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A note on strong law of large numbers of random variables*

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Abstract: In this paper, the Chung's strong law of large numbers is generalized to the random variables which do not need the condition of independence, while the sequence of Borel functions verifies some conditions weaker than that in Chung's theorem. Some convergence theorems for martingale difference sequence such as L_p martingale difference sequence are the particular cases of results achieved in this paper. Finally, the convergence theorem for ***A***-summability of sequence of random variables is proved, where ***A*** is a suitable real infinite matrix.

Key words: Strong law of large numbers (SLLN), Martingale difference sequence, ***A***-summable sequence
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INTRODUCTION

Chung (1947) proved the so-called "Chung's strong law of large numbers". Let $\{X_n, n\in\mathbb{N}\}$ be a sequence of independent random variables with $EX_n=0$ for all n and $0<a_n\uparrow\infty$, if φ is a positive even and continuous function such that either

$$\varphi(t)/|t|\downarrow \text{ as } |t|\uparrow$$

or

$$\varphi(t)/|t|\uparrow \text{ and } \varphi(t)/t^2\downarrow \text{ as } |t|\uparrow$$

and

$$\sum_{n=1}^{\infty} E[\varphi(X_n)]/\varphi(a_n)<\infty$$

holds, then

$$\sum_{n=1}^{\infty}(X_n/a_n) \text{ converges a.s.}$$

Jardas *et al.*(1998) extended the classical Chung's SLLN to a sequence of independent random variables $\{X_n, n\in\mathbb{N}\}$ weighted by a sequence of non-zero reals $\{a_n, n\in\mathbb{N}\}$ with $EX_n=0$ for all n, by using a sequence of $\{\varphi_n, n\in\mathbb{N}\}$ of Borel functions verifying some conditions weaker than Chung's condition.

We try to remove the independent condition of random variables. Let $\{X_n, n\in\mathbb{N}\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{F}_0=\{\varnothing, \Omega\}$, $\{\mathcal{F}_n, n\in\mathbb{N}\}$ be a sequence of σ-fields in $\mathcal{F}$ satisfying $\mathcal{F}_n\subseteq\mathcal{F}_{n+1}$, for all $n\in\mathbb{N}$. Suppose that $\{X_n, n\in\mathbb{N}\}$ is adapted to $\{\mathcal{F}_n, n\in\mathbb{N}\}$. This paper aims at studying the SLLN for stochastic sequence $\{X_n, \mathcal{F}_n, n\in\mathbb{N}\}$. As corollaries, some convergence theorems for martingale difference sequence are obtained. Chung's classical strong law of large numbers for sequence of independent random variables is a particular case of the result of this paper. The almost certain ***A***-summability for random variables is also considered.

MAIN RESULTS

Theorem 1 Let $\{X_n, \mathcal{F}_n, n\in\mathbb{N}\}$ be a stochastic sequence defined as before, and let $\{a_n, n\in\mathbb{N}\}$ be a sequence of non-zero reals. Let φ_n: $\mathbb{R}_+\to\mathbb{R}_+$ be Borel functions and let $\alpha_n\geq1$, $\beta_n\leq2$, $C_n>0$, $D_n>0$ $(n\in\mathbb{N})$ be

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constants satisfying

$$\nu \le \mu \Rightarrow C_n \frac{\mu^{\alpha_n}}{\nu^{\alpha_n}} \le \frac{\varphi_n(\mu)}{\varphi_n(\nu)} \le D_n \frac{\mu^{\beta_n}}{\nu^{\beta_n}}. \quad (1)$$

If

$$\sum_{n=1}^{\infty} A_n \frac{E[\varphi_n(|X_n|)|\mathcal{F}_{n-1}]}{\varphi_n(|a_n|)} < \infty \quad \text{a.s.} \quad (2)$$

where A_n=max($1/C_n$, D_n), then

$$\sum_{n=1}^{\infty} \frac{X_n - E[X_n|\mathcal{F}_{n-1}]}{a_n} \text{ converges a.s.} \quad (3)$$

Proof Let $X_n'=X_n I\{|X_n|\le|a_n|\}$, $n\in\mathbb{N}$. It follows from Eq.(1) that on the set $\{x: |x|>|a_n|\}$, we have

$$\frac{|x|}{|a_n|} \le \frac{|x|^{\alpha_n}}{|a_n|^{\alpha_n}} \le A_n \frac{\varphi_n(|x|)}{\varphi_n(|a_n|)}.$$

Thus we have

$$\left|\frac{E[X_n'|\mathcal{F}_{n-1}] - E[X_n|\mathcal{F}_{n-1}]}{a_n}\right| \le E\left[\frac{|X_n - X_n'|}{|a_n|}\Big|\mathcal{F}_{n-1}\right]$$
$$= E\left[\frac{|X_n|}{|a_n|} I\{|X_n|>|a_n|\}\Big|\mathcal{F}_{n-1}\right]$$
$$\le A_n E\left[\frac{\varphi_n(|X_n|)}{\varphi_n(|a_n|)} I\{|X_n|>|a_n|\}\Big|\mathcal{F}_{n-1}\right]$$
$$\le A_n \frac{E[\varphi_n(|X_n|)|\mathcal{F}_{n-1}]}{\varphi_n(|a_n|)} \quad \text{a.s.} \quad (4)$$

By Eqs.(2) and (4), we have

$$\sum_{n=1}^{\infty} \frac{E[X_n'|\mathcal{F}_{n-1}] - E[X_n|\mathcal{F}_{n-1}]}{a_n} \text{ converges a.s.} \quad (5)$$

Let

$$\Omega_k = \left\{\sum_{n=1}^{\infty} A_n \frac{E[\varphi_n(|X_n|)|\mathcal{F}_{n-1}]}{\varphi_n(|a_n|)} \le k\right\}, \quad (6)$$

and

$$n(k) = \max\left(n: n\ge 1,\ \sum_{i=1}^{n} A_i \frac{E[\varphi_i(|X_i|)|\mathcal{F}_{i-1}]}{\varphi_i(|a_i|)} \le k\right),$$
$$k\in\mathbb{Z}_+, \quad (7)$$

where $n(k)=+\infty$ if $\sum_{i=1}^{\infty} A_i \frac{E[\varphi_i(|X_i|)|\mathcal{F}_{i-1}]}{\varphi_i(|a_i|)} \le k$, then Ω_k= $\{n(k)=+\infty\}$. Since $I\{n(k)\ge n\}$ is measurable $\mathcal{F}_{n-1}$, $\varphi_n(|X_n|)/\varphi_n(|a_n|)$ is nonnegative, and

$$\sum_{n=1}^{n(k)} A_n \varphi_n(|X_n|)/\varphi_n(|a_n|)$$
$$= \sum_{n=1}^{\infty} A_n I\{n(k)\ge n\}\varphi_n(|X_n|)/\varphi_n(|a_n|),$$

we have

$$\sum_{n=1}^{\infty}\int_{\Omega_k} A_n \frac{\varphi_n(|X_n|)}{\varphi_n(|a_n|)} \mathrm{d}P_{X_n} = E\left[I\{\Omega_k\}\sum_{n=1}^{\infty} A_n \frac{\varphi_n(|X_n|)}{\varphi_n(|a_n|)}\right]$$
$$= E\left[I\{n(k)=\infty\}\sum_{n=1}^{\infty} A_n \frac{\varphi_n(|X_n|)}{\varphi_n(|a_n|)}\right]$$
$$\le E\left[\sum_{n=1}^{\infty} I\{n(k)\ge n\} A_n \frac{\varphi_n(|X_n|)}{\varphi_n(|a_n|)}\right]$$
$$= E\left\{\sum_{n=1}^{\infty} E\left[I\{n(k)\ge n\} A_n \frac{\varphi_n(|X_n|)}{\varphi_n(|a_n|)}\Big|\mathcal{F}_{n-1}\right]\right\}$$
$$= E\left\{\sum_{n=1}^{\infty} I\{n(k)\ge n\} A_n E\left[\frac{\varphi_n(|X_n|)}{\varphi_n(|a_n|)}\Big|\mathcal{F}_{n-1}\right]\right\}$$
$$= E\left\{\sum_{n=1}^{n(k)} A_n E\left[\frac{\varphi_n(|X_n|)}{\varphi_n(|a_n|)}\Big|\mathcal{F}_{n-1}\right]\right\} \le k. \quad (8)$$

Thus, we have

$$\sum_{n=1}^{\infty} P\{\Omega_k\{X_n' \ne X_n\}\} = \sum_{n=1}^{\infty}\int_{\Omega_k\{|X_n|>|a_n|\}} \mathrm{d}P_{X_n}$$
$$\le \sum_{n=1}^{\infty}\int_{\Omega_k} A_n \frac{\varphi_n(|X_n|)}{\varphi_n(|a_n|)} \mathrm{d}P_{X_n} \le k.$$

By the Borel-Cantelli lemma, we have

$$P\{\Omega_k\{X_n' \ne X_n\},\ \text{i.o.}\}=0.$$

Hence we have

$$\sum_{n=1}^{\infty}(X_n' - X_n)/a_n \text{ converges a.s. on } \Omega_k.$$

Since $\Omega=\cup_k\Omega_k$, it follows that

$$\sum_{n=1}^{\infty}(X_n' - X_n)/a_n \text{ converges a.s.} \quad (9)$$

Put $Y_n=(X'_n-E[X'_n\mid\mathcal{F}_{n-1}])/a_n$, $n\in\mathbb{N}$, then $|Y_n|\le 2$, $E[Y_n|\mathcal{F}_{n-1}]=0$ a.s. and $E[Y_n^2\mid\mathcal{F}_{n-1}]\le E[(X'_n)^2\mid\mathcal{F}_{n-1}]/a_n^2$ a.s. Let

$$\xi_n=\frac{\prod_{m=1}^{n}\exp(Y_m)}{\prod_{m=1}^{n}E[\exp(Y_m)\mid\mathcal{F}_{m-1}]},\tag{10}$$

$$\eta_n=\frac{\prod_{m=1}^{n}\exp(-Y_m)}{\prod_{m=1}^{n}E[\exp(-Y_m)\mid\mathcal{F}_{m-1}]},\ n\in\mathbb{N}.\tag{11}$$

It is easy to show that the sequences $\{\xi_n, n\in\mathbb{N}\}$ and $\{\eta_n, n\in\mathbb{N}\}$ are Martingales. Since $E|\xi_n|=E\xi_n=E\xi_1=1$, by Doob's Martingale convergence theorem, we have

$$\lim_{n\to\infty}\xi_n=\xi_\infty<\infty \text{ a.s.}\tag{12}$$

By inequality $0\le e^x-1-x\le 2x^2$, when $|x|\le 2$, we have

$$\begin{aligned}0&\le E[\exp(Y_n)\mid\mathcal{F}_{n-1}]-1\le E[2Y_n^2\mid\mathcal{F}_{n-1}]\\&\le(2/a_n^2)E[(X'_n)^2\mid\mathcal{F}_{n-1}].\end{aligned}\tag{13}$$

On the set $\{x: |x|\le|a_n|\}$, we have

$$\frac{|x|^2}{|a_n|^2}\le\frac{|x|^{\beta_n}}{|a_n|^{\beta_n}}\le A_n\frac{\varphi_n(|x|)}{\varphi_n(|a_n|)}.$$

Hence we have

$$\begin{aligned}0&\le E[\exp(Y_n)\mid\mathcal{F}_{n-1}]-1\\&\le(2/a_n^2)E[X_n^2 I\{|X_n|\le|a_n|\}\mid\mathcal{F}_{n-1}]\\&\le 2A_n\frac{E[\varphi_n(|X_n|)I\{|X_n|\le|a_n|\}\mid\mathcal{F}_{n-1}]}{\varphi_n(|a_n|)}\\&\le 2A_n\frac{E[\varphi_n(|X_n|)\mid\mathcal{F}_{n-1}]}{\varphi_n(|a_n|)}\ \text{ a.s.}\end{aligned}\tag{14}$$

It follows from Eq.(14) and Eq.(2) that

$$\sum_{n=1}^{\infty}(E[\exp(Y_n)\mid\mathcal{F}_{n-1}]-1)\ \text{ converges a.s.}$$

Or equivalently

$$\prod_{n=1}^{\infty}E[\exp(Y_n)\mid\mathcal{F}_{n-1}]\ \text{ converges a.s.}\tag{15}$$

By Eqs.(10), (12) and (15), we have

$$\lim_{n\to\infty}\prod_{m=1}^{n}\exp(Y_m)=\lim_{n\to\infty}\exp\left(\sum_{m=1}^{n}Y_m\right)<\infty\ \text{ a.s.}\tag{16}$$

Similarly, we can obtain that

$$\lim_{n\to\infty}\exp\left(-\sum_{m=1}^{n}Y_m\right)<\infty\ \text{ a.s.}\tag{17}$$

Thus, we have

$$\sum_{n=1}^{\infty}Y_n=\sum_{n=1}^{\infty}\frac{X'_n-E[X'_n\mid\mathcal{F}_{n-1}]}{a_n}\ \text{ converges a.s.}\tag{18}$$

By Eqs.(5), (9) and (18), Eq.(3) follows.

Remark 1 By putting $\mu=\nu$ (or by using a continuity argument, if φ_n is continuous), it is clear that $0<C_n\le 1$ and $D_n\ge 1$. Moreover, $\beta_n\ge\alpha_n$. The family of functions verifying Eq.(1) is wider than the family of functions verifying $(\varphi_n(x)/|x|)\uparrow$ and $(\varphi_n(x)/x^2)\downarrow$.

Corollary 1 Let $\{X_n, \mathcal{F}_n, n\in\mathbb{N}\}$ be a Martingale difference sequence, φ_n and a_n be as in Theorem 1. If Eq.(2) holds, then

$$\sum_{n=1}^{\infty}X_n/a_n\ \text{ converges a.s.}\tag{19}$$

Corollary 2 (Jardas *et al.*, 1998) Let $\{X_n, n\in\mathbb{N}\}$ be a sequence of independent random variables with $EX_n=0$ for all n, and let $\{a_n, n\in\mathbb{N}\}$ be a sequence of non-zero reals. Let φ_n: $\mathbb{R}_+\to\mathbb{R}_+$ be a Borel function and let $\alpha_n\ge 1$, $\beta_n\le 2$, $K_n\ge 1$, $M_n\ge 1$ $(n\in\mathbb{N})$ be constants satisfying

$$\nu\le\mu\Rightarrow\frac{\varphi_n(\nu)}{\nu^{\alpha_n}}\le K_n\frac{\varphi_n(\mu)}{\mu^{\alpha_n}}$$

and

$$\frac{\nu^{\beta_n}}{\varphi_n(\nu)}\le M_n\frac{\mu^{\beta_n}}{\varphi_n(\mu)}.\tag{20}$$

If

$$\sum_{n=1}^{\infty}K_n\frac{E[\varphi_n(|X_n|)]}{\varphi_n(|a_n|)}<\infty$$

and

$$\sum_{n=1}^{\infty}M_n\frac{E[\varphi_n(|X_n|)]}{\varphi_n(|a_n|)}<\infty,\tag{21}$$

then

$$\sum_{n=1}^{\infty} X_n / a_n \text{ converges a.s.}$$

Proof Let A_n=max(K_n, M_n), $\mathcal{F}_n=\sigma(X_1, \ldots, X_n)$ and $\mathcal{F}_0=\{\varnothing, \Omega\}$. By Eq.(21) and nonnegativeness of φ_n, we have

$$\sum_{n=1}^{\infty} A_n \frac{E[\varphi_n(|X_n|)|\mathcal{F}_{n-1}]}{\varphi_n(|a_n|)} < \infty \text{ a.s.} \quad (22)$$

and by the independence of $\{X_n\}$, we have

$$E[X_n | \mathcal{F}_{n-1}] = E[X_n] = 0 \text{ a.s.} \quad (23)$$

By Eq.(22), Eq.(23) and Theorem 1, this corollary follows:

Remark 2 Chung's theorem in (Chung, 1974; Petrov, 1975) is a special case of Corollary 2.

Corollary 3 (Chow and Teicher, 1998) Let $\{X_n, \mathcal{F}_n, n\in\mathbb{N}\}$ be an L_p Martingale difference sequence for $p\in[1,2]$, and let $0<a_n\uparrow\infty$. If

$$\sum_{n=1}^{\infty} \frac{E[|X_n|^p|\mathcal{F}_{n-1}]}{a_n^p} < \infty \text{ a.s.} \quad (24)$$

then

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{m=1}^{n} X_m = 0 \text{ a.s.} \quad (25)$$

Proof By letting $\varphi_n(x)=|x|^p$, $\alpha_n=1$, $\beta_n=2$, $C_n=D_n=A_n=1$ in Corollary 1, then we have $\sum_{n=1}^{\infty} X_n / a_n$ converges a.s. Since $0<a_n\uparrow\infty$, Eq.(25) follows from Corollary 1 and the Kronecker lemma.

Now we prove almost sure $\boldsymbol{A}$-summability of sequences of random variables satisfying the condition of Theorem 1, where $\boldsymbol{A}$ is a suitable infinite matrix.

Let $\boldsymbol{A}=[a_{nj}]$ $(n, j\in\mathbb{N})$ be a real infinite matrix and let $\{X_n, n\in\mathbb{N}\}$ be a sequence of real numbers. If all the series $\sum_{j=1}^{\infty} a_{nj}x_j$ $(n\in\mathbb{N})$ as well as the sequence $\left\{\sum_{j=1}^{\infty} a_{nj}x_j,\ n\in\mathbb{N}\right\}$ converges, we set

$$\boldsymbol{A} - \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sum_{j=1}^{\infty} a_{nj}x_j$$

and say that the sequence $\{X_n, n\in\mathbb{N}\}$ is $\boldsymbol{A}$-summable. A matrix $\boldsymbol{A}$ such that $\boldsymbol{A} - \lim_{n\to\infty} x_n$ exists whenever $\sum_{n=1}^{\infty} x_n$ converges is called a β-matrix.

Theorem 2 Let $\{X_n, \mathcal{F}_n, n\in\mathbb{N}\}$ be a stochastic sequence defined as Theorem 1. Let $\boldsymbol{A}=[a_{nj}]$ $(n, j\in\mathbb{N})$ be a real infinite matrix, and let $\{c_n, n\in\mathbb{N}\}$ be a sequence of non-zero reals such that

$$\lim_{n\to\infty} a_{nj} = 0,\ j\in\mathbb{N} \quad (26)$$

and

$$\sup_{n\in\mathbb{N}} \sum_{j=1}^{\infty} |c_j a_{nj} - c_{j+1}a_{n,j+1}| < \infty. \quad (27)$$

If Eq.(1) and Eq.(2) hold, then

$$\boldsymbol{A} - \lim_{n\to\infty} (X_n - E[X_n | \mathcal{F}_{n-1}]) = 0 \text{ a.s.} \quad (28)$$

Proof We shall use the idea of the proof of the Proposition in (Butković and Sarapa, 1981). Eq.(28) can be obtained by slight modification of the proof of Theorem 2 in (Jardas *et al.*, 1998) with $E[X_n|\mathcal{F}_{n-1}]$ instead of $E[X_n]$.

References

Butković, D., Sarapa, N., 1981. On the summability of sequence of independent random variables. *Glasnik Mat.*, **16**:157-166.

Chow, Y.S., Teicher, H., 1988. Probability Theory, 2nd Ed. Springer, New York, p.245-255.

Chung, K.L., 1974. A Course in Probability Theory, 2nd Ed. Academic Press, New York, p.109-130.

Jardas, C., Pečarić, J., Sarapa, N., 1998. A note on Chung's strong law of large numbers. *J. Math. Ana. Appl.*, **217**(1):328-334. [doi:10.1006/jmaa.1998.5740]

Petrov, V.V., 1975. Sums of Independent Random Variables. Springer, New York, p.263-268.