mu+8}{5}h^2-2l^2\right),\ u_0=-\frac{\mu ql}{2E},\ v_0=0,$$

$$\omega=-\frac{ql}{480EJ}\left[10l^2+3(5\mu+8)h^2\right]. \tag{31}$$

the obtained components of stress and displacement are

$$\sigma_x=\frac{q}{8J}xy(3l-4x)+\frac{q}{240J}y\left(80y^2-9(8+5\mu)h^2\frac{x}{l}-12h^2\right) \tag{32}$$

$$\sigma_y=-\frac{q}{24J}(4y^3-3h^2y+h^3) \tag{33}$$

$$\tau_{xy}=\frac{q}{16J}(3l-8x)\left(\frac{h^2}{4}-y^2\right)-\frac{3(8+5\mu)qh^2}{160Jl}\left(\frac{h^2}{4}-y^2\right) \tag{34}$$

$$u=\frac{q}{960EJl}\Big\{\left[160(2+\mu)xl+30\mu^2h^2-60\mu l^2+108\mu h^2+96h^2-120l^2\right]y^3-2\left[80x^3l+9(8h^2+5\mu h^2-10l^2)x^2+12(2+5\mu)lh^2x\right]y-\left[9(8+13\mu+5\mu^2)h^4+2(10l^2-21h^2-30h^2\mu)l^2\right]y\Big\}+\frac{q\mu}{2E}x-\frac{\mu ql}{2E} \tag{35}$$

$$v=\frac{qx}{48EJ}(2x^3-3lx^2+l^3)+\frac{(8+5\mu)qh^2}{160EJl}(l-x)^2x-\frac{(1+2\mu)q}{24EJ}y^4+\frac{q}{160EJl}\left[40\mu x^2l+3(5\mu h^2-10l^2+8h^2)\mu x+2(2\mu+5)h^2l\right]y^2-\frac{q}{2E}y \tag{36}$$

The components of stress and displacement obtained by using Euler-Bernoulli beam theory are

$$\sigma_x=\frac{q}{8J}xy(3l-4x) \tag{37}$$

$$\tau=\frac{q}{16J}(3l-8x)\left(\frac{h^2}{4}-y^2\right) \tag{38}$$

$$v=\frac{qx}{48EJ}(2x^3-3x^2+l^3) \tag{39}$$

The first part of stress σ_x in Eq.(32) coincides with Eq.(37), while the second part of Eq.(32) is the correction term. Also the first part in Eq.(34) coincides with Eq.(38), and the second part is correction term. Letting y=0 in Eq.(36) we then obtain the deflection of beam. Correspondingly, the first part of Eq.(36) coincides with Eq.(39), and the second part is correction term.

CANTILEVER BEAM SUBJECTED TO UNIFORM LOAD

The boundary conditions are, (1) $y=h/2$, $\sigma_y=0$; (2) $y=-h/2$, $\sigma_y=-q$; (3) $y=\pm h/2$, $\tau_{xy}=0$; (4) $x=0$, $N=0$, $M=0$, $Q=0$; (5) $x=l$, $y=0$ point, $u=v=0$, $\partial v/\partial x=0$,
in which

$$Q=\int_{-h/2}^{y}\tau_{xy}\,\mathrm{d}y=\frac{ah^3}{2}x-\frac{bh^3}{4}-2ehx-fh \quad (40)$$

With similar procedure, we obtain the undetermined constants

$$a=q/h^3,\ b=0,\ c=-q/10h,\ d=0,$$

$$e=3q/4h,\ f=0,\ g=-q/4,\ u_0=-\mu ql/2E$$

$$v_0=\frac{ql^2\left[10l^2-(8+5\mu)h^2\right]}{80EJ}$$

$$\omega=\frac{ql(20l^2-15\mu h^2-24h^2)}{120EJ} \quad (41)$$

Substituting these constants into Eqs.(5), (6), (7), (13) and (14), we obtain the stress and displacement expressions of a cantilever beam acted on uniform load, which coincide with degenerated forms in Lekhnitskii (1968).

References

Ahmed, S.R., Idris, B.M., Uddin, M.W., 1996. Numerical solution of both ends fixed deep beams. *Computer & Structures*, **61**(1):21-29.

Gere, J.M., Timoshenko, S.P., 1984. Mechanics of Materials. PWS-KENT Publishing Company, Boston.

Jiang, A.M., Ding, H.J., 2005. The analytical solutions for orthotropic cantilever beams (I): Subjected to surface-forces. *Journal of Zhejinag University SCIENCE*, **6A**(2):126-131.

Lekhnitskii, S.G., 1968. Anisotropic Plate. Gordon and Breach, New York.

Timoshenko, S.P., Goodier, J.N., 1970. Theory of Elasticity, 3rd Edition. McGraw Hill, New York.