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The Reissner-Sagoci problem for transversely isotropic piezoelectric half-space

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Abstract: Based on the general solution of piezoelectric media and the extended Cerruti solution for tangential point forces acted on the surface of transversely isotropic piezoelectric half-space (Ding and Chen, 2001), the electro-elastic fields in a transversely isotropic piezoelectric half-space caused by a circular flat bonded punch under torsion loading, which is called Reissner-Sagoci problem, are evaluated by first evaluating the displacement functions within the contact region and then differentiating them. All the coupling electro-elastic fields are expressed by elementary functions and are convenient to be used. Numerical results are finally presented.

Key words: Piezoelectric, Reissner-Sagoci problem, Cerruti solution

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INTRODUCTION

In isotropic elasticity a class of mixed boundary value problems for elastic half-space can be classified as axisymmetric torsion. In this case the elastic field in cylinder components takes a particularly simple form consisting of only one nonzero displacement and two nonzero stress components. For this loading case, Reissner and Sagoci (1944) were possibly the first to examine a mixed boundary problem between the tangential displacement and stress on the half-space surface. They considered the problem of a circular flat disk bonded to a half-space under the action of a torsional couple. Their analysis yielded the contact stress under the disk and the tangential displacement on the surface outside the disk. The elastic field was soon afterward given by Sneddon (1947) illustrating the use of Hankel transforms in developing complete solutions to mixed boundary value problems. The Reissner-Sagoci problem for transversely isotropic half-space was solved by Hanson and Puja (1997) with the use of the potential theory put forward by Fabrikant (1989; 1991).

The 6 mm crystal symmetry piezoelectric ceramics are transversely isotropic piezoelectric material that has found widespread applications because of their excellent piezoelectricity. But because of the intrinsic brittleness of piezoelectric ceramics, the stress concentration caused by inharmonious contact such as elliptical contact between the piezoelectric ceramics components and the other components could cause failure of the piezoelectric ceramics components. So from the viewpoint of electromechanical coupling, it is necessary to conduct theoretical analysis to obtain accurate quantitative description of the elastic and electric fields around the contact region of piezoelectric ceramics. Fan *et al.*(1996) studied the plane contact problem by Stroh formulae, and presented the solution for an anisotropic plane under contact loadings. And then, a series of solutions for three-dimensional contact problems were obtained by Ding *et al.*(1999; 2000), Chen (1999; 2000), Chen and Ding (1999), Chen *et al.*(1999), Giannakopoulos (2000) and Sridhar *et al.*(2000).

In this work, by virtue of the general solution of piezoelectric media and the extended Cerruti solution for tangential point forces acted on the surface of transversely isotropic piezoelectric half-space (Ding and Chen, 2001), the electro-elastic fields in a transversely isotropic piezoelectric half-space caused by a circular flat bonded punch under torsion loading are evaluated by first evaluating the displacement functions and then differentiating them. The displacement functions can be obtained by integrating the extended Cerruti solution in the contact region. The distributions of components of stress and electric displacement in PZT-4 half-space are shown in the figures of this paper.

GENERAL SOLUTION FOR TRANSVERSELY ISOTROPIC PIEZOELECTRIC MEDIA AND EXTENDED CERRUTI SOLUTION

Ding and Chen (2001) presented, in cylinder coordinate (r,ϕ,z), the following general solution of axisymmetric torsion for transversely isotropic media:

$$\begin{aligned}&u_r=u_z=0,\ \Phi=0,\\&\sigma_r=\sigma_\phi=\sigma_z=\tau_{zr}=0,\ D_r=D_z=0,\\&u_\phi=\frac{\partial\psi_0}{\partial r},\qquad \tau_{\phi z}=c_{44}\frac{\partial^2\psi_0}{\partial r\partial z},\\&\tau_{r\phi}=c_{66}\left(\frac{\partial^2}{\partial r^2}-\frac{\partial}{r\partial r}\right)\psi_0,\ D_\phi=e_{15}\frac{\partial^2\psi_0}{\partial r\partial z},\end{aligned}\tag{1}$$

where u_r, u_ϕ and u_z are displacements, σ_r, σ_ϕ, σ_z, $\tau_{r\phi}$, $\tau_{\phi z}$ and τ_{zr} are stresses, D_r, D_ϕ and D_z are electric displacements, c_{44} and c_{66} are elastic constants, e_{15} is piezoelectric constant. ψ_0 is displacement function described by the following equation:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z_0^2}\right)\psi_0=0,\tag{2}$$

and $z_0=s_0z$, $s_0=\sqrt{c_{66}/c_{44}}$.

Considering combined point forces $\boldsymbol{P}=P_x+\mathrm{i}P_y$ acting on a arbitrary point $(r_0, \phi_0, 0)$ of the surface $z=0$ of a transversely isotropic half-space $z\geq0$ with the isotropic plane perpendicular to z axis in cylindrical coordinates (r, ϕ, z), Ding and Chen (2001) obtained the extended Cerruti solution as

$$\psi_0=\mathrm{i}G_0(\boldsymbol{P}\bar{\Delta}-\bar{\boldsymbol{P}}\Delta)\chi(z_0),\tag{3}$$

where $\bar{\boldsymbol{P}}$ and $\bar{\Delta}$ are the complex conjugate of $\boldsymbol{P}$ and Δ, and

$$\begin{aligned}&G_0=\frac{1}{4\pi s_0c_{44}},\qquad \Delta=\mathrm{e}^{\mathrm{i}\phi}\left(\frac{\partial}{\partial r}+\mathrm{i}\frac{1}{r}\frac{\partial}{\partial\phi}\right),\\&\chi(z_0)=z_0\ln R_0^*-R_0,\ \ R_0^*=R_0+z_0,\\&R_0=\sqrt{r^2+r_0^2-2rr_0\cos(\phi-\phi_0)+z_0^2},\end{aligned}\tag{4}$$

Eq.(3) will be used as integrated function for the piezoelectric Reissner-Sagoci solution.

SOLUTION FOR REISSNER-SAGOCI PROBLEM

On the surface $z=0$ of half-space, the solution of Reissner-Sagoci problem should satisfy the following boundary conditions on the cylinder coordinate (r,ϕ,z)

$$\begin{aligned}&u_r=w=0,\ u_\phi=kr,\ \Phi=0,\ \ (r<a)\\&\sigma_z=\tau_{zr}=\tau_{\phi z}=0,\ D_z=0,\ \ (r>a)\end{aligned}\tag{5}$$

where k is a constant.

Firstly, we study the problem in which the following shear stress is acting on the contact region $(r<a, z=0)$.

$$\tau_{\phi z}(r,\phi)=-\frac{Cr}{\sqrt{a^2-r^2}}\ .\tag{6}$$

Obviously, this is a torsional problem. Substituting the following equation into Eq.(3),

$$\boldsymbol{P}=-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi_0}\tau_{\phi z}(r_0,\phi_0)r_0\mathrm{d}r_0\mathrm{d}\phi_0,\tag{7}$$

and integrating the result over $0\leq r_0\leq a$, $0\leq\phi_0\leq2\pi$, the displacement functions become

$$\begin{aligned}\psi_0(r,\phi,z)=-CG_0\{&\bar{\Delta}[z_0\varXi(r,\phi,z_0)-\Theta(r,\phi,z_0)]\\&+\Delta[z_0\bar{\varXi}(r,\phi,z_0)-\bar{\Theta}(r,\phi,z_0)]\},\end{aligned}\tag{8}$$

where

$$\Xi(r,\phi,z_0)=\int_0^{2\pi}\int_0^{a} r_0 \mathrm{e}^{\mathrm{i}\phi_0}\left(a^2-r_0^2\right)^{-1/2}\ln R_0^* r_0 \mathrm{d}r_0 \mathrm{d}\phi_0,$$

$$\Theta(r,\phi,z_0)=\int_0^{2\pi}\int_0^{a} r_0 \mathrm{e}^{\mathrm{i}\phi_0}(a^2-r_0^2)^{-1/2} R_0 r_0 \mathrm{d}r_0 \mathrm{d}\phi_0, \tag{9}$$

R_0^*, R_0 are defined in Eq.(4), and $\bar{\Xi}(r,\theta,z_0)$ and $\bar{\Theta}(r,\theta,z_0)$ can be obtained from Eq.(9) by replacing $\mathrm{e}^{\mathrm{i}\theta_0}$ with $\mathrm{e}^{-\mathrm{i}\phi_0}$.

The integrals in Eq.(9) can be extracted from the results in Hanson and Puja (1997) as

$$\Xi(r,\phi,z_0)=\pi r\mathrm{e}^{\mathrm{i}\phi}\left\{z_0\arcsin\frac{l_{10}(a)}{r} -\sqrt{a^2-l_{10}^2(a)}\left[1-\frac{2a^2+l_{10}^2(a)}{3r^2}\right]-\frac{2a^3}{3r^2}\right\},$$

$$\Theta(r,\phi,z_0)=\pi r\mathrm{e}^{\mathrm{i}\phi}\left\{\frac{1}{8}(4z_0^2+r^2-4a^2)\arcsin\frac{l_{10}(a)}{r} -\left[l_{10}^2(a)+2l_{20}^2(a)-\frac{3r^2}{2}\right]\frac{l_{10}(a)\sqrt{r^2-l_{10}^2(a)}}{4r^2}\right\}, \tag{10}$$

where

$$l_{10}(a)=\frac{1}{2}\left[\sqrt{(r+a)^2+z_0^2}-\sqrt{(r-a)^2+z_0^2}\right],$$

$$l_{20}(a)=\frac{1}{2}\left[\sqrt{(r+a)^2+z_0^2}+\sqrt{(r-a)^2+z_0^2}\right], \tag{11}$$

Substituting Eqs.(8) and (10) into the general solution Eq.(1), the electro-elastic fields in piezoelectric half-space can be obtained as

$$u_\theta=4\pi G_0 Cr\left[-\frac{1}{2}\arcsin\frac{l_{10}(a)}{r}+\frac{l_{10}(a)\sqrt{r^2-l_{10}^2}}{2r^2}\right],$$

$$\tau_{r\phi}=4\pi c_{66}G_0 C\frac{l_{10}^3(a)\sqrt{r^2-l_{10}^2(a)}}{r^2[l_{20}^2(a)-l_{10}^2(a)]},$$

$$\tau_{\phi z}=4\pi s_0 c_{44}G_0 C\frac{l_{10}^2(a)\sqrt{a^2-l_{10}^2(a)}}{r[l_{20}^2(a)-l_{10}^2(a)]},$$

$$D_\phi=4\pi s_0 e_{15}G_0 C\frac{l_{10}^2(a)\sqrt{a^2-l_{10}^2(a)}}{r[l_{20}^2(a)-l_{10}^2(a)]}, \tag{12}$$

From the first equation in Eq.(12), we obtain the displacement u_θ inside the contact circle (z=0, r<a) as follow.

$$u_\phi=-\pi^2 G_0 Cr, \tag{13}$$

If the constant k in Eq.(5) and constant C in Eq.(13) have following relation,

$$C=-k/(\pi^2 G_0), \tag{14}$$

Eq.(14) is the solution of Reissner-Sagoci problem for transversely isotropic piezoelectric media.

NUMERICAL RESULTS

Assuming the piezoelectric half-space is PZT-4, based Eq.(12), the shear stresses and electric displacement on the surface (z=0) and cylinder surface r=a in PZT-4 (Ding and Chen, 2001) half-space are showed in Fig.1 and Fig.2, respectively. The symbols in figures are defined as

$$\tilde{\tau}_{r\phi}=k\tau_{r\phi}/p_m,\quad \tilde{\tau}_{\phi z}=k\tau_{\phi z}/p_m,$$

$$p_m=10^{10}\ (\mathrm{N/m^2}),\quad \tilde{D}_\phi=kD_\phi, \tag{15}$$

From the figures, we can see that

(1) $\tau_{r\phi}$, $\tau_{\phi z}$ and D_ϕ are singular on the r=a contact circle and drops quickly on the points away from the

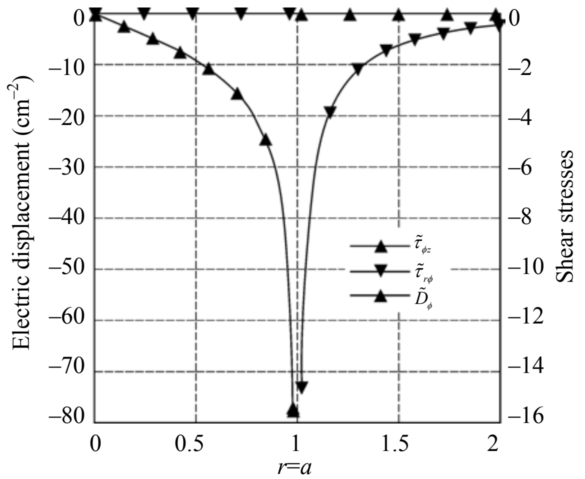

Fig.1 The electro-elastic fields on the surface of half-space z=0

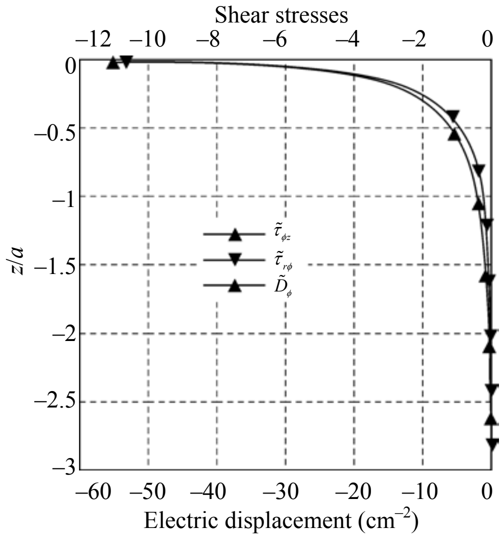

Fig.2 The electro-elastic fields on the *r*=*a* cylinder surface

circle (Fig.1 and Fig.2);

(2) The values of $\tau_{r\phi}$, $\tau_{\phi z}$ and D_ϕ are very near to zero at the depth of $z>2a$ on the $r=a$ cylinder surface (Fig.2);

(3) The curves of distributions of $\tilde{D}_\theta$ and $\tilde{\tau}_{\theta z}$ are coincident with each other as a result of $D_\theta/\tau_{\theta z}=e_{15}/c_{44}$ (Fig.1 and Fig.2).

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