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A generalized plane strain theory for transversely isotropic piezoelectric plates

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Abstract: Study of generalized plane strain has so far been limited to elasticity. The present is aimed at parallel development of transversely isotropic piezoelasticity. By assuming that the along depth distribution of electric potential is linear, and that commonly used Kane-Mindlin kinematical assumption is valid, two dimensional solution systems were deduced, for which, explicit solutions of the out-of-plane constraint factor, as well as the stress resultant concentration factor around a circular hole in a transversely isotropic piezoelectric plate subjected to remote biaxial tension are obtained. Comparisons of these formulas with their counterparts for elastic case yielded suggestions that whether the piezoelectric effect exacerbates or mitigates the stress resultant concentration greatly depends on material properties, particularly, the piezoelectric coefficients; the effect of plate thickness was extensively investigated.

Key words: Plane strain, Piezoelectric plate, Circular hole
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INTRODUCTION

For moderate thickness plate containing notches or holes, neither plane stress nor plane strain theory can describe its true deformation states (Jin and Hwang, 1989). This well known drawback has motivated many analytical and numerical studies carried out to determine the effect of plate thickness on the in-plane stress distribution. Pioneering work on the along depth stress components, i.e. the K-M generalized plane strain theory was proposed by Kane and Mindlin (1956) in their study of high-frequency extensional vibrations of plates, and used by Yang and Freund (1985), and Jin and Hwang (1989) in studying the three-dimensional stress distributions near a crack tip. By applying this theory, Kotousov and Wang (2002a) also obtained solutions for the stress distributions around a circular hole, and solutions for V-shaped notches with a circular tip. They deduced fundamental solutions for this theory (Kotousov and Wang, 2002b). In their later studies, the K-M generalized plane strain theory was extended to transversely isotropic materials (Kotousov and Wang, 2003; 2002c).

Since piezoelectric material is increasingly finding practical applications in many aerospace, mechanical-, civil-, and bio-engineering domains, it is very important to thoroughly understand the mechanical responses of such materials under various loading conditions. Consequently, it is very desirable to have analytical solutions available in the first place even for simple geometries and loading conditions. For the case of plane strain theory in piezoelectricity, Sosa (1991) with the help of complex variable theory, conducted a two-dimensional piezoelectric analysis on a transversely isotropic piezoelectric material containing defects. However, to the authors' best knowledge, analogous development of generalized plane strain theory in piezoelectricity has not been

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attempted.

The assumption that the along depth distribution of electric potential is linear, and the Kane-Mindlin kinematical assumption were used to derive two dimensional solution systems for which, explicit solutions of the out-of-plane constraint factor, as well as the stress resultant concentration factors around a circular hole in a transversely isotropic piezoelectric plate subjected to remote biaxial tension are presented. Comparisons of these formulas with their counterparts for elastic case were also investigated numerically.

We conclude this introduction by defining the notations used in this paper. A comma followed by an index i denotes partial differentiation with respect to x_i; a repeated index implies summation over the range of the index; Latin indices range from 1 to 3 and Greek indices from 1 to 2. The dependence of the functions and operators on x_i is not explicitly shown unless necessary.

GENERALIZED PLANE STRAIN THEORY FOR TRANSVERSELY ISOTROPIC PIEZOELECTRIC PLATES

Hereafter, the piezoelectric material is assumed, for simplicity, to possess the material symmetry of 6 mm. Now introduce a rectangular coordinate system (x_1, x_2, x_3), so that the x_1Ox_2 plane coincides with the mid-plane of the plate, while x_3 is along the axis of transverse isotropy, parallel to the poling direction.

For the convenience of subsequent derivation, governing equations of linear piezoelasticity are summarized as follows (Ding and Chen, 2001).

Constitutive relations:

$$\boldsymbol{\sigma}=\boldsymbol{C\varepsilon}-\boldsymbol{e}^{\mathrm{T}}\boldsymbol{E}, \quad (1.1)$$

$$\boldsymbol{D}=\boldsymbol{e\varepsilon}+\boldsymbol{gE}. \quad (1.2)$$

in which the stress and strain tensors, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, the displacement, electric displacement and electric field vectors, $\boldsymbol{u}$, $\boldsymbol{D}$ and $\boldsymbol{E}$, are denoted according to the usual Voigt's notation, in turn, by

$$\boldsymbol{\sigma}=[\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{23} \quad \sigma_{13} \quad \sigma_{12}]^{\mathrm{T}}, \quad (2.1)$$

$$\boldsymbol{\varepsilon}=[\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad 2\varepsilon_{23} \quad 2\varepsilon_{13} \quad 2\varepsilon_{12}]^{\mathrm{T}}, \quad (2.2)$$

$$\boldsymbol{u}=[u_1 \quad u_2 \quad u_3]^{\mathrm{T}}, \quad (2.3)$$

$$\boldsymbol{D}=[D_1 \quad D_2 \quad D_3]^{\mathrm{T}}, \boldsymbol{E}=-[\phi_{,1} \quad \phi_{,2} \quad \phi_{,3}]^{\mathrm{T}}. \quad (2.4)$$

where ϕ is electric potential. While $\boldsymbol{C}$, $\boldsymbol{e}$ and $\boldsymbol{g}$, being 6×6, 3×6 and 3×3 matrices, stand for the elasticity, piezoelectricity and dielectricity tensors respectively, and are given in Appendix A.

Strain-displacement relations:

$$\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2. \quad (3)$$

Equilibrium equations:

$$\sigma_{ij,j}=0, \quad (4.1)$$

$$D_{i,i}=0. \quad (4.2)$$

where the plate is free of body forces, inertia effects, as well as body electric charge.

Consider a plate bounded by planes $x_3=\pm h$. The displacement field and electric potential are assumed to be given by the usual K-M kinematical assumption and linear distribution assumption, respectively, viz.

$$u_\alpha=u_\alpha(x_\beta),\ u_3=x_3h^{-1}w(x_\beta),\ \phi=x_3h^{-1}\varphi(x_\beta). \quad (5)$$

Introduce the stress resultants and "pinching" shears, defined by

$$(N_{\alpha\beta},N_{33})=\int_{-h}^{h}(\sigma_{\alpha\beta},\sigma_{33})\mathrm{d}x_3,\quad R_\alpha=\int_{-h}^{h}x_3\sigma_{\alpha3}\mathrm{d}x_3, \quad (6)$$

respectively, which can be used to express the equilibrium equations, by integrating Eq.(4) with respect to x_3 between the limits $\pm h$ and then substituting Eq.(6) into the resulting equations, in the following form

$$N_{\alpha\beta,\beta}=0, \quad (7.1)$$

$$R_{\alpha,\alpha}=N_{33}. \quad (7.2)$$

where the stress-free boundary conditions along the plate surfaces, i.e. $\sigma_{\alpha3}=\sigma_{33}=0$ at $x_3=\pm h$, have been used.

Substitution of Eqs.(1) and (5) into Eq.(6) yields

$$N_{11}=2h(c_{11}\varepsilon_{11}+c_{12}\varepsilon_{22}+c_{13}h^{-1}w+e_{31}h^{-1}\varphi), \quad (8.1)$$

$$N_{22}=2h(c_{12}\varepsilon_{11}+c_{11}\varepsilon_{22}+c_{13}h^{-1}w+e_{31}h^{-1}\varphi), \quad (8.2)$$

$$N_{33}=2h[c_{13}(\varepsilon_{11}+\varepsilon_{22})+c_{33}h^{-1}w+e_{33}h^{-1}\varphi], \quad (8.3)$$

$$N_{12}=4hc_{66}\varepsilon_{12}, \quad (8.4)$$

$$R_\alpha=2h^2(c_{44}w_{,\alpha}+e_{15}\varphi_{,\alpha})/3. \quad (8.5)$$

Similar to Airy stress function, Eq.(7.1) can be automatically satisfied by the stress resultant function Φ

in the form of

$$N_{\alpha\beta}=\nabla^2\Phi\delta_{\alpha\beta}-\Phi_{,\alpha\beta}\,, \tag{9}$$

with $\nabla^2=()_{,\alpha\alpha}$ being a two-dimensional Laplace operator.

From Eq.(8), the normal stress resultant N_{33} can be expressed in terms of the mean in-plane stress resultant

$$N=N_{\alpha\alpha}/2, \tag{10}$$

w, and φ as

$$N_{33}=2[v'N+E'w+(e_{33}-2v'e_{31})\varphi], \tag{11}$$

The following relation among N, w and φ is now obtained by substituting Eqs.(8.5) and (11) into Eq.(7.2), in the form of

$$\left(\frac{h^2}{3}G'\nabla^2-E'\right)w+\left(\frac{h^2}{3}e_{15}\nabla^2-e_{33}+2v'e_{31}\right)\varphi=v'N. \tag{12}$$

Substituting Eqs.(1.2) and (5) into Eq.(4.2), yields the second governing equation

$$\nabla^2\varphi=\xi\nabla^2 w, \tag{13}$$

in which $\xi=e_{15}/g_{11}$.

Taking the strain compatibility condition

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}=2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2} \tag{14}$$

into consideration, the third and last governing equation is obtained in the form

$$\nabla^2 N=\zeta\nabla^2 w, \tag{15}$$

with $\zeta=[Ev'+(1-v-2\eta^2)e_{15}e_{31}/g_{11}]/(1-\eta^2)$, in which Eq.(13) had been used.

The two dimensional solution systems are completed by Eqs.(12), (13) and (15). In fact, following the same procedures in the works of Kotousov and Wang (2003; 2002c), this coupled solution system can be decoupled, so that the displacement w and stress-consultant function Φ are satisfied, respectively, by

$$\nabla^4 w-\lambda^2\nabla^2 w=0,\quad \nabla^6\Phi-\lambda^2\nabla^4\Phi=0. \tag{16}$$

with

$$\lambda^2=3\left\{\frac{E'}{1-\eta^2}+\left[e_{33}-\frac{1+v}{1-\eta^2}v'e_{31}\right]\frac{e_{15}}{g_{11}}\right\}\Big/\left[h^2\left(G'+\frac{e_{15}^2}{g_{11}}\right)\right]$$

AN INFINITE PLATE WITH A CIRCULAR HOLE UNDERGOING EXTENSIONAL DEFORMATIONS

As a closure for the theory developed in the previous section, a simple example is presented. To compare the results for elastic plate and for this one, we considered the same problem addressed by Kotousov and Wang (2003), i.e. a $2h$ thick infinite plate with radius a circular hole, and subjected to remote tensile stresses N_{11}^{∞} and N_{22}^{∞} parallel to the mid-plane of the plate, while electric displacement vanishes on the boundaries of $x_1, x_2\to\infty$. Since the system discussed here is assumed to be linear, solutions can be reached by superposing the following two decomposed cases

Case 1:

$$N_{11}=N_{22}=\bar{N}_1\equiv(N_{11}^{\infty}+N_{22}^{\infty})/2,\ \text{at infinity}. \tag{17}$$

Case 2:

$$N_{11}=-N_{22}=\bar{N}_2\equiv(N_{11}^{\infty}-N_{22}^{\infty})/2,\ \text{at infinity}. \tag{18}$$

For Case 1, the plane-stress or plane-strain solution in cylindrical coordinates given by (Kotousov and Wang, 2003)

$$N_{rr}=\bar{N}_1\left(1-\frac{a^2}{r^2}\right),\quad N_{\theta\theta}=\bar{N}_1\left(1+\frac{a^2}{r^2}\right),\quad D_r=0, \tag{19}$$

is still valid here.

For Case 2, when transformed into cylindrical coordinates, the boundary conditions are

$$N_{rr}=-\bar{N}_2\cos(2\theta), N_{\theta\theta}=\bar{N}_2\cos(2\theta), N_{r\theta}=\bar{N}_2\sin(2\theta),$$
$$D_r=0,\ \text{at infinity}. \tag{20}$$

which suggest that the following solutions for w, N, Φ and φ be chosen

$$w(r,\theta)=\left[-A_1\frac{\vartheta}{\lambda^2}r^{-2}-A_2\frac{\delta}{\lambda^2}r^{-2}+A_3K_2(\lambda r)\right]\cos(2\theta), \tag{21.1}$$

$$N(r,\theta)=\zeta\begin{bmatrix}A_1\left(1-\frac{\vartheta}{\lambda^2}\right)r^{-2}-A_2\frac{\delta}{\lambda^2}r^{-2}\\+A_3K_2(\lambda r)\end{bmatrix}\cos(2\theta), \tag{21.2}$$

$$\Phi(r,\theta)=-\frac{r^2}{2}\int_r^{\infty}\rho^{-1}N(\rho)\mathrm{d}\rho-\frac{r^{-2}}{2}\int_a^r\rho^3N(\rho)\mathrm{d}\rho$$
$$+(A_4r^{-2}+A_5r^2)\cos(2\theta), \tag{21.3}$$

$$\varphi(r,\theta)=\xi\begin{bmatrix}-A_1\frac{\vartheta}{\lambda^2}r^{-2}+A_2\left(1-\frac{\delta}{\lambda^2}\right)r^{-2}\\+A_3K_2(\lambda r)\end{bmatrix}\cos(2\theta), \tag{21.4}$$

where $\vartheta=3v'\zeta/[h^2(G'+e_{15}^2/g_{11})]$, $\delta=3(e_{33}-2v'e_{31})\xi/[h^2(G'+e_{15}^2/g_{11})]$, and K_i is the modified Bessel function of ith order. A_1, A_2, A_3, A_4, and A_5 are unknown constants to be determined.

Application of the prescribed boundary conditions at $r=a$ and $r\to\infty$ gives rise to

$$A_1=\frac{8\vartheta\lambda a}{\zeta\begin{Bmatrix}8\vartheta K_1(\lambda a)+(\lambda^2-\vartheta)\lambda^2a^2\\\cdot[K_1(\lambda a)+K_3(\lambda a)]\end{Bmatrix}}\overline{N}_2,$$
$$A_3=\frac{2\lambda^4a^4[K_1(\lambda a)+K_3(\lambda a)]}{\zeta\begin{Bmatrix}8\vartheta K_1(\lambda a)+(\lambda^2-\vartheta)\lambda^2a^2\\\cdot[K_1(\lambda a)+K_3(\lambda a)]\end{Bmatrix}}\overline{N}_2,$$
$$A_2=A_4=0,\quad A_5=\overline{N}_2/2. \tag{22}$$

Noticeably, the vanished transverse shear stress on the boundaries of the body has been utilized, which, based on Eq.(8), read

$$c_{44}w_{,r}+e_{15}\varphi_{,r}=0. \tag{23}$$

The final stress-resultants for the problem of an infinite transversely isotropic piezoelectric plate undergoing remote shear loading are obtained by substituting Eq.(22) into Eq.(21), then the results into Eq.(9), which, in the cylindrical coordinate system, read

$$N_{rr}=\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2},N_{\theta\theta}=\frac{\partial^2\Phi}{\partial r^2},N_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right). \tag{24}$$

Here we only present the quantities of particular interest, i.e., the stress resultant concentration factor under shear loading along the periphery of the hole

$$K_s=\frac{\max[N_{\theta\theta}(r=a)]}{\overline{N}_2}$$
$$=\frac{4\lambda^2a[(\lambda^2-\vartheta)aK_1(\lambda a)+2\lambda K_2(\lambda a)]}{[4\vartheta+\lambda^2a^2(\lambda^2-\vartheta)]K_1(\lambda a)+2\lambda a(\lambda^2-\vartheta)K_2(\lambda a)}. \tag{25}$$

According to the works of Kotousov and Wang (2002a; 2002c; 2003), the out-of-plane constraint factor is defined

$$C_{zs}=\frac{N_{zz}}{2v'N}. \tag{26}$$

The above solution's explicit expression is

$$C_{zs}=\frac{\chi_a[K_1(\lambda a)+K_3(\lambda a)]+\chi_rr^2K_2(\lambda r)}{\zeta v'g_{11}\begin{Bmatrix}(\lambda^2-\vartheta)\lambda a^3[K_1(\lambda a)+K_3(\lambda a)]\\+4\vartheta r^2K_2(\lambda r)\end{Bmatrix}}, \tag{27}$$

in which

$$\chi_a=[\zeta v'g_{11}(\lambda^2-\vartheta)-\vartheta E'g_{11}-\vartheta e_{15}(e_{33}-2v'e_{31})]\lambda a^3,$$
$$\chi_r=4\vartheta[(\zeta v'+E')g_{11}+e_{15}(e_{33}-2v'e_{31})].$$

In the case of uniaxial tension at infinity, we can obtain similar results by superposition, which are not presented here for brevity.

NUMERICAL RESULTS

To further illustrate the results and compare them with their counterparts for the elastic case, the stress resultant concentration factor and out-of-plane constraint factor are plotted in Figs.1~5 based on the material constants listed below.

For PZT-4 (Dunn and Taya, 1994), the elastic stiffness (10^{10} N/m^2) are, respectively, c_{11}=13.9, c_{12}=7.78, c_{13}=7.43, c_{33}=11.5, c_{44}=2.56. The piezoelectric coefficients (C/m^2) are, respectively, e_{15}=12.7, e_{31}=–5.2, e_{33}=15.1. The dielectric constants (10^{-10} F/m) are, respectively, g_{11}=64.64, g_{33}=56.22. The corresponding constants for ZnO (Deresiewicz and Royer, 1980) are c_{11}=20.97, c_{12}=12.11, c_{13}=7.43, c_{33}=10.51, c_{44}=21.09, e_{15}=–0.59, e_{31}=–0.61, e_{33}=1.14, g_{11}=0.738, g_{33}=0.783.

Fig.1 gives the variation of the stress resultant concentration factor K_s versus the ratio of the half-thickness to radius h/a for an infinite plate with a circular hole and subjected to shear loading. Both piezoelectric materials and their corresponding elastic material, which has the same elastic constants, are considered. The influence of the piezoelectric effect is clearly demonstrated in Fig.1 showing that for different materials, the effect of piezoelectric effect on K_s is different. The stress resultant concentration factor is smaller than its elastic counterpart for PZT-4 piezoelectric material. However, for ZnO, reverse tendency was observed. To further explore the piezoelectric effect's role in the calculation of the concentration factor, variations of K_s versus e_{15} and e_{31} are shown in Figs.2 and 3, respectively. The shape of the curve of K_s against e_{33}, which is omitted, is substantially similar to that of K_s against e_{15}. Figs.2 and 3 show that K_s decreases with increasing e_{15} and e_{33}, and decreasing e_{31}.

It is also interesting that K_s is not a monotonic function of thickness. K_s increases rapidly to its peak value at about h/a=3/2, then decreases gradually with increasing h/a. The tendency of variation agrees with the 3D FE result for elastic case (Li *et al.*, 2000). Fig.1 also shows that the discrepancies between K_s for piezoelectric plate with its elastic counterpart become more pronounced for K_s near its peak values.

Fig.4 displays the curve of the out-of-plane constraint factor C_{zs} versus the non-dimensional distance r/a. From which the influence of the piezoelectric effect is observed again. Noticeably, the results in Fig.4 are for PZT-4 piezoelectric material. As for piezoelectric material ZnO, the variation tendency of C_{zs} versus r/a is the same. The shape of curve C_{zs} versus r/a becomes flatter and flatter with increasing h/a. From Fig.4, we can also see that the out-of-plane constraint factor is always greater than the one corresponding to the elastic plate.

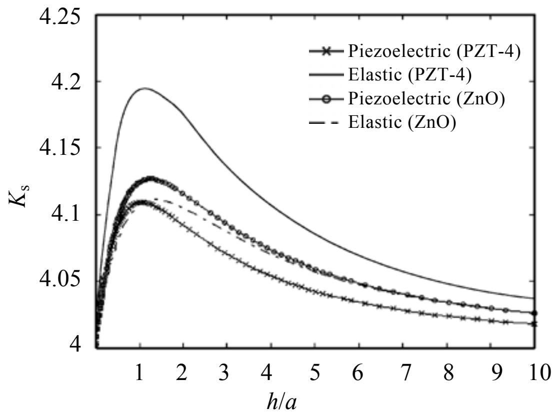

Fig.1 Variation of K_s versus h/a for an infinite plate with a circular hole undergoing shear loading

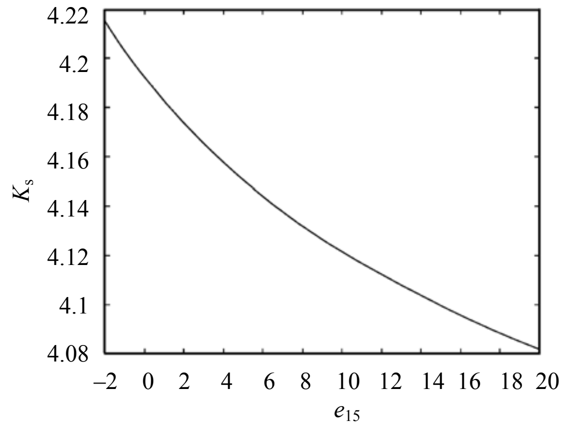

Fig.2 Variation of K_s versus e_{15}

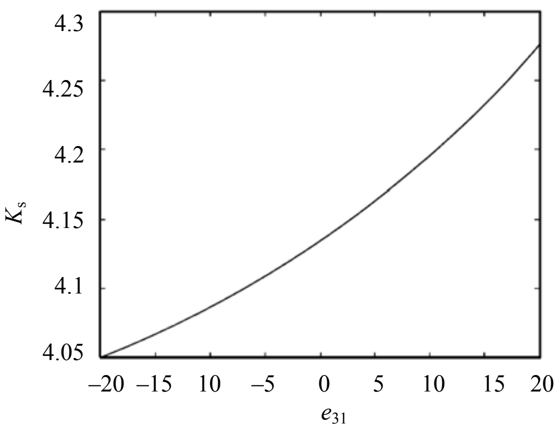

Fig.3 Variation of K_s versus e_{31}

In Fig.5, the out-of-plane constraint factor at the periphery of the hole C_{zsa} is plotted as a function of h/a. It is noted that C_{zsa} increases with increasing h/a, and approaches a saturated value when h/a is large enough. It can also be seen that the curves of C_{zsa} versus h/a are almost insensitive to those of piezoelectric material. Here, it is noteworthy that the transverse Poisson's ratios for PZT-4 (=0.34271) and ZnO (=0.31771) are almost the same too. Conclusion that the effect of plate thickness on C_{zs} prevails against that of piezoelectric effect can also be drawn from Figs.4 and 5.

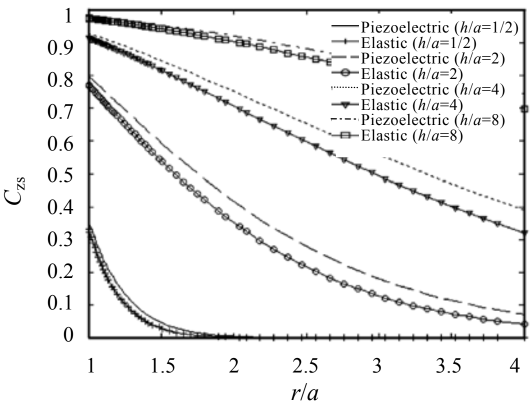

Fig.4 Variation of C_{zs} versus r/a

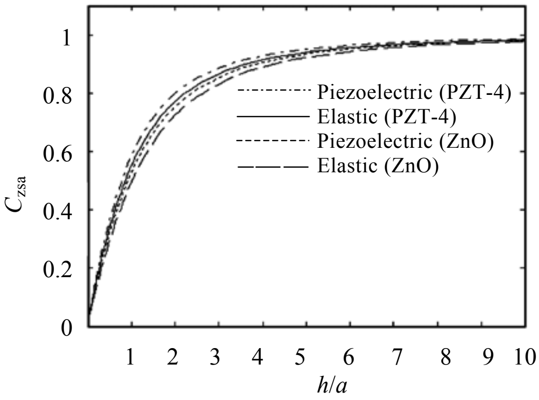

Fig.5 Variation of C_{zsa} versus h/a

The above conclusion does not mean that the piezoelectric effect is not important. Fig.6 shows clearly that analyses in the vicinity of a hole can produce incorrect results if the piezoelectric effect is not taken into account. In Fig.6, w_m is the maximum of transverse displacement w.

CONCLUSION

In this paper, a generalized plane strain theory for transversely isotropic piezoelectric plates is developed on the assumption that the along depth distribution of electric potential is linear, and on the Kane-Mindlin kinematical assumption. Two dimensional solution systems were obtained, with which explicit solutions of the stress resultant concentration factor around a circular hole and out-of-plane constraint factor in a transversely isotropic piezoelectric plate undergoing remote biaxial tension are presented in some detail, and the numerical results are also illustrated. Comparisons with their counterparts for elastic case revealed that piezoelectric coefficients dictate whether the piezoelectric effect exacerbates or mitigates the stress resultant concentration. The effect of plate thickness was extensively investigated.

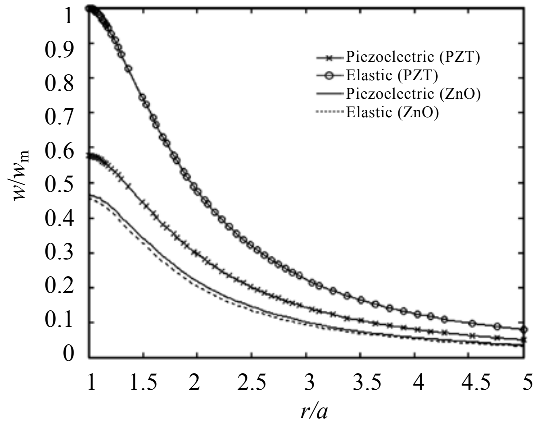

Fig.6 Variation of transverse displacement w/w_m versus r/a

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APPENDIX A

The matrices $\boldsymbol{C}$, $\boldsymbol{e}$ and $\boldsymbol{g}$ in Eq.(1) are given as

$$\boldsymbol{C}=\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}, \tag{A.1}$$

$$\boldsymbol{e}=\begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix}, \tag{A.2}$$

$$\boldsymbol{g}=\begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{11} & 0 \\ 0 & 0 & g_{33} \end{bmatrix}. \tag{A.3}$$

with $2c_{66}=c_{11}-c_{12}$, in which the coefficients c_{ij} also can be expressed with the conventional Young's modulus, Poisson's ratio, and shear modulus (Kotousov and Wang, 2003)

$$\begin{aligned} c_{11}&=\frac{1-\eta^2}{(1+v)(1-v-2\eta^2)}E, \\ c_{12}&=\frac{v+\eta^2}{(1+v)(1-v-2\eta^2)}E, \\ c_{13}&=\frac{v'}{1-v-2\eta^2}E, \\ c_{33}&=\frac{1-v}{1-v-2\eta^2}E', \\ c_{44}&=G', \\ \eta&=v'\sqrt{E/E'}. \end{aligned} \tag{A.4}$$