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A symplectic eigensolution method in transversely isotropic piezoelectric cylindrical media*

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Abstract: This paper reports establishment of a symplectic system and introduces a 3D sub-symplectic structure for transversely isotropic piezoelectric media. A complete space of eigensolutions is obtained directly. Thus all solutions of the problem are reduced to finding eigenvalues and eigensolutions, which include zero-eigenvalue solutions and all their Jordan normal form of the corresponding Hamiltonian matrix and non-zero-eigenvalue solutions. The classical solutions are described by zero-eigen-solutions and non-zero-eigensolutions show localized solutions. Numerical results show some rules of non-zero-eigenvalue and their eigensolutions.

Key words: Symplectic method, Hamiltonian system, Transverse isotropic, Piezoelectric media, Eigensolution
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INTRODUCTION

Since the phenomenon of piezoelectricity in natural crystals was discovered in 1880, it has been applied in many fields, such as smart structure, sensing devices and active control, because of its special properties. Combining electrical behavior with mechanical deformation is the special property which attracted great interest of researchers. Many models have been developed for studying piezoelectric effects. Dunn and Wienecke (1996) used Green's functions for transversely isotropic piezoelectric solids and solved some problems. Ding *et al*.(1996; 2000; 2002), and Ding and Chen (2001), and Chen (1999) conducted a series of exact analyses of 3D piezoelasticity problems, studied the mechanical behaviors of cylinder under the effects of piezoelectricity and different conditions and obtained some solutions. The researches above are under the Lagrange system. Zhong (1995) introduced a symplectic space method based on the conservative Hamiltonian system to solve the elastic problem which is different from the traditional semi-inverse solution method. The complete solutions space can be obtained and a satisfactory solution can be obtained under the boundary conditions. Leung and Xu (2005) obtained 2D solutions for transversely isotropic media with the symplectic method. This paper introduces a symplectic method to a three dimensional problem.

SYMPLECTIC SYSTEM AND HAMILTONIAN DUAL EQUATIONS

Consider an anisotropic piezoelectric cylinder, which is transversely isotropic and anisotropic in the longitudinal direction. The circular cylindrical coordinate (γ, θ, z) is selected such that the z-axis is along the longitudinal direction with the origin is located at

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the central point of the cross section. Let σ_{ij}, D_i, and $\boldsymbol{q}=\{u, v, \omega, \varphi\}^{\mathrm{T}}$ be the components of stress, electric displacement and displacements (φ is the electrical potential function) respectively. Relations of stress-displacement and electric displacement-electrical potential function can be described as

$$\begin{cases}\sigma_{rr}=c_{11}\partial_r u+c_{12}(\partial_\theta v+u)/r+c_{13}\dot{w}+e_{31}\dot{\varphi};\\ \sigma_{r\theta}=c_{66}(\partial_r v-v/r+\partial_\theta u/r);\\ \sigma_{\theta\theta}=c_{12}\partial_r u+c_{11}(\partial_\theta v+u)/r+c_{13}\dot{w}+e_{31}\dot{\varphi};\\ D_r=e_{15}(\partial_r w+\dot{u})-\varepsilon_{11}\partial_r\varphi;\\ \sigma_{zz}=c_{13}\partial_r u+c_{13}(\partial_\theta v+u)/r+c_{33}\dot{w}+e_{33}\dot{\varphi};\\ D_\theta=e_{15}(\partial_\theta w/r+\dot{v})-\varepsilon_{11}\partial_\theta\varphi/r;\\ \sigma_{rz}=c_{44}(\partial_r w+\dot{u})+e_{15}\partial_r\varphi;\\ D_z=e_{31}\partial_r u+e_{31}(\partial_\theta v+u)/r+e_{33}\dot{w}-\varepsilon_{33}\dot{\varphi};\\ \sigma_{\theta z}=c_{44}(\partial_\theta w/r+\dot{v})+e_{15}\partial_\theta\varphi/r\end{cases}\tag{1}$$

where c_{ij}, e_{ij} and ε_{ij} are elastic stiffness and piezoelectric and dielectric constants, respectively, and the over-dot represents differential with respect to z, namely $(\dot{\ })=(\partial/\partial z)()$, which z coordinate is looked upon phonily as time coordinate. The potential energy density is

$$\begin{aligned}U=r\Big\{&c_{11}(\partial_r u)^2+c_{11}(\partial_\theta v+u)^2/r^2+c_{33}\dot{w}^2\\&+2c_{12}(\partial_r u)(\partial_\theta v+u)/r+2c_{13}(\partial_r u)\dot{w}\\&+2c_{13}\dot{w}(\partial_\theta v+u)/r+c_{44}(\partial_\theta w/r+\dot{v})^2\\&+c_{44}(\partial_r w+\dot{u})^2+c_{66}(\partial_r v-v/r+\partial_\theta u/r)^2\\&+2e_{31}(\partial_r u)\dot{\varphi}+2e_{31}\dot{\varphi}(\partial_\theta v+u)/r+2e_{33}\dot{w}\dot{\varphi}\\&+2e_{15}(\partial_\theta w/r+\dot{v})\partial_\theta\varphi/r+2e_{15}(\partial_r w+\dot{u})(\partial_r\varphi)\\&-\varepsilon_{11}(\partial_r\varphi)^2-\varepsilon_{11}(\partial_\theta\varphi)^2/r^2-\varepsilon_{33}\dot{\varphi}^2\Big\}/2\end{aligned}\tag{2}$$

The Lagrange function is

$$L(u,v,w,\varphi)=U(u,v,w,\varphi)-u\,f_r-v\,f_\theta-w\,f_z+\varphi q\tag{3}$$

where $\{f_r, f_\theta, f_z\}^{\mathrm{T}}$ represent the external body forces and q the density of free charges. The governing equations and conditions of the lateral boundary in the Lagrange system can be obtained by the principle of minimum potential energy of Eq.(3) where the equations are expressed in terms of displacements. The displacement method has only one kind of variable and belongs to the Lagrange formulation. For introducing Hamiltonian system, the dual vector $\boldsymbol{p}$ can be given as

$$\boldsymbol{p}=\frac{\partial L}{\partial\dot{\boldsymbol{q}}}=\begin{Bmatrix}r[c_{44}(\partial_r w+\dot{u})+e_{15}\partial_r\varphi]\\ r[c_{44}(w'/r+\dot{v})+e_{15}\varphi'/r]\\ r[c_{13}(\partial_r u+v'/r+u/r)+c_{33}\dot{w}+e_{15}\dot{\varphi}]\\ r[e_{31}(\partial_r u+v'/r+u/r)+e_{33}\dot{w}-\varepsilon_{33}\dot{\varphi}]\end{Bmatrix}=[r\sigma_{rz}\quad r\sigma_{\theta z}\quad r\sigma_{zz}\quad rD_z]^{\mathrm{T}}\tag{4}$$

The physical meaning of the dual variable $\boldsymbol{p}$ is related to explicitly the stresses and electric displacement in the z-direction on the basis of the mutually dual vectors $\boldsymbol{q}$ and $\boldsymbol{p}$, the Hamiltonian function can be introduced as

$$H(\boldsymbol{q},\boldsymbol{p})=\boldsymbol{p}^{\mathrm{T}}\dot{\boldsymbol{q}}-L(\boldsymbol{q},\dot{\boldsymbol{q}})\tag{5}$$

The variational equation has the form

$$\delta\int_\Omega[\boldsymbol{p}^{\mathrm{T}}\dot{\boldsymbol{q}}-H(\boldsymbol{q},\boldsymbol{p})]\,\mathrm{d}\Omega=0\tag{6}$$

The dual equations for Hamiltonian system can be obtained directly as

$$\begin{cases}\dot{u}=-\partial_r w-a_1\partial_r\varphi+a_2p_1/r\\ \dot{v}=-w'/r-a_1\varphi'/r+a_2p_2/r\\ \dot{w}=-a_3(\partial_r u+u/r+v'/r)+a_4p_3/r+a_5p_4/r\\ \dot{\varphi}=a_6(\partial_r u+u/r+v'/r)+a_5p_3/r-a_7p_4/r\\ \dot{p}_1=-a_8(r\partial_r^2u+\partial_r u-u/r)+a_9u''/r\\ \qquad+a_{10}(\partial_r v'-v'/r)-a_3(\partial_r p_3-p_3/r)\\ \qquad+a_6(\partial_r p_4-p_4/r)+rf_r\\ \dot{p}_2=-a_{10}\partial_r u'-a_8v''/r-a_{11}\partial_r u'/r-a_9(r\partial_r^2v\\ \qquad+\partial_r v-v/r)-a_3p_3'+a_6p_4'+rf_\theta\\ \dot{p}_3=-\partial_r p_1-p_2'/r+rf_z\\ \dot{p}_4=a_{12}(r\partial_r^2\varphi+\partial_r\varphi+\varphi/r)-a_1\partial_r p_1-a_1p_2'/r+rq\end{cases}\tag{7}$$

where $()'=\partial_\theta()$, $a_0=1/(e_{33}^2+c_{33}\varepsilon_{33})$, $a_1=e_{15}/c_{44}$, $a_2=1/c_{44}$, $a_3=(c_{13}\varepsilon_{33}+e_{31}e_{33})a_0$, $a_4=\varepsilon_{33}a_0$, $a_5=e_{33}a_0$, $a_6=(e_{13}c_{33}+c_{31}e_{33})a_0$, $a_7=c_{33}a_0$, $a_8=c_{11}-(c_{13}^2\varepsilon_{33}+2e_{31}e_{33}c_{13}-e_{31}^2c_{33})a_0$, $a_9=c_{66}$, $a_{10}=a_8-c_{11}+c_{12}+a_9$, $a_{11}=a_{10}+c_{11}-c_{12}$, $a_{12}=\varepsilon_{11}+e_{15}^2a_2$ and $a_{13}=a_{10}-a_9$.

At the same time, the corresponding conditions of the lateral accompanying boundary can be given as

$$\begin{cases} [a_8 r\partial_r u + a_{12}u + a_{12}\partial_\theta v + a_3 p_2 - a_6 p_3]l \\ \qquad + a_9[\partial_\theta u + r\partial_r v - v]m = 0, \\ a_9[\partial_\theta u + r\partial_r v - v]l + [a_{13} r\partial_r u + a_8 u \\ \qquad + a_8\partial_\theta v + a_3 p_2 - a_6 p_3]m = 0, \\ p_1 l + p_2 m = 0, \\ \dfrac{\mathrm{d}\varphi}{\mathrm{d}n}\Big|_{\partial\Omega} = 0. \end{cases} \tag{8}$$

In fact, it is the external normal stress, shear stress and electric displacement in the surface traction at the lateral boundary where the outward normal $\boldsymbol{n}$ of its boundary $\partial\Omega$ has direction cosines (l, m).

ZERO EIGENVALUE SOLUTIONS

Let the state vetor $\boldsymbol{\psi}=\{\boldsymbol{q},\boldsymbol{p}\}^{\mathrm{T}}$, Eq.(7) are rewritten as

$$\dot{\boldsymbol{\psi}} = \boldsymbol{H}\boldsymbol{\psi} + \boldsymbol{f} \tag{9}$$

The solution of Eq.(9) can be divided into two parts: the general solution of the homogeneous equations and a special solution of non-homogeneous equations. In discussing the general problem, the homogeneous equations and the traction free natural boundary conditions are considered only below. Suppose there are neither external body forces nor surface traction at the lateral boundary, except at the two ends z=0,l. The solution of the homogeneous equation Eq.(9) can be solved as

$$\boldsymbol{\psi} = \boldsymbol{\psi}_j(r,\theta)\,\mathrm{e}^{\mu_j z} \tag{10}$$

where $\boldsymbol{\psi}_j$ and μ_j are eigenpair, or eigenvector and eigenvalue, which have a specified quality or characteristic, for example, adjoint symplectic orthogonality relationship and so on (Zhong, 1995).

Consider the problem of zero-eigenvalue, or μ=0. Besides direct eigensolutions, $\boldsymbol{\psi}_i^{(0)}$, of $\boldsymbol{H}\boldsymbol{\psi}$=0, Jordan form principal vectors can be determinated by $\boldsymbol{H}\boldsymbol{\psi}_i^{(n+1)}=\boldsymbol{\psi}_i^{(n)}$ and the solution of the original problem can be expressed as

$$\boldsymbol{\eta}_i^{(n+1)} = \boldsymbol{\psi}_i^{(n+1)} + z\boldsymbol{\psi}_i^{(n)} + z^2\boldsymbol{\psi}_i^{(n-1)}/2 + z^3\boldsymbol{\psi}_i^{(n-2)}/6 \\ + \cdots + z^{(n+1)}\boldsymbol{\psi}_i^{(0)}/(n+1)! \tag{11}$$

Direct zero-eigenvalue eigensolutions and Jordan form solutions can be obtained as

$$\begin{cases} \boldsymbol{\eta}_1^{(0)}=\boldsymbol{\psi}_1^{(0)}=\{\sin\theta \quad \cos\theta \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^{\mathrm{T}}, \\ \boldsymbol{\eta}_2^{(0)}=\boldsymbol{\psi}_2^{(0)}=\{-\cos\theta \quad \sin\theta \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^{\mathrm{T}}, \\ \boldsymbol{\eta}_3^{(0)}=\boldsymbol{\psi}_3^{(0)}=\{0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^{\mathrm{T}}, \\ \boldsymbol{\eta}_4^{(0)}=\boldsymbol{\psi}_4^{(0)}=\{0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0\}^{\mathrm{T}}, \\ \boldsymbol{\eta}_5^{(0)}=\boldsymbol{\psi}_5^{(0)}=\{0 \quad r \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^{\mathrm{T}}. \end{cases} \tag{12}$$

$$\begin{cases} \boldsymbol{\psi}_1^{(1)}=\{0 \quad 0 \quad -r\sin\theta \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^{\mathrm{T}}, \\ \boldsymbol{\psi}_2^{(1)}=\{0 \quad 0 \quad -r\cos\theta \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^{\mathrm{T}}, \\ \boldsymbol{\psi}_3^{(1)}=\{-a_{15}r \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad a_{16}r \quad a_{17}r\}^{\mathrm{T}}, \\ \boldsymbol{\psi}_4^{(1)}=\{a_{18}r \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad a_{19}r \quad -a_{20}r\}^{\mathrm{T}}, \\ \boldsymbol{\psi}_5^{(1)}=\{0 \quad 0 \quad \phi \quad 0 \quad r\partial_r\phi/a_2 \quad (\partial_\theta\phi + r^2)/a_2 \quad 0 \quad 0\}^{\mathrm{T}}. \end{cases} \tag{13}$$

$$\begin{cases} \boldsymbol{\psi}_1^{(2)}=\{a_{15}r^2\cos\theta \quad a_{15}r^2\sin\theta \quad 0 \quad 0 \quad 0 \quad 0 \\ \qquad -a_{16}r^2\cos\theta \quad -a_{17}r^2\cos\theta\}^{\mathrm{T}} \\ \boldsymbol{\psi}_2^{(2)}=\{a_{15}r^2\sin\theta \quad a_{15}r^2\cos\theta \quad 0 \quad 0 \quad 0 \quad 0 \\ \qquad -a_{16}r^2\sin\theta - a_{17}r^2\sin\theta\}^{\mathrm{T}} \end{cases} \tag{14}$$

$$\begin{cases} \boldsymbol{\psi}_1^{(3)}=\{0 \quad 0 \quad \varPsi_1 - a_1\varPhi_1 + a_{21}r^3\cos\theta \\ \qquad \varPhi_1 + a_{22}r^3\cos\theta r \quad \partial_r\varPsi_1 + a_{23}r^2\cos\theta \\ \qquad \partial_\theta\varPsi_1 + a_{24}r^2\sin\theta \quad 0 \quad 0\}^{\mathrm{T}} \\ \boldsymbol{\psi}_2^{(3)}=\{0 \quad 0 \quad \varPsi_1 - a_1\varPhi_1 + a_{21}r^3\sin\theta \\ \qquad \varPhi_1 + a_{22}r^3\sin\theta r\partial_r\varPsi_1 + a_{23}r^2\sin\theta \\ \qquad \partial_\theta\varPsi_1 + a_{24}r^2\cos\theta \quad 0 \quad 0\}^{\mathrm{T}} \end{cases} \tag{15}$$

where ϕ, $\varPhi_i$ and $\varPsi_i$ are determinate harmonic functions, $a_{14}=2a_3^2a_7 + 2a_6^2a_4 + 4a_3a_5a_6 - (a_5^2 + a_4a_7)(a_8 + a_{13})$, $a_{15}=2(a_5a_6-a_3a_7)/a_{14}$, $a_{16}=(a_7a_8+a_7a_{13}+a_6^2)/a_{14}$, $a_{17}=(2a_3a_6+a_5a_8+a_5a_{13})/a_{14}$, $a_{18}=2(a_3a_5+a_4a_6)/a_{14}$, $a_{19}=[2a_3a_6+a_5(a_8+a_{13})]/a_{14}$, $a_{20}=(a_4a_6 - a_3^2)/a_{14}$,

$a_{21}=(a_2a_{12}a_{16}+a_1a_{17}-a_1^2a_{16}-2a_{12}a_{15})/(6a_{12})$, $a_{22}=(a_1a_{16}-a_{17})/(6a_{12})$, $a_{23}=(a_2a_{16}-a_{15})/(2a_2)$ and $a_{24}=a_{15}/(2a_2)$.

The geometrical interpretation of these solutions is the rigid translation of coordinates, the rigid translation of electric displacement, the simple extension deformations, which the external electric displacement does not act on the ends but forces and the displacement induced by uniform electric field in which there does not exist any external force, free torsion, pure bendings and shearing-bendings. These solutions satisfy relationships of the adjoint symplectic orthogonal.

SUB-SYMPLECTIC SYSTEM AND NON-ZERO EIGENVALUE SOLUTIONS

Consider a circular cylinder, with radius a. For solving the eigenequation of non-zero eigenvalue problem, a Hamiltonian system sub-symplectic structure is introduced again. Let θ coordinate be taken in analogy to the time coordinate also. The Lagrange function and the dual variables can be obtained similarly. In sub-system, eigensolutions can be expressed as

$$\zeta=\zeta_n(r)\mathrm{e}^{\beta_n\theta} \tag{16}$$

Based on continuous conditions of the boundary (θ=0 and θ=2π), eigenvalue $\beta_n=ni$. With the aid of dual Eq.(7), eigensolutions, $\zeta_n(r)$, can be shown by Bessel functions. The eigensolutions of Eq.(7) are

$$\bar{u}=-\frac{in}{r}C_0J_n\left(\frac{r\mu}{s_0}\right)+\sum_{j=1}^{3}C_j\left[J_{n-1}\left(\frac{r\mu}{s_j}\right)-\frac{n}{r}J_n\left(\frac{r\mu}{s_j}\right)\right],$$

$$\bar{v}=C_0\left[\frac{\mu}{s_0}J_{n-1}\left(\frac{r\mu}{s_0}\right)-\frac{n}{r}J_n\left(\frac{r\mu}{s_0}\right)\right]+\frac{in}{r}\sum_{j=1}^{3}C_jJ_n\left(\frac{r\mu}{s_j}\right),$$

$$\bar{w}=\mu\sum_{j=1}^{3}C_jK_{1j}J_n\left(\frac{r\mu}{s_j}\right),\qquad \bar{\varphi}=\mu\sum_{j=1}^{3}C_jK_{2j}J_n\left(\frac{r\mu}{s_j}\right),$$

$$\bar{p}_1=-\mu a_{25}\frac{in}{r}C_0J_n\left(\frac{r\mu}{s_0}\right)+\mu\sum_{j=1}^{3}C_jK_{3j}\left[\frac{\mu}{s_j}J_{n-1}\left(\frac{r\mu}{s_j}\right)-\frac{n}{r}J_n\left(\frac{r\mu}{s_j}\right)\right],$$

$$\bar{p}_2=\mu a_{25}C_0\left[\frac{\mu}{s_0}J_{n-1}\left(\frac{r\mu}{s_0}\right)-\frac{n}{r}J_n\left(\frac{r\mu}{s_0}\right)\right]+\mu\frac{in}{r}\sum_{j=1}^{3}C_jK_{3j}J_n\left(\frac{r\mu}{s_j}\right),$$

$$\bar{p}_3=\mu^2\sum_{j=1}^{3}C_jK_{4j}J_n\left(\frac{r\mu}{s_j}\right),\ \bar{p}_4=\mu^2\sum_{j=1}^{3}C_jK_{5j}J_n\left(\frac{r\mu}{s_j}\right) \tag{17}$$

where $a_{25}=1/a_2$, $s_0=a_2a_9$, $K_{1i}=\alpha_i\beta_i$, $K_{2i}=\alpha_i\gamma_i$, $K_{3i}=a_{25}(1+K_{1i})+e_{15}K_{2i}$, $K_{4i}=c_{33}K_{1i}+e_{33}K_{2i}-c_{13}/s_i^2$, $K_{5i}=e_{33}K_{1i}+\varepsilon_{33}K_{2i}-e_{31}/s_i^2$, $\alpha_i=1/(a_{26}s_i^2-a_{27}s_i^4)$, $\beta_i=c_{11}\varepsilon_{11}-a_{28}s_i^2+c_{44}\varepsilon_{33}s_i^4$, $\gamma_i=c_{11}e_{15}-a_{29}s_i^2+c_{44}e_{33}s_i^4$ (i=1,2,3), $a_{26}=\varepsilon_{11}a_{30}+e_{15}a_{31}$, $a_{27}=\varepsilon_{33}a_{30}+e_{33}a_{31}$, $a_{28}=c_{11}\varepsilon_{33}+c_{44}\varepsilon_{11}+a_{31}^2$, $a_{29}=c_{11}e_{33}+c_{44}e_{15}-a_{30}a_{31}$, $a_{30}=c_{13}+c_{44}$, $a_{31}=e_{31}+e_{15}$, C_i (i=0,1,2,3) are undetermined constants and s_i^2 $(i=1,2,3)$ are three roots of the equation, $a_{32}s^3+a_{33}s^2+a_{34}s+a_{35}=0$, in which $a_{32}=a_{25}/a_0$, $a_{33}=c_{33}(a_{25}\varepsilon_{11}+a_{31}^2)+\varepsilon_{33}(c_{11}c_{33}+a_{25}^2-a_{30}^2)+e_{33}(2a_{25}e_{15}+c_{11}e_{33}-2a_{30}a_{31})$, $a_{34}=a_{25}\times(c_{11}\varepsilon_{33}+a_{31}^2)+\varepsilon_{11}(c_{11}c_{33}+a_{25}^2-a_{30}^2)+e_{15}(a_{25}e_{15}+2c_{11}e_{33}-2a_{30}a_{31})$ and $a_{35}=c_{11}(\varepsilon_{11}a_{25}+e_{15}^2)$. Substituting the solutions Eq.(17) into conditions of the lateral boundary Eq.(8), yields

$$\boldsymbol{A}\boldsymbol{c}=0 \tag{18}$$

where $\boldsymbol{c}=\{C_0\ C_1\ C_2\ C_3\}^{\mathrm{T}}$ and components of the matrix $\boldsymbol{A}$ are respectively $A_{21}=0$, $A_{41}=0$, $A_{11}=(a_{12}-a_8)ni\times[\mu J_{n-1}(a\mu/s_0)/s_0-nJ_n(a\mu/s_0)]$, $A_{1(m+1)}=a_8\mu A_{2(m+1)}+[(a_8-a_{12})n^2+\mu^2(a_3K_{4m}-a_6K_{5m})]J_n(a\mu/s_m)]$, $A_{2(m+1)}=\mu J_{n-2}(a\mu/s_m)/s_m^2+(1-2n)J_{n-1}(a\mu/s_m)/s_m$, $A_{31}=-a_{25}niJ_n(a\mu/s_0)$, $A_{l(m+1)}=K_{(6-l)m}[\mu J_{n-1}(a\mu/s_m)/s_m-nJ_n(a\mu/s_m)]$ (l=3,4; m=1,2,3). Consider the condition of non-zero-solution of Eq.(18), the eigenvalue μ representing the decay rates can be determined from

$$|\boldsymbol{A}|=0 \tag{19}$$

The eigenvalues, which are obtained by Eq.(19), are of infinite number and denoted by μ_m (m=1,2,3,…) and eigensolutions are given by Eqs.(18) and (17). Finally, the solutions of the problem can be linear combinations of eigenfunctions of zero-eigenvalues and of non-zero-eigenvalues respectively.

NUMERICAL RESULTS

For numerical solutions, the non-dimensional form is considered. Let $R=r/a$, $Z=z/a$, $U=u/a$, $V=v/a$, $W=w/a$, $\Phi=\varphi/(a\times10^{10}$ N/c), $P_1=p_1/c_{11}$, $P_2=p_2/c_{11}$, $P_3=p_3/c_{11}$ and $P_4=p_4/(c_{11}\times10^{-10}$ c/N). The material parameters, show some proportional relations, for example $\varepsilon_{11}:\varepsilon_{33}\approx1:1$ and $-e_{31}:e_{15}:e_{33}\approx1:2.5:3$. We are interested in the piezoelectric characteristic and consider transversely isotropic elastic parameters are nearly constant, or $c_{11}=12\times10^{10}$ N/m^2, $c_{13}=0.625c_{11}$, $c_{33}=0.917c_{11}$ and $c_{44}=0.250c_{11}$. Introduce piezoelectric and dielectric characteristic constants, with e being 3 to 8 and ε 30 to 150. So the piezoelectric constants are

$$e_{31}=-0.083e[c_{11}/(a\times10^{10}\text{ N/c})],$$
$$e_{15}=0.208e[c_{11}/(a\times10^{10}\text{ N/c})],$$
$$e_{33}=0.250e[c_{11}/(a\times10^{10}\text{ N/c})]$$

and dielectric constants are

$$\varepsilon_{11}=\varepsilon_{33}=0.083\varepsilon(c_{11}\text{ Fm/c}^2)$$

respectively. We discuss the real parts of eigenvalue $-\mu$, which is the decaying coefficient, since the eigenvalues μ and $-\mu$ come in pairs. The effects of the piezoelectric parameter (e) and dielectric characteristic constant (ε) are described in Fig.1 and Fig.2 respectively, in which the solid line, dash line and dot dashed line show n=0, n=1 and n=2. The figures give curves of numerical eigenvalues (decay rates).

In terms of eigenvalues are obtained by Eq.(19), the corresponding eigensolutions can be shown by Eqs.(17) and (18). Since the electrical potential function φ and the electric displacement $p_4=D_z$ are important in the problem, Fig.3 and Fig.4 give two components of eigensolutions respectively. In the figures, (a)~(c) correspond to the first three eigenvalues. Graph of eigensolutions are depicted clearly.

CONCLUSION

High order equations of partial differentiation in

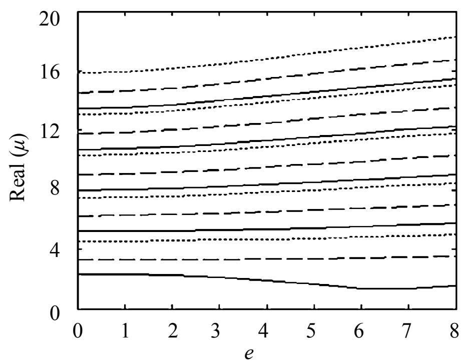


Fig.1 The effects of the piezoelectric parameter

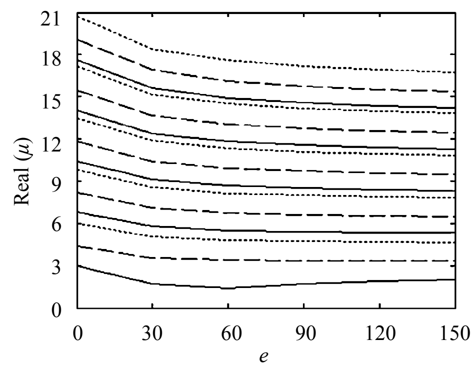
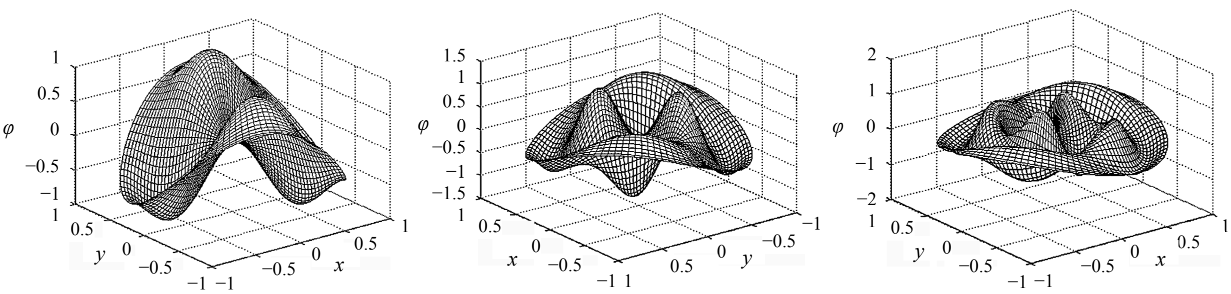


Fig.2 The effects of the dielectric parameter



(a) (b) (c)

Fig.3 Electrical potential function φ (n=2)

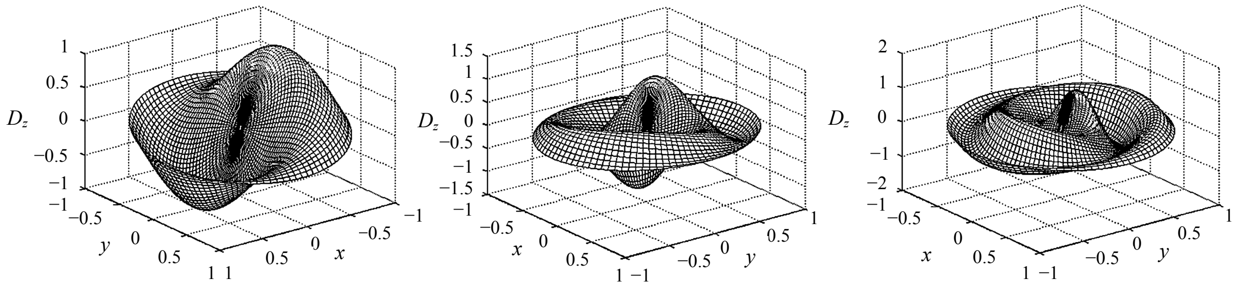


Fig.4 Electric displacement $p_4=D_z$ (n=1)

the symplectic space of the Hamiltonian system are reduced in order and dimension with the aid of the symplectic system and the sub-symplectic system. The eigensolutions of transversely isotropic piezoelectric media can be obtained and symplectic solutions space is complete. The problem is reduced to the zero eigenvalues with their Jordan chance and the non-zero eigenvalue solutions, which are important in application. The symplectic method is effective for mixed boundary conditions and can be generalized to others fields.

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