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Artificial perturbation for solving the Korteweg-de Vries equation

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Abstract: A perturbation method is introduced in the context of dynamical system for solving the nonlinear Korteweg-de Vries (KdV) equation. Best efficiency is obtained for few perturbative corrections. It is shown that, the question of convergence of this approach is completely guaranteed here, because a limited number of term included in the series can describe a sufficient exact solution. Comparisons with the solutions of the quintic spline, and finite difference are presented.

Key words: Perturbation, Taylor series, Quintic spline, Korteweg-de Vries (KdV) equation
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INTRODUCTION

The famous Korteweg-de Vries (KdV) equation (Korteweg and de Vries, 1895; Freeman and Johnson, 1970; Zerarka, 1996) is paid particular attention to in recent years because of its wide use in the area of engineering, physical systems, laser theory, observation of a solitary wave, description of the solitary waves propagation in shallow water, etc. (Boiti *et al*., 1988; Hirota, 1971; Gardner *et al*., 1967; El-Zoheiry *et al*., 1994; Scott *et al*., 1973; Iskandar, 1989). It is also shown that the solitons can occur in hydrodynamics, plasma theory, collision process, and nonlinear optics. Many powerful methods have also been developed (Fornberg and Whitham, 1978; Tang *et al*., 2001; Cherruault *et al*., 1992; Fang and Yao, 1992).

This paper presents a different approach to construct solutions in an explicit manner by using the perturbation method to separate the nonlinear term. In order to illustrate the efficiency of this approach, we examine a test numerical example. Comparison of the results with the finite difference (Iskandar, 1989), the quintic spline (El-Zoheiry *et al*., 1994) and the exact solutions are presented.

APPLICATION OF PERTURBATION METHOD

The generalized KdV equation can be written in the form

$$\begin{gathered} U_t + \eta(h(U))_x + \gamma U_{xxx} = 0, \\ x \in [-b, b],\ t > 0,\ U(x,0) = g(x), \end{gathered} \tag{1}$$

η and γ are constants. We start with our development with $h(U)=U^2/2$.

The separation of the nonlinear term allows us to obtain a system of linear differential equations which may be recursively solved.

This approach consists of introducing an auxiliary parameter ε ($0\le\varepsilon\le1$) and replacing Eq.(1) with the following analogous equation

$$v_t + \varepsilon\eta v \cdot v_x + \gamma v_{xxx} = 0, \tag{2}$$

with the solution $U(x,t)$ being of the form

$$U(x,t) = \lim_{\varepsilon\to1} v(x,t,\varepsilon), \tag{3}$$

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(After all calculations we set ε=1).

Assuming the existence of the solutions of the differential equations at all values of ε defined above, we may see that U will be modified and depend now on ε.

The solutions v in Eq.(2) may be made soluble by expansion of the solution as a Taylor series in the parameter ε. In the practical calculations we keep only the M-terms first in the series to describe the solution v, noted v_M as

$$v_M(x,t,\varepsilon)=\sum_{i=0}^{M}\varepsilon^i v^{(i)}, \tag{4}$$

where

$$v^{(0)}=v_M(x,t,0);\ v^{(n)}=\frac{1}{n!}\frac{\partial n_{v_M}}{\partial\varepsilon^n}\bigg|_{\varepsilon=0};\ n=1,\ \ldots,\ M,$$

where $\frac{\partial n_{v_M}}{\partial\varepsilon^n}\Big|_{\varepsilon=0}$ represents the nth derivative of v_M evaluated at 0.

When we input the form of the solution Eq.(4) into the differential Eq.(2) and subsequently by equating like powers of ε, the following systems of equations are obtained

$$v_t^{(0)}+\gamma v_{xxx}^{(0)}=0, \tag{5}$$

$$v_t^{(1)}+\eta v^{(0)}\cdot v_x^{(0)}+\gamma v_{xxx}^{(1)}=0, \tag{6}$$

$$v_t^{(2)}+\eta(v^{(0)}\cdot v_x^{(1)}+v^{(1)}\cdot v_x^{(0)})+\gamma v_{xxx}^{(2)}=0, \tag{7}$$

$$\ldots,$$

where $v_\beta^{(\alpha)}=\frac{\partial v^{(\alpha)}}{\partial\beta}$.

It is now obvious that, from Eq.(5), the solution $v^{(0)}$ may be obtained. For example the linearized Eq.(5) may be resolved by using the Fourier transform:

$$v^{(0)}(x,t,0)=\int_{-\infty}^{+\infty}a(\omega,t)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega, \tag{8}$$

we obtain

$$a(\omega,t)=\mathrm{e}^{\mathrm{i}\gamma\omega^3x}a(\omega,0). \tag{9}$$

We get the solution $a(\omega,0)$ of the relation Eq.(9) by using the inverse Fourier transform for a given initial value $g(x)$. The solution of the original problem Eq.(5) is

$$v^{(0)}(x,t,0)=\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\omega(x+\gamma\omega^2t)}\left(\int_{-\infty}^{+\infty}g(y)\mathrm{e}^{-\mathrm{i}\omega y}\mathrm{d}y\right)\mathrm{d}\omega. \tag{10}$$

By introducing $v^{(0)}$ in Eq.(6) we obtain the solution $v^{(1)}$ which will be incorporated in Eq.(7) to get the solution $v^{(2)}$ and so on. It may be noted however that the separation of the nonlinear term from the initial equation enables us then to obtain a system of equations which is soluble and that the reconstruction of the final solution is then possible.

EXAMPLE

Let us consider as a testbed the case of the nonlinear dynamics of the solutions of Eq.(1).

Eq.(1) has the boundary conditions

$$U(0,t)=0 \text{ and } U(X,t)=0,$$

where X is chosen to be large

$$g(x)=3\mu^{22}\,\mathrm{sech}^2(\mu x/2).$$

The exact solution takes the form (Scott *et al.*, 1973)

$$U(x,t)=3\mu^2\,\mathrm{sech}^2[(\mu x-\mu^3t)/2+\Phi].$$

We take X=400, μ^2=0.55, and Φ=0.

The results of this approach, the exact solution, the finite difference calculations (Iskandar, 1989) and the quintic spline method (El-Zoheiry *et al.*, 1994) are displayed in Figs.1 and 2, for the cases t=75 and t=150.

Results in Figs.1 and 2 show satisfactory agreement with those cited in references, but are more accurate compared with those of (Iskandar, 1989). It is noted that the discrepancy between the present results and exact results is not surprising but may be expected from the number of terms inserted in the series (M=4 for t=75 and t=150). On the other hand, note that more exact results may be expected from M>4. For convenience, we also show in Fig.3, the soliton solution with respect to the variables (x,t).

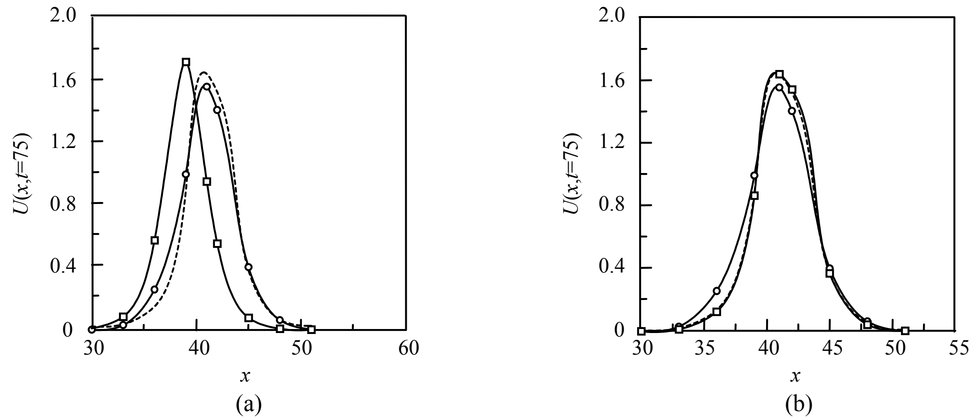

Fig.1 Representation of $U(x,t=75)$, case M=4. Dashed line is original curve, full circle is present work. (a) Full square is finite difference solution (Iskandar, 1989); (b) Full square is quintic spline solution (El-Zoheiry *et al.*, 1994)

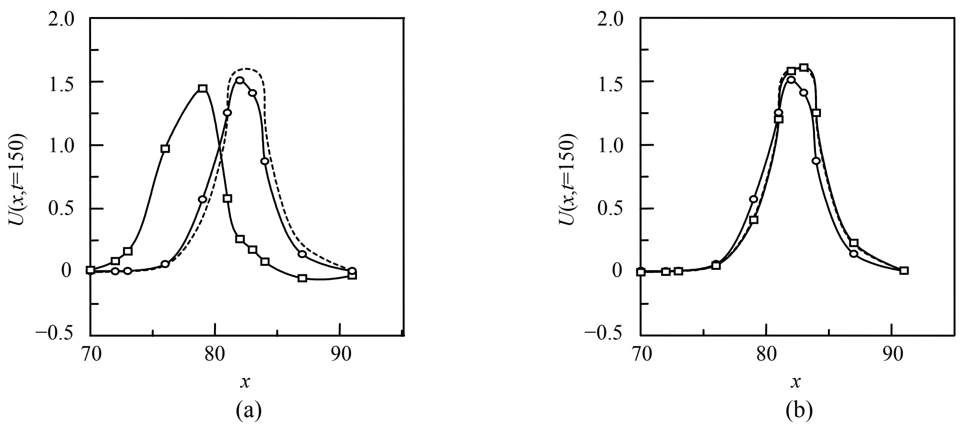

Fig.2 Representation of $U(x,t=150)$, case M=4. Dashed line is original curve, full circle is present work. (a) Full square is finite difference solution (Iskandar, 1989); (b) Full square is quintic spline solution (El-Zoheiry *et al.*, 1994)

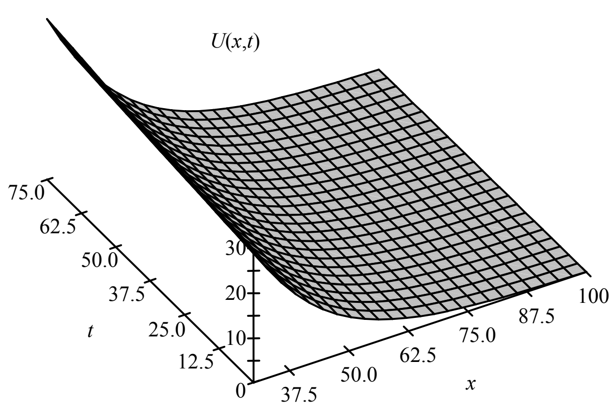

Fig.3 Representation of $U(x,t)$×150, case M=4

CONCLUSION

We have presented a Taylor series based perturbation method enabling getting the order-by-order correction terms of the original problem's approximated solution. In summary, we may conclude that, within the number of terms used in the series Eq.(4) (M=4), the solution suggested in this work can satisfactorily reproduce the function $U(x,t)$. The maximum accuracy in the approximate solution at minimum computational cost can be obtained after a small number of terms. For the equations of KdV type, it seems to us that one of the merits of this approach is that it provides a system of equations governing the complete separation of the KdV equations' nonlinear term. The use of this approach together with mathematical arguments may give very accurate results. Also, it is suggested that an extension of the current approach to coupled nonlinear differential equations will be interesting.

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