Journal of Zhejiang University SCIENCE A
ISSN 1009-3095 (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Two-order Hermite vector-interpolating subdivision schemes

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Received Apr. 20, 2006; revision accepted June 4, 2006

Abstract: A family of two-order Hermite vector-interpolating subdivision schemes is proposed and its convergence and continuity are analyzed. The iterative level can be estimated for given error. The sufficient conditions of C^2 continuity are proved. Geometric features of subdivision curves, such as line segments, cusps and inflection points, are obtained by appending some conditions to initial vectorial Hermite sequence. An algorithm is presented for generating geometric features. For an initial sequence of two-order Hermite elements from unit circle, the numerical error of the 4th subdivided level is $O(10^{-4})$.

Key words: Two-order vectorial Hermite element, Hermite-interpolating subdivision schemes, Geometric features
doi:10.1631/jzus.2006.A1566 **Document code:** A **CLC number:** TP391

INTRODUCTION

Along with the great progress in 3D data measurement device and the popularization of complex solid modelling, recursive subdivision has been a research focus in CAGD and CG (Wang *et al*., 2001). However, most interpolating subdivision schemes are Lagrange-type schemes whose smoothness is not easily verified. The geometric features of this kind of subdivided curves and surfaces are difficult to be generated, which restricts the application of subdivision methods.

From an initial sequence of Hermite elements (i.e. elements containing function values and associated derivatives), Hermite-type subdivision schemes recursively generate refined sequences of Hermite elements (Merrien, 1992; 1999). Dyn and Levin (1995; 1999) and Jüttler and Schwanecke (2002) studied them and gave sufficient and necessary conditions of convergence and C^k continuity. Hermite vector-interpolating subdivision scheme is a useful tool for spacial curve design. Curves with geometric features are required in geometric design (Zhang, 2003). In this paper we construct the initial sequence of two-order vectorial Hermite elements from a sequence of spacial points, and propose two-order Hermite vector-interpolating subdivision schemes. The convergence, C^2 smoothness and geometric properties are analyzed.

TWO-ORDER HERMITE VECTOR-INTERPOLATING SUBDIVISION SCHEMES

The sequence $\{\boldsymbol{P}_i\}_{i=0}^n$ is obtained by 3D data measurement device. The tangential vector associated to $\boldsymbol{P}_i$ is denoted as $\boldsymbol{D}_i^1$ (Shi, 2001), estimated by

$$\boldsymbol{D}_i^1 = (1-\alpha)\Delta\boldsymbol{P}_{i-1} + \alpha\Delta\boldsymbol{P}_i, \tag{1}$$

where $\Delta\boldsymbol{P}_i=\boldsymbol{P}_{i+1}-\boldsymbol{P}_i$, $A=|\Delta\boldsymbol{P}_{i-2}\times\Delta\boldsymbol{P}_{i-1}|$, $B=|\Delta\boldsymbol{P}_i\times\Delta\boldsymbol{P}_{i+1}|$, $\alpha=A/(A+B)$.

Specially, if $\boldsymbol{P}_0=\boldsymbol{P}_n$, then $\boldsymbol{P}_i=\boldsymbol{P}_{i+1}$ ($i=-2, -1, \ldots, 2$).

According to Bessel boundary condition,

$$\Delta\boldsymbol{P}_{-1} = 2\Delta\boldsymbol{P}_0 - \Delta\boldsymbol{P}_1,\ \Delta\boldsymbol{P}_{-2} = 2\Delta\boldsymbol{P}_{-1} - \Delta\boldsymbol{P}_0,$$
$$\Delta\boldsymbol{P}_n = 2\Delta\boldsymbol{P}_{n-1} - \Delta\boldsymbol{P}_{n-2},\ \Delta\boldsymbol{P}_{n+1} = 2\Delta\boldsymbol{P}_n - \Delta\boldsymbol{P}_{n-1}.$$

The tangential vectors sequence $\{\boldsymbol{D}_i^2\}_{i=0}^n$ is derived from $\{\boldsymbol{D}_i^1\}_{i=0}^n$ by the above method. So $\{\boldsymbol{D}_i^2\}_{i=0}^n$ is the estimative second derivative vectors sequence of $\{\boldsymbol{P}_i\}_{i=0}^n$.

Thus, we construct initial two-order vectorial Hermite sequence $\{\boldsymbol{P}_i, \boldsymbol{D}_i^1, \boldsymbol{D}_i^2\}_{i=0}^n$.

Then, we parameterize $\{\boldsymbol{P}_i, \boldsymbol{D}_i^1, \boldsymbol{D}_i^2\}_{i=0}^n$ by length of chord as follows

$$\begin{cases} t_0 = 0, \\ t_i = t_{i-1} + \|\Delta \boldsymbol{P}_{i-1}\|, \ i = 1,\ 2,\ ...,\ n \end{cases} \tag{2}$$

We define two-order Hermite vector-interpolating subdivision schemes.

Definition 1 Given $\{\boldsymbol{P}_i, \boldsymbol{D}_i^1, \boldsymbol{D}_i^2\}_{i=0}^n$ and $\{t_i\}_{i=0}^n$, the sequence $\{\boldsymbol{P}_i^{k+1}, \boldsymbol{D}_i^{1,k+1}, \boldsymbol{D}_i^{2,k+1}\}$ at the $(k+1)$th refinement level is recursively defined as follows:

$$\begin{cases} \boldsymbol{P}_{2i}^{k+1} = \boldsymbol{P}_i^k, \\ \boldsymbol{D}_{2i}^{1,k+1} = \boldsymbol{D}_i^{1,k}, \\ \boldsymbol{D}_{2i}^{2,k+1} = \boldsymbol{D}_i^{2,k}, \end{cases}$$
$$i = 0,\ ...,\ 2^k n,$$

$$\begin{cases} \boldsymbol{P}_{2i+1}^{k+1} = \dfrac{1}{2}(\boldsymbol{P}_i^k + \boldsymbol{P}_{i+1}^k) + \lambda h_j^k (\boldsymbol{D}_i^{1,k} - \boldsymbol{D}_{i+1}^{1,k}) \\ \qquad + \dfrac{8\lambda - 1}{16}(h_j^k)^2 (\boldsymbol{D}_i^{2,k} + \boldsymbol{D}_{i+1}^{2,k}), \\ \boldsymbol{D}_{2i+1}^{1,k+1} = (1-\mu)\dfrac{(\boldsymbol{P}_{i+1}^k - \boldsymbol{P}_i^k)}{h_j^k} + \mu \dfrac{(\boldsymbol{D}_i^{1,k} + \boldsymbol{D}_{i+1}^{1,k})}{2} \\ \qquad + \gamma \cdot h_j^k (\boldsymbol{D}_{i+1}^{2,k} - \boldsymbol{D}_i^{2,k}), \\ \boldsymbol{D}_{2i+1}^{2,k+1} = (1-\omega)\dfrac{(\boldsymbol{D}_{i+1}^{1,k} - \boldsymbol{D}_i^{1,k})}{h_j^k} + \omega \dfrac{(\boldsymbol{D}_i^{2,k} + \boldsymbol{D}_{i+1}^{2,k})}{2}, \end{cases}$$
$$j=0,\ \ldots,\ n-1;\ i=2^k j,\ \ldots,\ 2^k(j+1)-1, \tag{3}$$

where $h_j^k = \dfrac{t_{j+1} - t_j}{2^k}$, $\boldsymbol{P}_i^0 = \boldsymbol{P}_i$, $\boldsymbol{D}_i^{1,0} = \boldsymbol{D}_i^1$, $\boldsymbol{D}_i^{2,0} = \boldsymbol{D}_i^2$, λ, μ, γ and ω are four control factors.

Different from the standard vector schemes, the scheme Eq.(3) has non-stationary rules, which depend on the refinement level k. $\boldsymbol{P}(t)$, $\boldsymbol{D}^1(t)$ and $\boldsymbol{D}^2(t)$ are defined as the limits of Eq.(3). They are all defined on the dyadic set $B = \bigcup_{k=0}^{\infty} B_k$, where

$$B_k = \left\{ t_i + \frac{j(t_{i+1} - t_i)}{2^k},\ j = 0,\ ...,\ 2^k \right\},\ i=0,\ \ldots,\ n.$$

Example 1 When λ=5/32, μ=−7/8, γ=1/32, ω=−1/2, the limit curve is parametric Hermite interpolatory spline of fifth degree.

CONVERGENCE AND CONTINUITY

Definition 2 Scheme Eq.(3) is C^2 if there exists a vectorial function $\boldsymbol{P}(t) \in C^2[t_0, t_n]$ satisfying that

$$\boldsymbol{\Phi}_i^k = (\boldsymbol{P}_i^k, \boldsymbol{D}_i^{1,k}, \boldsymbol{D}_i^{2,k})^{\mathrm{T}} = (\boldsymbol{P}(t_{ij}^k), \boldsymbol{P}'(t_{ij}^k), \boldsymbol{P}''(t_{ij}^k))^{\mathrm{T}},$$

where $j=0,\ \ldots,\ n-1$; $k \in \mathbb{Z}_+$; $i=2^k j,\ \ldots,\ 2^k(j+1)-1$.

Parameters corresponding to the elements $\{\boldsymbol{P}_i^k, \boldsymbol{D}_i^{1,k}, \boldsymbol{D}_i^{2,k}\}_{i=2^k j}^{2^k(j+1)}$ are $t_j + \dfrac{(i-2^k j)(t_{j+1} - t_j)}{2^k}$, marked as t_{ij}^k. Connect $\boldsymbol{P}_i^k$ successively to make a piecewise linear vectorial function $\boldsymbol{P}^k(t)$, when $t \in [t_0, t_n]$, $\boldsymbol{P}^k(t_{ij}^k) = \boldsymbol{P}_i^k$. The derivative functions are $\boldsymbol{D}^{1,k}(t)$=$(\boldsymbol{P}^k(t))'$, $\boldsymbol{D}^{2,k}(t)$=$(\boldsymbol{D}^{1,k}(t))'$=$(\boldsymbol{P}^k(t))''$, where $\boldsymbol{D}^{1,k}(t_{ij}^k)$=$\boldsymbol{D}_i^{1,k}$ and $\boldsymbol{D}^{2,k}(t_{ij}^k) = \boldsymbol{D}_i^{2,k}$. So the limits of Eq.(3) are $\boldsymbol{P}(t)$= $\lim_{k\to\infty} \boldsymbol{P}^k(t)$, $\boldsymbol{D}^1(t) = \lim_{k\to\infty} \boldsymbol{D}^{1,k}(t)$ and $\boldsymbol{D}^2(t) = \lim_{k\to\infty} \boldsymbol{D}^{2,k}(t)$.

Matrix $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are defined as follows:

$$\boldsymbol{A}_1 = \begin{pmatrix} 4\lambda + 4\gamma + \frac{1}{2} & 4\gamma - \mu + \frac{1}{2} & -4\lambda + 4\gamma + \frac{1}{2} \\ -8\lambda - 4\gamma + 1 & -4\gamma + \mu + 1 & 8\lambda - 4\gamma - 1 \\ 4\lambda - \frac{\omega}{2} - \frac{1}{2} & -1 & -4\lambda + \frac{\omega}{2} + \frac{1}{2} \end{pmatrix},$$

$$\boldsymbol{A}_2 = \begin{pmatrix} -4\lambda + \frac{\omega}{2} + \frac{1}{2} & -1 & 4\lambda - \frac{\omega}{2} - \frac{1}{2} \\ 8\lambda - 4\gamma - 1 & -4\gamma + \mu + 1 & -8\lambda - 4\gamma + 1 \\ -4\lambda + 4\gamma + \frac{1}{2} & 4\gamma - \mu + \frac{1}{2} & 4\lambda + 4\gamma + \frac{1}{2} \end{pmatrix}.$$

Theorem 1 If λ, μ, γ and ω are all fixed, and there exists a positive integer L, satisfying

$$\|\boldsymbol{A}_1^{\rho_1} \boldsymbol{A}_2^{\rho_2}\|_\infty < 1,\ \forall \rho_1,\ \rho_2 \in \mathbb{Z}_+,\ \rho_1 + \rho_2 = L, \tag{4}$$

then the scheme Eq.(3) is C^2.

Proof The kth generation Hermite elements are $\{\boldsymbol{P}_i^k, \boldsymbol{D}_i^{1,k}, \boldsymbol{D}_i^{2,k}\}_{i=2^k j}^{2^k(j+1)}$ (j=0, …, n−1). Assume

$$\boldsymbol{\Psi}_i^k = (\boldsymbol{D}_i^{2,k}, d_1\boldsymbol{D}_i^{1,k}, d_2\boldsymbol{D}_i^{1,k})^{\mathrm{T}},$$

where

$$d_1\boldsymbol{D}_i^{1,k} = \frac{2}{h_j^k}\left(\boldsymbol{D}_{i+1}^{1,k} - \frac{\boldsymbol{P}_{i+1}^k - \boldsymbol{P}_i^k}{h_j^k}\right),$$

$$d_2\boldsymbol{D}_i^{1,k} = \frac{2}{h_j^k}\left(\frac{\boldsymbol{P}_{i+1}^k - \boldsymbol{P}_i^k}{h_j^k} - \boldsymbol{D}_i^{1,k}\right).$$

$\boldsymbol{\Psi}_i^k$ is defined as

$$\boldsymbol{\Psi}_{2i}^{k+1} = \begin{pmatrix} 1 & 0 & 0 \\ -4\lambda - 4\gamma + 1/2 & 4\lambda + \mu & 4\lambda - \mu \\ 4\lambda - 1/2 & -4\lambda & -4\lambda + 2 \end{pmatrix} \boldsymbol{\Psi}_i^k + \begin{pmatrix} 0 & 0 & 0 \\ -4\lambda + 4\gamma + 1/2 & 0 & 0 \\ 4\lambda - 1/2 & 0 & 0 \end{pmatrix} \boldsymbol{\Psi}_{i+1}^k,$$

$$\boldsymbol{\Psi}_{2i+1}^{k+1} = \begin{pmatrix} \omega/2 & 1/2 - \omega/2 & 1/2 - \omega/2 \\ 4\lambda - 1/2 & -4\lambda + 2 & -4\lambda \\ -4\lambda + 4\gamma + 1/2 & 4\lambda - \mu & 4\lambda + \mu \end{pmatrix} \boldsymbol{\Psi}_i^k + \begin{pmatrix} \omega/2 & 0 & 0 \\ 4\lambda - 1/2 & 0 & 0 \\ -4\lambda - 4\gamma + 1/2 & 0 & 0 \end{pmatrix} \boldsymbol{\Psi}_{i+1}^k,$$

where i=0, …, $2^k n$−1.

Setting $\boldsymbol{u}_{3i}^k = \boldsymbol{D}_i^{2,k}$, $\boldsymbol{u}_{3i+1}^k = d_1\boldsymbol{D}_i^{1,k}$, $\boldsymbol{u}_{3i+2}^k = d_2\boldsymbol{D}_i^{1,k}$, we define a subdivision scheme $\boldsymbol{u}^{k+1}=\boldsymbol{S}\boldsymbol{u}^k$. That is

$$[\boldsymbol{u}_{3i}^{k+1} \quad \boldsymbol{u}_{3i+1}^{k+1} \quad \boldsymbol{u}_{3i+2}^{k+1}]^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -4\lambda - 4\gamma + 1/2 & 4\lambda + \mu & 4\lambda - \mu & -4\lambda + 4\gamma + 1/2 \\ 4\lambda - 1/2 & -4\lambda & -4\lambda + 2 & 4\lambda - 1/2 \end{pmatrix} \cdot [\boldsymbol{u}_l^k \quad \boldsymbol{u}_{l+1}^k \quad \boldsymbol{u}_{l+2}^k \quad \boldsymbol{u}_{l+3}^k]^{\mathrm{T}},$$

when i is even;

$$[\boldsymbol{u}_{3i}^{k+1} \quad \boldsymbol{u}_{3i+1}^{k+1} \quad \boldsymbol{u}_{3i+2}^{k+1}]^{\mathrm{T}} = \begin{pmatrix} \omega/2 & (1-\omega)/2 & (1-\omega)/2 & \omega/2 \\ 4\lambda - 1/2 & -4\lambda + 2 & -4\lambda & 4\lambda - 1/2 \\ -4\lambda + 4\gamma + 1/2 & 4\lambda - \mu & 4\lambda + \mu & -4\lambda - 4\gamma + 1/2 \end{pmatrix} \cdot [\boldsymbol{u}_l^k \quad \boldsymbol{u}_{l+1}^k \quad \boldsymbol{u}_{l+2}^k \quad \boldsymbol{u}_{l+3}^k]^{\mathrm{T}},$$

when i is odd,

where $l = \left\lfloor \frac{3i}{2} \right\rfloor$.

We find that $\boldsymbol{S}$ does not depend on the subdivision level k, so $\boldsymbol{S}$ is the stationary Lagrange-type subdivision schemes.

The difference scheme $\Delta\boldsymbol{S}$ associated to the difference sequence $\Delta\boldsymbol{u}^k = \boldsymbol{u}_{+1}^k - \boldsymbol{u}^k$ is

$$\begin{pmatrix} \Delta\boldsymbol{u}_{3i}^{k+1} \\ \Delta\boldsymbol{u}_{3i+1}^{k+1} \\ \Delta\boldsymbol{u}_{3i+2}^{k+1} \end{pmatrix} = \boldsymbol{A}_1 \begin{pmatrix} \Delta\boldsymbol{u}_l^k \\ \Delta\boldsymbol{u}_{l+1}^k \\ \Delta\boldsymbol{u}_{l+2}^k \end{pmatrix}, \text{ for even } i;$$

$$\begin{pmatrix} \Delta\boldsymbol{u}_{3i}^{k+1} \\ \Delta\boldsymbol{u}_{3i+1}^{k+1} \\ \Delta\boldsymbol{u}_{3i+2}^{k+1} \end{pmatrix} = \boldsymbol{A}_2 \begin{pmatrix} \Delta\boldsymbol{u}_l^k \\ \Delta\boldsymbol{u}_{l+1}^k \\ \Delta\boldsymbol{u}_{l+2}^k \end{pmatrix}, \text{ for odd } i.$$

According to the uniform convergent theory of stationary Lagrange-type subdivided schemes (Dyn *et al*., 1991), for some λ, μ, γ and ω, if there exists a positive integer L such that

$$\| \boldsymbol{A}_1^{\rho_1} \boldsymbol{A}_2^{\rho_2} \|_\infty < 1, \ \forall \rho_1, \ \rho_2 \in \mathbb{Z}_+ \text{ and } \rho_1 + \rho_2 = L,$$

then the difference scheme $\Delta\boldsymbol{S}$ uniformly converges to zero vectorial function for arbitrary initial data, $\boldsymbol{S}$ is uniformly convergent.

Hence, we have verified that $\{\boldsymbol{D}_i^{2,k}\}_{i=0}^{2^k n-1}$, $\{d_1\boldsymbol{D}_i^{1,k}\}_{i=0}^{2^k n-1}$ and $\{d_2\boldsymbol{D}_i^{1,k}\}_{i=0}^{2^k n-1}$ all converge to $\boldsymbol{D}^2(t)$. In terms of Definition 2, the scheme Eq.(3) is C^2.

Remark 1 Two specific cases are given to illustrate Theorem 1 more concretely:

(1) When λ=5/32, μ=−7/8, γ=1/32, and ω=−1/2, L=5 is the minimal positive integer that makes the condition Eq.(4) hold;

(2) When λ=1/8, γ=1/32, ω and μ are in the shadow, the condition Eq.(4) holds for the minimal L=2 (Fig.1). The boundary curves of the shadow are:

$$\mu^2 + 11/8\mu + 1/8\omega - 5/4 = 0,$$
$$\mu^2 + 15/8\mu + 1/8\omega - 5/16 = 0,$$
$$1/2\omega\mu - 13/16\omega - \mu - 7/8 = 0,$$
$$1/2\omega^2 - 1/2\omega\mu + 21/16\omega + \mu - 1/8 = 0.$$

Theorem 2 For given ε, suppose λ, μ, γ and ω satisfy Eq.(4), when the subdivision level k≥1+

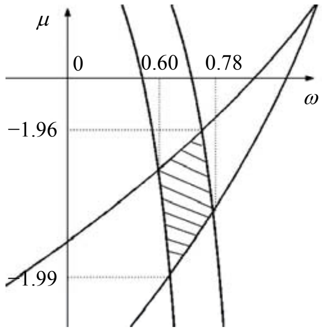


Fig.1 L=2, the range of values of ω and μ

$\frac{\ln(d\cdot s)-\ln\varepsilon}{\ln 2}$, the iteration can stop. Both d and s are constants.

Proof Since the scheme Eq.(3) interpolates the sequence of Hermite elements, the maximum distance between $\boldsymbol{P}^k(t)$ and $\boldsymbol{P}^{k+1}(t)$ is only attained at $t^{k+1}_{2i+1,j}$.

Let

$$a_0=|\lambda|,\ b_0=|\lambda|(|\mu|+1)+\frac{|(8\lambda-1)(1-\omega)|}{16},$$

$$c_0=2|\lambda\gamma|+\frac{|8\lambda-1|(|\omega|+1)}{32},$$

$$\begin{cases} a_l=a_{l-1}+b_{l-1},\\ b_l=4|\lambda(1-\mu)|a_{l-1}+(|\mu|+1)b_{l-1}+|1-\omega|c_{l-1},\\ c_l=\frac{1}{4}|(1-\mu)(8\lambda-1)|a_{l-1}+2|\gamma|b_{l-1}+\frac{1}{2}(|\omega|+1)c_{l-1},\end{cases}$$

$$l=1,\ \ldots,\ k-1.$$

Using the scheme Eq.(3), we get

$$\begin{aligned}
&\left\|\boldsymbol{P}^{k+1}(t)-\boldsymbol{P}^k(t)\right\|_\infty\\
&=\max_{0\le j\le n-1}\left\{\max_{2^k j\le i\le 2^k(j+1)-1}\left\|\boldsymbol{P}^{k+1}(t^{k+1}_{2i+1,j})-\boldsymbol{P}^k(t^{k+1}_{2i+1,j})\right\|\right\}\\
&=\max_{0\le j\le n-1}\left\{\max_{2^k j\le i\le 2^k(j+1)-1}\left\|\boldsymbol{P}^{k+1}_{2i+1}-\frac{1}{2}(\boldsymbol{P}^k_i+\boldsymbol{P}^k_{i+1})\right\|\right\}\\
&=\max_{0\le j\le n-1}\left\{\max_{2^k j\le i\le 2^k(j+1)-1}\left\|\lambda h^k_j(\boldsymbol{D}^{1,k}_i-\boldsymbol{D}^{1,k}_{i+1})\right.\right.\\
&\quad\left.\left.+\frac{8\lambda-1}{16}(h^k_j)^2(\boldsymbol{D}^{2,k}_i+\boldsymbol{D}^{2,k}_{i+1})\right\|\right\}\\
&\le 2\max_{0\le j\le n-1}\left\{h^k_j\left[\max_{2^k j\le i\le 2^k(j+1)}\left(|\lambda|\cdot\left\|\boldsymbol{D}^{1,k}_i\right\|\right)+\frac{|8\lambda-1|}{16}h^k_j\left\|\boldsymbol{D}^{2,k}_i\right\|\right]\right\}\\
&\le 2\max_j\left\{h^k_j\left(\frac{|1-\mu|}{h^{k-1}_j}a_0\max_i\left\|\boldsymbol{P}^{k-1}_{i+1}-\boldsymbol{P}^{k-1}_i\right\|\right.\right.\\
&\quad\left.\left.+b_0\max_i\left\|\boldsymbol{D}^{1,k-1}_i\right\|+c_0h^{k-1}_j\max_i\left\|\boldsymbol{D}^{2,k-1}_i\right\|\right)\right\}\\
&\le 2\max_j\left\{h^k_j\left(\frac{|1-\mu|}{h^{k-2}_j}a_1\max_i\left\|\boldsymbol{P}^{k-2}_{i+1}-\boldsymbol{P}^{k-2}_i\right\|\right.\right.\\
&\quad\left.\left.+b_1\max_i\left\|\boldsymbol{D}^{k-2}_i\right\|+c_1h^{k-2}_j\max_i\left\|\boldsymbol{D}^{2,k-2}_i\right\|\right)\right\}\\
&\le\ldots\\
&\le 2\max_{0\le j\le n-1}\left\{h^k_j\left(|1-\mu|/h^0_j\cdot a_{k-1}\left\|\boldsymbol{P}^0_{j+1}-\boldsymbol{P}^0_j\right\|\right.\right.\\
&\quad\cdot\left(+b_{k-1}\max\left\{\left\|\boldsymbol{D}^{1,0}_j\right\|,\left\|\boldsymbol{D}^{1,0}_{j+1}\right\|\right\}\right.\\
&\quad\left.\left.+c_{k-1}h^0_j\max\left\{\left\|\boldsymbol{D}^{2,0}_j\right\|,\left\|\boldsymbol{D}^{2,0}_{j+1}\right\|\right\}\right)\right\}\\
&\le 2^{1-k}sd,
\end{aligned}$$

where

$$s=\max_{0\le j\le n-1}\left(\left\|\boldsymbol{P}_{j+1}-\boldsymbol{P}_j\right\|+h^0_j\max\left\{\left\|\boldsymbol{D}^1_j\right\|,\left\|\boldsymbol{D}^1_{j+1}\right\|\right\}\right.$$
$$\left.+(h^0_j)^2\max\left\{\left\|\boldsymbol{D}^2_j\right\|,\left\|\boldsymbol{D}^2_{j+1}\right\|\right\}\right)$$

$$d=\max\{|1-\mu|a_{k-1},b_{k-1},c_{k-1}\}.$$

When λ, μ, γ and ω satisfy Theorem 1, the scheme Eq.(3) is C^2.

If $k\ge 1+\frac{\ln(d\cdot s)-\ln\varepsilon}{\ln 2}$, then $\|\boldsymbol{P}^{k+1}(t)-\boldsymbol{P}^k(t)\|_\infty\le 2^{1-k}sd\le\varepsilon$. The refinement process can stop.

GEOMETRIC PROPERTIES AND ALGORITHM

Property 1 If there exist two elements whose tangential vectors are both unit and in the direction of $\boldsymbol{P}_j\boldsymbol{P}_{j+1}$, and $\boldsymbol{D}^2_j=\boldsymbol{D}^2_{j+1}=\boldsymbol{0}$, then the limit curve of Eq.(3) includes a line segment $\boldsymbol{P}_j\boldsymbol{P}_{j+1}$.

Property 2 If we append some conditions to a certain element: $\boldsymbol{P}'_j=\boldsymbol{P}_j$, $(\boldsymbol{D}^1_j)'\ne\boldsymbol{D}^1_j$, $(\boldsymbol{D}^2_j)'=\boldsymbol{0}$, and generate subdivision curves on $[t_{j-1},t_j]$ and $[t_j,t_{j+1}]$ respectively from $\{\boldsymbol{P}_{j-1},\boldsymbol{D}^1_{j-1},\boldsymbol{D}^2_{j-1}\}$, $\{\boldsymbol{P}_j,\boldsymbol{D}^1_j,\boldsymbol{D}^2_j\}$ and $\{\boldsymbol{P}'_j,(\boldsymbol{D}^1_j)',(\boldsymbol{D}^2_j)'\}$, $\{\boldsymbol{P}_{j+1},\boldsymbol{D}^1_{j+1},\boldsymbol{D}^2_{j+1}\}$, the limit curve is only C^0 at $\boldsymbol{P}_j$ that is a cusp.

Property 3 If there are some Hermite elements of the initial sequence, which subject to (1) $\boldsymbol{P}_{j-1}$, $\boldsymbol{P}_j$, $\boldsymbol{P}_{j+1}$ are not co-linear; (2) $\boldsymbol{D}_j^1$ and $\boldsymbol{D}_{j-1}^1$ are both unit and in the direction of $\boldsymbol{P}_{j-1}\boldsymbol{P}_j$, and $\boldsymbol{D}_{j-1}^2=\boldsymbol{D}_j^2=\boldsymbol{0}$; (3) $\boldsymbol{D}_{j+1}^1$ is unit and in the direction of $\boldsymbol{P}_j\boldsymbol{P}_{j+1}$, and $\boldsymbol{D}_{j+1}^2=\boldsymbol{0}$; and we set $\boldsymbol{P}_j'=\boldsymbol{P}_j$, $(\boldsymbol{D}_j^1)'=\boldsymbol{D}_{j+1}^1$, $(\boldsymbol{D}_j^2)'=\boldsymbol{0}$, then the limit curve includes a corner that is just at $\boldsymbol{P}_j$.

Property 4 For the plane initial sequence $\{\boldsymbol{P}_i,\boldsymbol{D}_i^1,\boldsymbol{D}_i^2\}_{i=0}^n$, we let $\|\boldsymbol{D}_i^1\|=1$, If there exists $\boldsymbol{D}_j^2=\boldsymbol{0}$, $j\in(1, \ldots, n-1)$, then the limit curve includes an inflection point at $\boldsymbol{P}_j$; if we append conditions, $\boldsymbol{D}_j^2=\begin{pmatrix}0 & -\kappa\\ \kappa & 0\end{pmatrix}\boldsymbol{D}_j^1$, then the curvature is κ at $\boldsymbol{P}_j$.

Property 5 For the plane initial sequence $\{\boldsymbol{P}_i,\boldsymbol{D}_i^1,\boldsymbol{D}_i^2\}_{i=0}^n$, the scheme Eq.(3) recursively generates the refinement Hermite elements and corresponding offset elements simultaneously.

In terms of above properties, we give an algorithm to generate the different geometric features.

Algorithm:

Step 1: Choose the suitable factors λ, μ, γ and ω, according to Theorem 1;

Step 2: By Eqs.(1) and (2), construct $\{\boldsymbol{P}_i,\boldsymbol{D}_i^1,\boldsymbol{D}_i^2\}_{i=0}^n$ and $\{t_i\}_{i=0}^n$;

Step 3: According to Theorem 2, calculate the subdivision level k for given error ε;

Step 4: Set up flags for all elements, denoted as *flag*[j] (j=1, …, $n-1$), six cases are given:

(1) *flag*[j]=*flag*[j+1]=1, when a line segement is generated between $\boldsymbol{P}_j$ and $\boldsymbol{P}_{j+1}$;

(2) *flag*[j]=2, when a cusp is generated at $\boldsymbol{P}_j$;

(3) *flag*[j]=3, when a corner is generated at $\boldsymbol{P}_j$;

(4) *flag*[j]=4, when the curvature at $\boldsymbol{P}_j$ is κ;

(5) *flag*[j]=5, when its associated offset element is generated;

(6) *flag*[j]=0, for else;

Step 5: According to Step 4, append corresponding conditions to the initial elements;

Step 6: Apply the scheme Eq.(3) to generate the refinement elements;

Step 7: Connect points at the kth generation to form the subdivided curve.

EXPERIMENTS AND ANALYSIS OF NUMERICAL ERROR

Figs.2~6 show that subdivided curves are generated by the above algorithm, where k=4, λ=0.125, μ=−1/2, γ=1/32, and ω=−1/2. The dots in figures denote the initial points. The numerical errors of subdivided curves for the initial sequence sampled from unit circle are listed in Tables 1 and 2.

(a) (b)

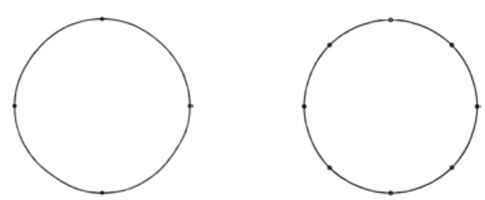
Fig.2 (a) The second generation of sequence; (b) The associated offset of sequence

(a) (b)

Fig.3 (a) The first generation of sequence; (b) The curve with line segments

Fig.4 The recursive curve with a cusp

Fig.5 The refinement curve with a corner



(a) (b)

Fig.6 The subdivision unit circle from (a) 4 elements and (b) 8 elements

Table 1 Numerical errors of different number of initial elements

Number of initial elements n	Error between generation points and associated ones in unit circle
4	$O(10^{-2})$
5	$O(10^{-3})$
6	$O(10^{-3})$
7	$O(10^{-4})$
8	$O(10^{-4})$

Refinement level: k=4; Control factors: λ=1/8, μ=−1, γ=1/32, and ω=−1/2

Table 2 Numerical errors of different refinement level

Refinement level k	Error between generation points and associated ones in unit circle
3	$O(10^{-4})$
4	$O(10^{-4})$
5	$O(10^{-4})$
6	$O(10^{-4})$
7	$O(10^{-4})$

Number of initial elements: n=7; Control factors: λ=1/8, μ=−1, γ=1/32, and ω=−1/2

CONCLUSION

In this paper a family of non-stationary two-order Hermite vector-interpolating subdivision schemes with four control factors is presented. Because the schemes can design both spacial points and associated derivative vectors, the shape control of subdivision curves becomes more convenient and the limit curves have better approximation property. When we choose the factors satisfying continuous conditions in Theorem 1, subdivision curves can be C^2. The subdivision level can be pre-estimated for a given error. Compared with other schemes, this kind subdivision can generate curves with geometric features only by adding auxiliary conditions to the initial sequence instead of changing the scheme itself. For the initial sequence sampled from unit circle, the numerical error of the 4th subdivided level can be $O(10^{-4})$. We extend the application of Hermite interpolatory subdivision schemes. Furthermore, the Hermite vector-interpolating subdivision schemes of higher-order continuity need more research in the future.

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