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Multi-objective robust controller synthesis for discrete-time systems with convex polytopic uncertain domain*

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Abstract: Multi-objective robust state-feedback controller synthesis problems for linear discrete-time uncertain systems are addressed. Based on parameter-dependent Lyapunov functions, the Gl_2 and GH_2 norm expressed in terms of LMI (Linear Matrix Inequality) characterizations are further generalized to cope with the robust analysis for convex polytopic uncertain system. Robust state-feedback controller synthesis conditions are also derived for this class of uncertain systems. Using the above results, multi-objective state-feedback controller synthesis procedures which involve the LMI optimization technique are developed and less conservative than the existing one. An illustrative example verified the validity of the approach.

Key words: Gl_2 and GH_2 performance, Multi-objective optimization, Robust controller synthesis, Parameter-dependent Lyapunov functions, Convex polytopic uncertain system

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INTRODUCTION

de Oliverira *et al*.(1999a) proposed a new LMI condition with a non-symmetric matrix ***G*** to test the Lyapunov stability for precisely known discrete-time systems and the quadratic stability for discrete-time uncertain systems. Based on this new stability condition, many different control synthesis problems can be restated so that the resulting controller no longer depends on the symmetric matrix ***P***, which leads to less conservativeness, and furthermore, many successful extensions have been accomplished. For instance, de Oliverira *et al*.(1999b; 2002) has extended LMI characterizations for H_2 and H_∞ norm computations which are coped with state-feedback as well as output-feedback controller parameterizations to transform the synthesis problems to LMI optimization problems. Yan and Zhang (2004) have been further extended this idea to the mixed Gl_2/GH_2 optimization problem with pole placement constraints.

Yan and Zhang (2004) only considered controller synthesis for the known discrete-time system, did not include the uncertain system. Although de Oliverira *et al*.(1999b; 2002) considered both multi-objective and robust controller synthesis for uncertain system respectively, the combination of both sides was not accomplished. Mibar *et al*.(2002) dealt with multi-objective output-feedback synthesis for Linear Time Invariant (LTI) polytopic uncertain systems, but the problem emerging from its synthesis procedure is that the controller parameterization is given through a matrix inequality which is not linear but bilinear with respect to system matrices and involves complex computations. That is, the tractable multi-objective multi-channels robust controller synthesis procedure for discrete-time systems with convex polytopic uncertainty still poses open and challenging questions.

In this work, the G-shaping paradigm (de Oliverira *et al*., 1999a; 2002) was further extended to actual multi-objective multi-channels robust controller synthesis for discrete-time systems with convex polytopic uncertain domain. The derived synthesis

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conditions turned the synthesis into more tractable LMI optimization problem. The design objectives can be combinations of Gl_2 performance, GH_2 performance, constraint on the closed-loop pole location and other performance that can be achieved in the extended LMI based on the new robust stability condition.

This paper is organized as follows: In Section 2, the extended LMI analysis conditions for Gl_2, GH_2 norm and constraint on the closed-loop pole location based on the new Lyapunov stability condition is generalized to cope with state-feedback synthesis problem. In Section 3, robust controller synthesis conditions for state-feedback control of uncertain discrete-time system are proposed. In Section 4, robust controller synthesis is further extended to multi-objective multi-channels for uncertain discrete-time system, which is one of the main results of this work. In Section 5, an illustrational example is given to verify this paper's proposed approach and compare it with state feedback synthesis based on Lyapunov shaping paradigm (Scherer *et al.*, 1997). A brief summary of the proposed method is given in Section 6.

EXTENDED LMI ANALYSIS CONDITIONS

Consider a linear discrete time-invariant system in the state-space form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \tilde{\boldsymbol{A}}\boldsymbol{x}(k) + \tilde{\boldsymbol{B}}\boldsymbol{w}(k), \\ \boldsymbol{z}(k) &= \tilde{\boldsymbol{C}}\boldsymbol{x}(k) + \tilde{\boldsymbol{D}}\boldsymbol{w}(k). \end{aligned} \tag{1}$$

where the state vector $\boldsymbol{x} \in \mathbb{R}^{n_x}$, the exogenous input $\boldsymbol{w} \in \mathbb{R}^{n_w}$, the controlled output $\boldsymbol{z} \in \mathbb{R}^{n_z}$, and the corresponding matrices have appropriate dimensions.

Also consider a linear discrete time-invariant uncertain system $\boldsymbol{G}(\alpha)$ in the state-space form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \tilde{\boldsymbol{A}}(\alpha)\boldsymbol{x}(k) + \tilde{\boldsymbol{B}}(\alpha)\boldsymbol{w}(k), \\ \boldsymbol{z}(k) &= \tilde{\boldsymbol{C}}(\alpha)\boldsymbol{x}(k) + \tilde{\boldsymbol{D}}(\alpha)\boldsymbol{w}(k). \end{aligned} \tag{2}$$

Assume that the matrices characterizing the dynamic system belong to a convex bounded set defined by matrix $\boldsymbol{M}$

$$\boldsymbol{M} = \begin{bmatrix} \tilde{\boldsymbol{A}} & \tilde{\boldsymbol{B}} \\ \tilde{\boldsymbol{C}} & \tilde{\boldsymbol{D}} \end{bmatrix} \tag{3}$$

belonging to a convex bounded polyhedron $\boldsymbol{\Phi}$. That is, each uncertain matrix in this domain may be written as an unknown convex combination of N given extreme matrices $\boldsymbol{M}_1,\ldots,\boldsymbol{M}_N$ such that

$$\begin{aligned} &\boldsymbol{\Phi} := \left\{ \boldsymbol{M}(\alpha) : \boldsymbol{M}(\alpha) = \sum_{1}^{N} \alpha_i \boldsymbol{M}_i \right\}, \\ &\alpha \in \boldsymbol{\Psi} := \left\{ \alpha \mid \alpha_i \ge 0, \sum_{i=1}^{N} \alpha_i = 1 \right\}. \end{aligned} \tag{4}$$

Assume that the uncertain parameter α is time-invariant. The symbols

$$\boldsymbol{T}_{zw} := \left\{ \tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}, \tilde{\boldsymbol{C}}, \tilde{\boldsymbol{D}} \right\}, \tag{5}$$

$$\boldsymbol{T}_{zw}(\alpha) := \left\{ \tilde{\boldsymbol{A}}(\alpha), \tilde{\boldsymbol{B}}(\alpha), \tilde{\boldsymbol{C}}(\alpha), \tilde{\boldsymbol{D}}(\alpha) \right\} \tag{6}$$

denote respectively transfer function matrices from the input $\boldsymbol{w}$ to the output $\boldsymbol{z}$ for the nominal system and uncertain system.

Extension of Gl_2 norm

For Gl_2 control problem, the disturbance sets $\boldsymbol{W}$ and criterion sets $\boldsymbol{Z}$, which are associated with the input from the uncertainty $\boldsymbol{\Delta}$ and the cost criterion respectively, will be introduced firstly. Their definitions and properties were described and explored by D'Andrea (1999) in detail. For the sake of simplicity, reconstruct these sets as

$$\boldsymbol{W} := \{ \boldsymbol{w}_k \in l_2 : \|\boldsymbol{w}_k\| \le 1, k \in [1, n_w] \} \tag{7}$$

$$\boldsymbol{Z} := \{ z_l \in l_2 : \|z_l\| \le 1, l \in [1, n_z] \} \tag{8}$$

here $\boldsymbol{w}_k$ and $\boldsymbol{z}_l$ are the elements of $\boldsymbol{w}$ and $\boldsymbol{z}$ in system Eq.(1). Some notations are defined as follows.

$$\begin{aligned} &\boldsymbol{X}_w := x_1 \boldsymbol{I}_{w_1} \oplus x_2 \boldsymbol{I}_{w_2} \oplus \cdots \oplus x_m \boldsymbol{I}_{w_m} > 0, \\ &\sum_{i=1}^{m} x_i \le \gamma; \end{aligned} \tag{9}$$

$$\begin{aligned} &\boldsymbol{Y}_z := y_1 \boldsymbol{I}_{z_1} \oplus y_2 \boldsymbol{I}_{z_2} \oplus \cdots \oplus y_n \boldsymbol{I}_{z_n} > 0, \\ &\sum_{i=1}^{n} y_i \le \gamma. \end{aligned} \tag{10}$$

where x_k, $y_l \in \mathbb{R}^+$, $\oplus$ stands for the direct sum of matrices, i.e. $\boldsymbol{A}\oplus\boldsymbol{B}:=\begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{B}\end{bmatrix}$, $\boldsymbol{I}_{w_m}$ and $\boldsymbol{I}_{z_n}$ are the identity matrices with the same dimensions as $\boldsymbol{w}_k\boldsymbol{w}_k^{\mathrm{T}}$ and $\boldsymbol{z}_l\boldsymbol{z}_l^{\mathrm{T}}$ respectively.

The following Lemma is used to extend the Gl_2 norm conditions to uncertain system analysis problem, based on extended LMI characterizations

Lemma 1 [Extended Gl_2 norm (Yan and Zhang, 2004)] Let the system Eq.(1) be a minimal discrete time state space representation, then $\|\boldsymbol{T}_{zw}\|_{\mathrm{G}l_2}<\gamma$ holds if and only if $\boldsymbol{X}_w$ and $\boldsymbol{Y}_z$ satisfy Eqs.(9) and (10), and there exist a general matrix $\boldsymbol{G}$ and a symmetric matrix $\boldsymbol{P}$ such that

$$\begin{bmatrix}\boldsymbol{P} & \tilde{\boldsymbol{A}}\boldsymbol{G} & \tilde{\boldsymbol{B}} & \boldsymbol{0}\\ (\cdot)^{\mathrm{T}} & \boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{G}^{\mathrm{T}}\tilde{\boldsymbol{C}}^{\mathrm{T}}\\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{X}_w & \tilde{\boldsymbol{D}}^{\mathrm{T}}\\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{Y}_z\end{bmatrix}>0 \qquad (11)$$

is feasible.

The proof was presented in detail in Yan and Zhang (2004).

Let us consider plant Eq.(2), based on this Lemma, the guaranteed Gl_2 cost LMI conditions of Eq.(2) based on this Lemma can be derived according to parameter-dependent Lyapunov function. The following theorem characterizes guaranteed Gl_2 cost can be computed using LMI optimization.

Theorem 1 (Extended guaranteed Gl_2 norm) If $\boldsymbol{X}_w$ and $\boldsymbol{Y}_z$ satisfy Eqs.(9) and (10) and there exist a general matrix $\boldsymbol{G}$ and symmetric matrices $\boldsymbol{P}_i$, i=1, …, N, such that

$$\begin{bmatrix}\boldsymbol{P}_i & \tilde{\boldsymbol{A}}_i\boldsymbol{G} & \tilde{\boldsymbol{B}}_i & \boldsymbol{0}\\ (\cdot)^{\mathrm{T}} & \boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}_i & \boldsymbol{0} & \boldsymbol{G}^{\mathrm{T}}\tilde{\boldsymbol{C}}_i^{\mathrm{T}}\\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{X}_w & \tilde{\boldsymbol{D}}_i^{\mathrm{T}}\\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{Y}_z\end{bmatrix}>0 \qquad (12)$$

holds for all i=1, …, N, where the matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $\boldsymbol{C}_i$ and $\boldsymbol{D}_i$ define the extreme matrices $\boldsymbol{M}_i$, i=1, …, N, then inequality $\|\boldsymbol{T}_{zw}(\alpha)\|_{\mathrm{G}l_2}<\gamma$ holds for all matrices $\boldsymbol{M}$ in the domain $\boldsymbol{\Phi}$.

This theorem can be proved by introducing the affine parameter-dependent Lyapunov matrix, which is defined as the convex combination of the uncertain parameters

$$\boldsymbol{P}(\alpha)=\sum_{i=1}^{N}\alpha_i\boldsymbol{P}_i \qquad (13)$$

The key point of the proof is that Eq.(12) are affine in the extreme matrices $\boldsymbol{M}_i$ and in the variables $\boldsymbol{P}_i$, i=1, …, N. For instance, to prove Theorem 1 assume that all inequalities in Eq.(12) hold. Multiply each of these inequalities by α_i>0, and sum them up to obtain

$$\begin{bmatrix}\boldsymbol{P}(\alpha) & \tilde{\boldsymbol{A}}(\alpha)\boldsymbol{G} & \tilde{\boldsymbol{B}}(\alpha) & \boldsymbol{0}\\ (\cdot)^{\mathrm{T}} & \boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}(\alpha) & \boldsymbol{0} & \boldsymbol{G}^{\mathrm{T}}\tilde{\boldsymbol{C}}(\alpha)^{\mathrm{T}}\\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{X}_w & \tilde{\boldsymbol{D}}(\alpha)^{\mathrm{T}}\\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{Y}_z\end{bmatrix}>0 \qquad (14)$$

Since the above inequality holds for every α such that $\boldsymbol{M}$ belongs to a convex bounded polyhedron $\boldsymbol{\Phi}$, Theorem 1 is true.

Extension of GH_2

The following Lemma is used to extend the GH_2 conditions to uncertain system analysis problem based on extended LMI characterizations.

Lemma 2 [Extended GH_2 (Yan and Zhang, 2004)] The inequality $\|\boldsymbol{T}_{zw}\|^2_{\mathrm{G}H_2}<\beta$ holds if, and only if, there exist a matrix $\boldsymbol{G}$ and a symmetric matrix $\boldsymbol{P}$ such that

$$\begin{bmatrix}\beta\boldsymbol{I} & \tilde{\boldsymbol{C}}\boldsymbol{G}\\ (\cdot)^{\mathrm{T}} & \boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}\end{bmatrix}>0, \qquad (15)$$

$$\begin{bmatrix}\boldsymbol{P} & \tilde{\boldsymbol{A}}\boldsymbol{G} & \tilde{\boldsymbol{B}}\\ (\cdot)^{\mathrm{T}} & \boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P} & \boldsymbol{0}\\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{I}\end{bmatrix}>0 \qquad (16)$$

are feasible.

This proof is given in detail in Yan and Zhang (2004).

Consider plant Eq.(2), being similar to Theorem 1, the following theorem characterizes guaranteed GH_2 cost can be computed using LMI optimization.

Theorem 2 (Extended guaranteed GH_2 norm) If there exist a general matrix $\boldsymbol{G}$ and symmetric matrices $\boldsymbol{P}_i$, i=1, …, N, such that

$$\begin{bmatrix} \beta \boldsymbol{I} & \tilde{\boldsymbol{C}}_i \boldsymbol{G} \\ (\cdot)^{\mathrm{T}} & \boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}_i \end{bmatrix} > 0 \tag{17}$$

$$\begin{bmatrix} \boldsymbol{P}_i & \tilde{\boldsymbol{A}}_i \boldsymbol{G} & \tilde{\boldsymbol{B}}_i \\ (\cdot)^{\mathrm{T}} & \boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}_i & \boldsymbol{0} \\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{I} \end{bmatrix} > 0 \tag{18}$$

hold for all i=1, …, N, where the matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, and $\boldsymbol{C}_i$ define the extreme matrices $\boldsymbol{M}_i$, i=1, …, N, then inequality $\|\boldsymbol{T}_{zw}\|_{\mathrm{G}H_2}^2 < \beta$ holds for all matrices $\boldsymbol{M}$ in the domain $\boldsymbol{\Phi}$.

The proof of this theorem is similar to that for Gl_2 in Theorem 1.

Extension of pole placement constraints

For a discrete-time system, a simple but useful pole placement sub-region, i.e. disc region, can be represented as follows.

$$\boldsymbol{C}_D(z_0, \rho) := \{\lambda \in \mathbb{C}, |\lambda + z_0| < \rho\}$$

where $\mathbb{C}$ denotes the complex plane, ρ>0 and $z_0 \in \mathbb{C}$.

Lemma 3 (Root-Clustering condition (Yan and Zhang, 2004)) Assuming $\lambda \in \boldsymbol{C}_D$ to be the poles of system Eq.(1), it is equivalent to

$$\begin{bmatrix} \rho \boldsymbol{P} & \tilde{\boldsymbol{A}}\boldsymbol{G} + z_0 \boldsymbol{G} \\ (\cdot)^{\mathrm{T}} & \rho(\boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}) \end{bmatrix} > 0 \tag{19}$$

for some $\boldsymbol{P}$ and $\boldsymbol{G}$.

Similarly, the extended root-clustering condition in Yan and Zhang (2004) can be extended to uncertain system.

Theorem 3 (Extended Root-Clustering condition) If there exist a general matrix $\boldsymbol{G}$ and symmetric matrices $\boldsymbol{P}_i$, i=1, …, N, such that

$$\begin{bmatrix} \rho \boldsymbol{P}_i & \tilde{\boldsymbol{A}}_i\boldsymbol{G} + z_0 \boldsymbol{G} \\ (\cdot)^{\mathrm{T}} & \rho(\boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}_i) \end{bmatrix} > 0 \tag{20}$$

holds for all i=1, …, N, where the matrices $\boldsymbol{A}_i$ define the extreme matrices $\boldsymbol{M}_i$, i=1, …, N, then $\lambda_i \in \boldsymbol{C}_D$ are the poles of system Eq.(2).

The proof of this theorem is easy and similar to that for Gl_2 in Theorem 1.

ROBUST CONTROL SYNTHESIS FOR UNCERTAIN SYSTEM

In the preceding sections, we extended LMI characterizations for Gl_2, GH_2 performances and for the D-stability constraints to uncertain system. These extended LMI characterizations are applied in this section to robust state-feedback control synthesis for discrete-time with convex polytopic uncertain domain.

For the purpose of synthesis, consider the linear discrete time-invariant plant with convex polytopic uncertain domain described by

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}(\alpha)\boldsymbol{x}(k) + \boldsymbol{B}_w(\alpha)\boldsymbol{w}(k) + \boldsymbol{B}_u(\alpha)\boldsymbol{u}(k) \\ \boldsymbol{z}(k) &= \boldsymbol{C}_z(\alpha)\boldsymbol{x}(k) + \boldsymbol{D}_{zw}(\alpha)\boldsymbol{w}(k) + \boldsymbol{D}_{zu}(\alpha)\boldsymbol{u}(k) \end{aligned} \tag{21}$$

where the state vector $\boldsymbol{x} \in \mathbb{R}^{n_x}$, the exogenous input $\boldsymbol{w} \in \mathbb{R}^{n_w}$, the controlled output $\boldsymbol{z} \in \mathbb{R}^{n_z}$, the control input $\boldsymbol{u} \in \mathbb{R}^{n_u}$ and all other matrices and vectors have appropriate dimensions.

Being similar to Eq.(3), a matrix $\boldsymbol{M}$ which belongs to a convex bounded polyhedron $\boldsymbol{\Phi}$ for open-loop can be defined as

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}_w & \boldsymbol{B}_u \\ \boldsymbol{C}_z & \boldsymbol{D}_{zw} & \boldsymbol{D}_{zu} \end{bmatrix} \tag{22}$$

with the transfer function $\boldsymbol{T}_{zw}(a)$ changing accordingly.

Connecting this system with the linear controllers to be defined will always provide linear systems with closed-loop state-space representation

$$\begin{aligned} \tilde{\boldsymbol{x}}(k+1) &= \boldsymbol{A}_{\mathrm{cl}}(\alpha)\tilde{\boldsymbol{x}}(k) + \boldsymbol{B}_{\mathrm{cl}}(\alpha)\boldsymbol{w}(k) \\ \boldsymbol{z}(k) &= \boldsymbol{C}_{\mathrm{cl}}(\alpha)\tilde{\boldsymbol{x}}(k) + \boldsymbol{D}_{\mathrm{cl}}(\alpha)\boldsymbol{w}(k) \end{aligned} \tag{23}$$

Considering plant Eq.(21), the linear static state-feedback control law

$$\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x} \tag{24}$$

is sought for. The closed-loop transfer function $\boldsymbol{T}_{zw}(a)$ represented in Eq.(5) or Eq.(6) has the system matrices below.

$$\boldsymbol{A}_{\mathrm{cl}i}:=\boldsymbol{A}_i+\boldsymbol{B}_{ui}\boldsymbol{K},\quad \boldsymbol{B}_{\mathrm{cl}i}:=\boldsymbol{B}_{wi} \tag{25}$$

$$\boldsymbol{C}_{\mathrm{cl}i}:=\boldsymbol{C}_{zi}+\boldsymbol{D}_{zui}\boldsymbol{K},\quad \boldsymbol{D}_{\mathrm{cl}i}:=\boldsymbol{D}_{zw} \tag{26}$$

After placing Eqs.(25)~(26) into the inequalities of Theorems 1~3 and using non-linear transformation

$$\boldsymbol{L}:=\boldsymbol{K}\boldsymbol{G}, \tag{27}$$

yields the following state-feedback controller synthesis version of Theorems 1~3.

Theorem 4 (Gl_2 state-feedback synthesis) There exists a controller in Eq.(24) such that the inequality $\|\boldsymbol{T}_{zw}\|_{\mathrm{G}l_2}<\gamma$ holds, if the LMI

$$\begin{bmatrix} \boldsymbol{P}_i & \boldsymbol{A}_i\boldsymbol{G}+\boldsymbol{B}_{ui}\boldsymbol{L} & \boldsymbol{B}_{wi} & \boldsymbol{0} \\ (\cdot)^{\mathrm{T}} & \boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}_i & \boldsymbol{0} & \boldsymbol{G}^{\mathrm{T}}\boldsymbol{C}_{zi}^{\mathrm{T}}+\boldsymbol{L}^{\mathrm{T}}\boldsymbol{D}_{zui}^{\mathrm{T}} \\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{X}_w & \boldsymbol{D}_{zwi}^{\mathrm{T}} \\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{Y}_z \end{bmatrix}>0 \tag{28}$$

hold, where the matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_{wi}$, $\boldsymbol{B}_{ui}$, $\boldsymbol{C}_{zi}$, $\boldsymbol{D}_{zwi}$, and $\boldsymbol{D}_{zui}$ define the extreme matrices $\boldsymbol{M}_i$, i=1, …, N, and the matrices $\boldsymbol{G}$ and $\boldsymbol{L}$ and the symmetric $\boldsymbol{P}_i$, i=1, …, N, are variables.

Theorem 5 (GH_2 state-feedback synthesis) There exists a controller in Eq.(24) such that the inequality $\|\boldsymbol{T}_{zw}(\alpha)\|_{\mathrm{G}H_2}^2<\beta$ holds, if the following LMI

$$\begin{bmatrix} \beta\boldsymbol{I} & \boldsymbol{C}_{zi}\boldsymbol{G}+\boldsymbol{D}_{zui}\boldsymbol{L} \\ (\cdot)^{\mathrm{T}} & \boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}_i \end{bmatrix}>0 \tag{29}$$

$$\begin{bmatrix} \boldsymbol{P}_i & \boldsymbol{A}_i\boldsymbol{G}+\boldsymbol{B}_{ui}\boldsymbol{L} & \boldsymbol{B}_{wi} \\ (\cdot)^{\mathrm{T}} & \boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}_i & \boldsymbol{0} \\ (\cdot)^{\mathrm{T}} & (\cdot)^{\mathrm{T}} & \boldsymbol{I} \end{bmatrix}>0 \tag{30}$$

hold, where the matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_{wi}$, $\boldsymbol{B}_{ui}$, $\boldsymbol{C}_{zi}$, $\boldsymbol{D}_{zwi}$, and $\boldsymbol{D}_{zui}$ define the extreme matrices $\boldsymbol{M}_i$, i=1, …, N, and the matrices $\boldsymbol{G}$ and $\boldsymbol{L}$ and the symmetric $\boldsymbol{P}_i$, i=1,…, N, are variables.

Theorem 6 (Root-Clustering state-feedback synthesis) There exists a controller in Eq.(24) such that the $\lambda_i\in\boldsymbol{C}_D$ to be the poles of system Eq.(21), if the following LMI

$$\begin{bmatrix} \rho\boldsymbol{P}_i & \boldsymbol{A}_i\boldsymbol{G}+\boldsymbol{B}_{ui}\boldsymbol{L}+z_0\boldsymbol{G} \\ (\cdot)^{\mathrm{T}} & \rho(\boldsymbol{G}+\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{P}_i) \end{bmatrix}>0 \tag{31}$$

holds, where the matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_{ui}$ define the extreme matrices $\boldsymbol{M}_i$, i=1, …, N, and the matrices $\boldsymbol{G}$ and $\boldsymbol{L}$ and the symmetric $\boldsymbol{P}_i$, i=1, …, N, are variables.

It is easy to prove Theorems 4~6. Here, the proof is omitted because of limited paper space.

Note Eq.(27), the controller is given by $\boldsymbol{K}:=\boldsymbol{L}\boldsymbol{G}^{-1}$.

If we let $\boldsymbol{P}_i=\boldsymbol{P}$, i=1, …, N and $\boldsymbol{G}=\boldsymbol{P}$ in the LMIs Eqs.(28)~(31), they are reduced to LMI formulations based on quadratic stability concept for comparison.

MULTI-OBJECTIVE ROBUST CONTROLLER SYNTHESIS FOR UNCERTAIN SYSTEM

The multi-objective problem to be dealt with in this section is defined as the problem of determining a state-feedback controller for the plant Eq.(21) such that closed-loop Gl_2 performance, GH_2 performance, and constraints on the closed-loop pole location are satisfied for all vertices of the polytope.

The motivations of using such mix of performance measures are as follows:

(1) The Gl_2 specification combines concepts such as H_∞ optimization, linear matrix inequalities (LMI's), and integral quadratic constraints, so a convex characterization of the solution to a large class of robust and optimal control problems is presented. The problem can be represented in LMI formulation (Wang and Wilson, 2001) with the potential to get solutions less conservative than those obtained via H_∞ control synthesis approach. The computations involved in solving a Gl_2-optimization problem are more tractable than those required by the μ-optimization.

(2) The GH_2 specification is a desirable way to keep the peak amplitude of the output below a certain level to avoid actuator saturation if the input is measured in energy (Scherer *et al*., 1997).

(3) Suitable pole placement can effectively prevent the rapid system dynamics important for the digital controller implementation.

Typically, those specifications are defined for particular channels or channel combinations. It is assumed that these specifications are imposed on closed-loop transfer functions of the form

$$\boldsymbol{T}^i_{z_j w_j} := \boldsymbol{L}_j \boldsymbol{T}^i_{zw} \boldsymbol{R}_j \tag{32}$$

where the matrices $\boldsymbol{L}_j$, $\boldsymbol{R}_j$ select the appropriate input/output channels or channel combinations, i denotes the ith vertex in set $\boldsymbol{\Psi}$, j denotes the jth specification of closed-loop system.

Employing the matrices $\boldsymbol{L}_j$, $\boldsymbol{R}_j$, the associated pairs of signals is defined as Eq.(33).

$$w_j := \boldsymbol{R}_j \boldsymbol{w}, \quad z_j := \boldsymbol{L}_j \boldsymbol{z} \tag{33}$$

The closed-loop subsystems $\boldsymbol{T}^i_{z_j w_j}$ represented in Eq.(5) or Eq.(6) have the following system matrices.

$$\boldsymbol{A}^j_{\mathrm{cl}i} := \boldsymbol{A}_i + \boldsymbol{B}_{ui}\boldsymbol{K}, \qquad \boldsymbol{B}^j_{\mathrm{cl}i} := \boldsymbol{B}_{wi}\boldsymbol{R}_j \tag{34}$$

$$\boldsymbol{C}^j_{\mathrm{cl}i} := \boldsymbol{L}_j \boldsymbol{C}_{zi} + \boldsymbol{L}_j \boldsymbol{D}_{zui}\boldsymbol{K}, \quad \boldsymbol{D}^j_{\mathrm{cl}i} := \boldsymbol{L}_j \boldsymbol{D}_{zwi}\boldsymbol{R}_j \tag{35}$$

In this form, closed-loop performance and robustness may be ensured by constraining the Gl_2 and GH_2 norms of transfer functions from exogenous input w_j to regulated output z_j for all vertices of the convex polytopic uncertain domain.

Suppose that a controller meeting M closed-loop specifications (constraints) is sought for. Applying Theorems 4~6 to the M closed-loop transfer functions, one has to deal with two kinds of variables: the controller parameters and instrumental matrices $\boldsymbol{G}_j$, $\boldsymbol{P}_{ij}$, i=1, …, N, j=1, …, M. It is not difficult to see from Theorems 4~6 that this approach employs M affine parameter-dependent Lyapunov functions

$$\boldsymbol{P}_j(\alpha) = \sum_{i=1}^{N} \alpha_i \boldsymbol{P}_{ij}, \; j=1, \ldots, M \tag{36}$$

to establish the robust multi-objective performance, where these Lyapunov variables are non-common for the multi-specification and at the same time parameter-dependent over the whole uncertain domain. Because there exist different $\boldsymbol{G}_j$ in different LMI, directly applying the results of Theorems 4~6 to each closed-loop system with convex polytopic uncertain domain will lead to different controllers. Therefore, it is necessary to impose the additional constraint $\boldsymbol{G}_j$=$\boldsymbol{G}$, j=1, …, M. These constraints ensure that the controllers of different channels to be identical. Meanwhile, it guarantees that the joint problem is an LMI optimization readily solved efficiently by a convex optimization program.

Eqs.(32)~(36) show that G-shaping paradigm used to deal with multi-objective multi-channels synthesis problem for known discrete-time system has been generalized to discrete-time uncertain system. It is one main result of this work. In the next section, an illustrative example is used to show that the synthesis procedure developed in this paper is prior to the one based on Lyapunov-shaping paradigm (Scherer *et al*., 1997).

ILLUSTRATIVE EXAMPLE

Consider the following example of a linear discrete time-invariant system with convex polytopic uncertainty, which had been investigated in de Oliverira *et al*.(2002) for output feedback control. Here, it is modified properly to interpret the approach developed in this paper.

$$\boldsymbol{x}(k+1) = \alpha_1 \begin{bmatrix} 1.9 & 0 & 1 \\ 1 & 0.5 & 0 \\ 0 & 1 & -0.5 \end{bmatrix} \boldsymbol{x}(k) + \alpha_1 \begin{bmatrix} 0.9 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u}(k)$$
$$+ \alpha_2 \begin{bmatrix} 2.1 & 0 & 1 \\ 1 & 0.5 & 0 \\ 0 & 1 & -0.5 \end{bmatrix} \boldsymbol{x}(k) + \alpha_2 \begin{bmatrix} 1.1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u}(k)$$

where α_1, α_2>0 are the uncertain parameters and $\alpha_1+\alpha_2=1$. The following inputs and outputs are added to the above system

$\boldsymbol{w}(k)=[w_1(k), w_2(k), w_3(k), w_4(k), w_5(k)]$,
$\boldsymbol{z}(k)=[z_1(k), z_2(k), z_3(k), z_4(k), z_5(k), z_6(k), z_7(k)]$,

so that the original system is modified as follows

$$\boldsymbol{x}(k+1) = \alpha_1 \boldsymbol{A}_1 \boldsymbol{x}(k) + \alpha_2 \boldsymbol{A}_2 \boldsymbol{x}(k)$$
$$+\boldsymbol{B}_{u1}\boldsymbol{u}(k) + \boldsymbol{B}_{u2}\boldsymbol{u}(k) + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1(k) \\ w_2(k) \\ w_4(k) \end{bmatrix}$$

and outputs

$$\begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{z}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{u}(k)$$

$$z_4(k)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}x(k)+\begin{bmatrix}0\\0\\0\\1\end{bmatrix}u(k)$$

Partition inputs and outputs as two sets of channels, the symbols $\boldsymbol{T}_{z_1w_1}^1$, $\boldsymbol{T}_{z_1w_1}^2$ denote respectively the transfer function from the input $[w_1(k), w_2(k), w_3(k)]$ to the output $[z_1(k), z_2(k), z_3(k)]$ for two extreme matrices of convex polytopic domain. The symbols $\boldsymbol{T}_{z_2w_2}^1$, $\boldsymbol{T}_{z_2w_2}^2$ respectively denote the transfer function from the input $[w_4(k), w_5(k)]$ to the output $[z_4(k), z_5(k), z_6(k), z_7(k)]$ of two extreme matrices of convex polytopic domain.

The problem is to design a state feedback controller such that

$$\min k_1\gamma+k_2\beta$$

where $k_1>0$, $k_2>0$ are the weight coefficients, and $k_1+k_2=1$, under constraints $\|\boldsymbol{T}_{z_1w_1}^1\|_{\mathrm{G}l_2}<\gamma$, $\|\boldsymbol{T}_{z_1w_1}^2\|_{\mathrm{G}l_2}<\gamma$, $\|\boldsymbol{T}_{z_2w_2}^1\|_{\mathrm{G}H_2}<\beta$, $\|\boldsymbol{T}_{z_2w_2}^2\|_{\mathrm{G}H_2}<\beta$ and that the closed-loop system poles locate at the circle in complex plane, which centre $z_0=0$ and radius $r=0.9$.

Employing Theorems 4~6, and gathering the appropriate LMI for each Gl_2, GH_2 and pole placement constraint, this control problem can be recast into a multi-channel and multi-objective state feedback control synthesis problem that can be efficiently solved by the LMI convex procedure. Setting different k_1, k_2 values yields different trade-off solutions.

If we let $k_1=0.9$, $k_2=0.1$ and the number of input and output uncertain block for the first channel are both equal to one, the optimal performance is

$$\gamma=2.1, \beta=1.71$$

and controller

$$\boldsymbol{K}=[-2.2091 \quad -0.4391 \quad -0.8423]$$

If Lyapunov shaping paradigm is used, the results are

$$\gamma=2.15, \beta=2.06$$
$$\boldsymbol{K}=[-2.1923 \quad -0.3992 \quad -0.8473]$$

From the example, we can see that the approach proposed in this paper is accurate and less conservative than the existing one based on Lyapunov shaping paradigm.

CONCLUSION

The proposed multi-objective multi-channel robust controller synthesis procedure for discrete-system with convex polytopic uncertain system is based on the results of Yan and Zhang (2004)'s work in extended Gl_2 and GH_2 norm condition and on the results of de Oliverira *et al*.(1999b; 2002)'s work on the G-shaping paradigm. The new synthesis method can deal with multi-objective state-feedback controller synthesis for convex polytopic uncertain system in unified tractable LMI form rather than bilinear from involving in complex computations. Controller parameterization based on the extended G-shaping paradigm, does not depend on the Lyapunov matrix but general matrix, which results in a reduction of the conservation involved in a standard design framework.

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